A FRIENDLY INTRODUCTION TO THE BORDERED CONTACT INVARIANT

AKRAM ALISHAHI, JOAN LICATA, INA PETKOVA, AND VERA VÉRTESI

ABSTRACT. We give a short introduction to the contact invariant in bordered Floer homology defined in [AFH+21]. The construction relies on a special class of foliated open books. We discuss a procedure to obtain such a foliated open book and present a definition of the contact invariant. We also provide a "local proof", through an explicit bordered computation, of the vanishing of the contact invariant for overtwisted structures.

1. INTRODUCTION

Many of the recent advances in classifying tight contact structures were made possible by the advent of Heegaard Floer homology in the early 2000s and the subsequent development of Floer-theoretic contact invariants. Using open books, Ozsváth and Szabó defined an invariant of closed contact three-manifolds [OS05]. Given a closed, contact manifold (M, ξ), this invariant is a class $c(\xi)$ in the Heegaard Floer homology $\widehat{HF}(-M)$. In [HKM09b], Honda, Kazez, and Matić gave an alternative description of $c(\xi)$, again using open books. This "contact class" was used to show that knot Floer homology detects both genus [OS04] and fiberedness [Ghi08, Ni07]. It also gives information about overtwistedness: if ξ is overtwisted, then $c(\xi) = 0$, whereas if ξ is Stein fillable, then $c(\xi) \neq 0$ [OS05]. The contact class was also used to distinguish notions of fillability: Ghiggini used it to construct examples of strongly symplectically fillable contact three-manifolds which do not have Stein fillings [Ghi05].

In [AFH+21], a contact invariant was defined in the bordered sutured Floer homology of a foliated contact three-manifold $(M, \xi, \mathcal{F})$ (that is, a contact manifold with a certain type of singular foliation on the boundary). We associate to a foliated contact three-manifold a bordered sutured manifold $(M, \Gamma, \mathcal{Z})$. The resulting sutures are particularly simple, so one can think of $(M, \Gamma, \mathcal{Z})$ as a bordered manifold $(M, \mathcal{Z})$ of a type slightly more general than in [LOT18] – the parametrization on the boundary uses n zero-handles and n two-handles, where n is half the number of elliptic points in the foliation $\mathcal{F}$. Below, we rephrase the main results of [AFH+21], translating from "bordered sutured" to "multipointed" language. Section 2 explores the correspondence between these two viewpoints in more detail.

Using a special decomposition of $(M, \xi, \mathcal{F})$ called a *sorted foliated open book*, one can construct an admissible multipointed bordered Heegaard diagram for the manifold $(M, \mathcal{Z})$ and identify a preferred generator. This preferred generator is an invariant of the contact structure.

Theorem 1 (cf. [AFH+21, Theorem 1])**.** *Let $(M, \xi, \mathcal{F})$ be a foliated contact three-manifold with associated bordered manifold $(M, \mathcal{Z})$. Then there are invariants $c_D(M, \xi, \mathcal{F})$ and $c_A(M, \xi, \mathcal{F})$ of the contact structure which are well defined homotopy equivalence classes in the multipointed bordered Floer homologies $\widetilde{CFD}(-M, \overline{\mathcal{Z}})$ and $\widetilde{CFA}(-M, \overline{\mathcal{Z}})$, respectively.*

Further, this generator "vanishes" for overtwisted manifolds, in the following sense.

2010 *Mathematics Subject Classification.* 57M27.

Key words and phrases. Heegaard Floer homology, open book, TQFT, contact topology.

AA was supported by NSF Grant DMS-2019396.

VV was supported by the ANR grant "Quantum topology and contact geometry".

Theorem 2 (cf. [AFH+21, Corollary 4]). *If $(M, \xi, \mathcal{F})$ is overtwisted, then the classes $c_D(M, \xi, \mathcal{F})$ and $c_A(M, \xi, \mathcal{F})$ are zero in $H_*(\widetilde{CFD}(-M, \overline{\mathcal{Z}}))$ and $H_*(\widetilde{CFA}(-M, \overline{\mathcal{Z}}))$, respectively.*

Given a pair of foliated contact three-manifolds $(M^L, \xi^L, \mathcal{F}^L)$ and $(M^R, \xi^R, \mathcal{F}^R)$ whose boundaries agree in an appropriate sense, there is a natural way to glue them to obtain a closed contact three-manifold (M, ξ). The contact invariants of the two foliated contact three-manifolds pair to recover the contact invariant of (M, ξ).

Theorem 3 (cf. [AFH+21, Theorem 2]). *The tensor product $c_A(M^L, \xi^L, \mathcal{F}^L) \boxtimes c_D(M^R, \xi^R, \mathcal{F}^R)$ recovers the contact invariant $c(M, \xi)$.*

In this short paper we give a hands-on introduction to the construction of the bordered contact invariant. Section 2 lays out the language of multipointed bordered Floer homology as a special case of bordered sutured Floer homology, laying the groundwork for a simplified description of the construction of the bordered contact invariant. In Section 3 we introduce foliated open books and develop an important example. We start with a relatively simple foliated open book for a neighborhood of an overtwisted disk. We then execute the stabilization process to produce a more complicated, but better-behaved, foliated open book, called *sorted*. In Section 4 we describe how to construct a Heegaard diagram from a sorted foliated open book and define a generator on it that represents the contact invariant. Finally, in Section 5 we use the sorted foliated open book from our earlier example to construct a Heegaard diagram, and we study the associated contact invariant via a local computation. We then apply this local computation, in conjunction with Theorem 3, to recover the following vanishing result.

Corollary 4. *Let (M, ξ) be a closed contact three-manifold. If ξ is overtwisted, then $c(\xi) = 0$.*

Note that in [HKM09a], vanishing of the sutured contact class is established for a neighborhood of an overtwisted disk. A sutured argument for the vanishing of $c(\xi)$ for overtwisted closed manifolds can then be obtained with the TQFT gluing map from [HKM08]. Our local construction explicitly constructs the "contact compatible" layer needed in the sutured setting, giving a bordered counterpart to the argument.

Acknowledgements. We are grateful to BIRS for hosting the workshop *Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4.*

2. Multipointed bordered Floer homology

As a stepping stone for defining link Floer homology, Ozsváth and Szabó defined a *multipointed* version of $\widehat{HF}$ denoted by $\widetilde{HF}$ [OS08]. This version is defined using Heegaard diagrams with multiple basepoints, and, given a closed, oriented three-manifold M, it is related to $\widehat{HF}(M)$ by the isomorphism

$$\widetilde{HF}(M, n) \cong \widehat{HF}(M) \otimes V^{n-1}.$$

Here, n is the number of basepoints and V is a 2-dimensional graded $\mathbb{Z}/2\mathbb{Z}$-vector space with generators in gradings 1 and 0; i.e., $V \cong H_*(S^1)$. Note that both $\widehat{HF}(M)$ and $\widetilde{HF}(M, n)$ are sutured Floer homologies of specific sutured manifolds corresponding to M. Specifically, let $M(n)$ be the sutured manifold obtained from M by removing n pairwise disjoint balls and adding as a suture one oriented simple closed curve on each resulting sphere boundary component. Then, we have $\widehat{HF}(M) \cong SFH(M(1))$ while $\widetilde{HF}(M, n) \cong SFH(M(n))$.

Lipshitz, Ozsváth, and Thurston define bordered Floer homology as an extension of $\widehat{HF}$ for three-manifolds with parametrized boundary [LOT18]. First, they associate a differential graded algebra $\mathcal{A}(\partial M)$ to the parametrization. Then, they define an $\mathcal{A}_\infty$-module, or *type A structure*,

$\widehat{CFA}(M)$ over $\mathcal{A}(\partial M)$, or equivalently a *type D structure* (roughly, a dg module) $\widehat{CFD}(M)$ over $\mathcal{A}(-\partial M)$. These invariants are constructed to satisfy a nice gluing formula which recovers $\widehat{HF}$. Specifically, if M is a closed three-manifold obtained by a gluing $M_1 \cup_\partial M_2$, then the derived tensor product $\widehat{CFA}(M_1)\widetilde{\otimes}_{\mathcal{A}(\partial M_1)}\widehat{CFD}(M_2)$ (which often has a smaller model denoted $\boxtimes$) is homotopy equivalent to $\widehat{CF}(M)$.

A generalization of bordered Floer homology, called bordered sutured Floer homology, was defined by Zarev [Zar09]. It is an invariant of three-manifolds whose boundary is "part sutured, part parametrized" and satisfies a gluing formula which recovers sutured Floer homology.

In this section, we introduce a *multipointed* theory for bordered Floer homology as a special case of bordered sutured Floer homology. First, we recall the definition of the boundary parametrization in bordered Floer homology. Let M be a three-manifold with boundary of genus k. A parametrization for ∂M consists of a disk $D \subset \partial M$; a basepoint $z \in \partial D$; and $2k$ pairwise disjoint properly embedded arcs $\amalg_{i=1}^{2k}\delta_i$ in $\partial M \setminus \mathrm{Int}(D)$ such that $M \setminus \left(D \cup \amalg_{i=1}^{2k}\delta_i\right)$ is an open disk. The parametrization data is recorded by a pointed matched circle $\mathcal{Z} = (Z, a, m)$, where $Z = \partial D$ with $z \in Z$, $a = \partial(\amalg_{i=1}^{2k}\delta_i)$ union of $4k$ points on Z, and M is a matching on a that pairs endpoints of the same arc δ_i.

Definition 2.1. A *pointed matched multicircle* is a triple $\mathcal{Z} = (Z, a, m)$ where $Z = \amalg_{i=1}^{n} Z_i$ is a union of n circles with a basepoint z_i on each Z_i, $a \subset Z$ is a set of an even number of points, and $m\colon a \to a$ is a matching. Given a three-manifold M with boundary of genus k, a *(multipointed) parametrization* of ∂M is a pointed matched multicircle $\mathcal{Z}$ with $|a| = 4n + 4k - 4$, along with an embedding of Z and of pairwise disjoint arcs $\delta = \amalg_{i=1}^{2n+2k-2}\delta_i$ into M, satisfying the following:

(1) The image of each Z_i bounds a disk D_i in ∂M whose interior is disjoint from the arcs δ_i for all i.
(2) $\partial\delta = a$ and each $\partial\delta_i$ is a pair of points matched by m.
(3) $\partial M \setminus \left((\amalg_{i=1}^{n} D_i) \cup (\amalg_{i=1}^{2n+2k-2}\delta_i)\right)$ is the union of n open disks such that each disk contains exactly one of the marked points z_i for $i = 1, \dots, n$.

We call the three-manifold with multipointed parametrized boundary a *bordered manifold*, as in [LOT18], and denote it by $(M, \mathcal{Z})$, omitting from the notation the implicit data of how the arcs δ_i are embedded on ∂M.

It is easy to see that a three-manifold with multipointed parametrized boundary $(M, \mathcal{Z})$ can be reinterpreted as a bordered sutured manifold $(M, \Gamma, \mathcal{Z}^\circ)$ where $\amalg_{i=1}^{n} D_i$ is the sutured part while its complement is the parametrized part, and $\mathcal{Z}^\circ$ is the arc diagram obtained from $\mathcal{Z}$ by removing neighborhoods of the basepoints. Thus, Zarev's construction associates a type A structure $\widehat{BSA}(M)$ over $\mathcal{A}(\mathcal{Z}) \coloneqq \mathcal{A}(\mathcal{Z}^\circ)$, or equivalently a type D structure $\widehat{BSD}(M)$ over $\mathcal{A}(-\mathcal{Z})$. We will denote these structures $\widetilde{CFA}(M)$ and $\widetilde{CFD}(M)$, respectively. The construction uses a Heegaard diagram presentation $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z}^\circ)$ for the bordered sutured manifold. The arc diagram $\mathcal{Z}^\circ$ is embedded on $\partial\mathcal{H}$ so that there is one interval on each component of $\partial\mathcal{H}$, and can be extended to an identification of $\mathcal{Z}$ with $\partial\mathcal{H}$, by placing back the basepoints, one in each component of $\partial\mathcal{H} \setminus \mathcal{Z}^\circ$. The result is a *multipointed bordered Heegaard diagram* for $(M, \mathcal{Z})$. Since there is no loss of information from one perspective to the other, we denote $\mathcal{Z}^\circ$ simply by $\mathcal{Z}$ in this paper. The gluing formula for bordered sutured Floer homology implies that if the closed three-manifold M with multiple basepoints is obtained by gluing multipointed bordered three-manifolds $M_1 \cup_\partial M_2$, with M_1 parametrized by $\mathcal{Z}$ and M_2 by $-\mathcal{Z}$, then $\widetilde{CFA}(M_1) \boxtimes_{\mathcal{A}(\mathcal{Z})} \widetilde{CFD}(M_2)$ is homotopy equivalent to $\widetilde{CF}(M, n)$.

3. Foliated open books

3.1. **Foliated open books.** An abstract open book for a closed three-manifold is a pair (S, h), where S is a surface with boundary and h and element of its mapping class group. This data suffices to construct an S-bundle over S^1, and after collapsing the boundary in a controlled way, yields a closed three-manifold. A foliated open book adapts this approach to the setting of a manifold with boundary. This time, the data consist of a sequence of $2k$ topologically distinct surfaces and the maps identifying one surface with the next. Analogously, this determines a manifold with foliated boundary.

Definition 3.1. [LV20, cf. Definition 3.12] An *abstract foliated open book* is a tuple $(\{S_i\}_{i=0}^{2k}, h)$ where S_i is a surface with boundary $\partial S_i = B \cup A_i$ [1] and corners at $E = B \cap A_i$ such that

(1) for all i, A_i is a union of intervals;
(2) the surface S_i is obtained from S_{i-1} by either
  - attaching a 1-handle along two points on A_{i-1}, or
  - cutting S_{i-1} along a properly embedded arc γ_i with endpoints in A_{i-1} and then smoothing.[2]

Furthermore, $h\colon S_{2k} \to S_0$ is a diffeomorphism between cornered surfaces that preserves B pointwise.

To construct a manifold from this data, one concatenates the elementary cobordisms associated to successive pages and glues S_{2k} to S_0 by the map h. Each abstract page S_i yields a product $S_i \times (\frac{\pi i}{k}, \frac{\pi(i+1)}{k})$, and two such products join smoothly along a singular page which is surface with two points on its boundary identified. After collapsing each component $B \times S^1$ to a curve called the binding and still denoted by B, the remaining boundary is decorated by the non-binding boundaries of the pages. The result of this decoration is a singular foliation $\mathcal{F}$ whose regular leaves are copies of A_i, oriented as the boundary of the pages, and whose singular leaves each contain a single four-pronged hyperbolic point. The elliptic points E and the hyperbolic points H each come with signs: each interval component of A_i is oriented from a positive elliptic point towards a negative one. Hyperbolic points associated to attaching a one-handle are negative, while hyperbolic points associated to cutting along an arc are positive. See Figure 3.1.

With the above decomposition in mind, define a function $\pi\colon M \setminus B \to S^1$ so that on each piece $S_i \times (\frac{\pi i}{k}, \frac{\pi(i+1)}{k})$ it is given by projection to the second coordinate; here S^1 is identified with $[0, 2\pi]/(0 \sim 2\pi)$. Below, we abuse notation a couple of times and write S_t for $\pi^{-1}(t)$.

We denote the resulting smooth object by the triple $(M, \partial M, \mathcal{F})$ and call it a *manifold with foliated boundary*. We remember the identification of leaves with intervals on the boundary of abstract pages, and in particular, the foliation has a distinguished 0-leaf which is always regular. Because there are no S^1 leaves and the singular points come with signs, we may associate a *dividing set* to the foliation: a dividing set Γ is the boundary of a neighborhood of the positive separatrices of positive hyperbolic points. With this in hand, we recall the compatibility between foliated open books and contact structures.

Definition 3.2. [LV20, Definition 3.7] The abstract foliated open book $(\{S_i\}, h)$ *supports* the contact structure ξ on $(M, \partial M, \mathcal{F})$ if ξ is the kernel of some one-form α on M satisfying the following properties:

(1) $\alpha(TB) > 0$;
(2) for all t, $d\alpha|_{\pi^{-1}(t)}$ is an area form; and

[1] By a slight abuse of notation we denote the "constant" part of the boundary of S_i by B for all i.

[2] The indices of γ_i in this paper are shifted compared to [LV20], where the cutting arcs were denoted by γ_{i-1}.

(3) there is a topological isotopy of ∂M taking $\mathcal{F}$ to the characteristic foliation $\mathcal{F}_\xi$ such that some Γ is a dividing set for the foliation throughout.

We will often want to consider a manifold with foliated boundary $(M, \partial M, \mathcal{F})$ together with a contact structure ξ supported by a corresponding foliated open book; we call this a *foliated contact three-manifold* and denote it by the triple $(M, \xi, \mathcal{F})$.

Foliated open books were developed to provide easy tools for cutting and gluing contact manifolds. Cutting a manifold with a classical open book decomposition along a suitably generic surface produces two foliated open books whose boundary foliations match, but with the signs of all singular points removed. Conversely, any two foliated open books with matching, sign-reverse boundary foliations may be glued to produce a closed manifold with an open book structure. These cutting and gluing results also respect the supported contact structures of each of the open books involved.

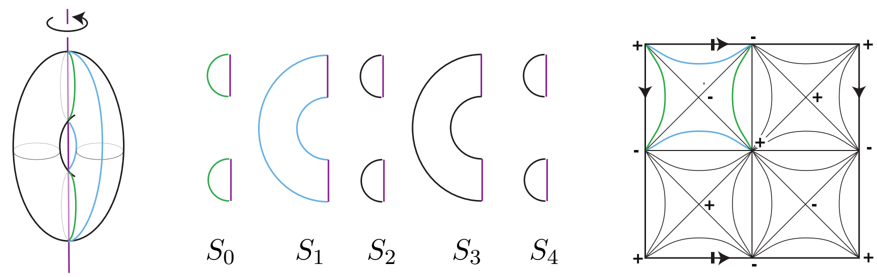

FIGURE 3.1. Left: A foliated open book for the solid torus in $\mathbb{R}^3$. Middle: Some of the pages. Right: The singular foliation on the boundary torus.

To a foliated open book that obeys a certain technical condition called *sorted*, one can associate a multipointed bordered Heegaard diagram, along with a preferred generator.

3.2. **Sorted foliated open books.** In order to associate a Heegaard diagram to a foliated open book, we will need to impose a further condition. Since the notation to verify this condition is somewhat involved, we pause to motivate it first. As above, the definition of a foliated open book requires pages to evolve by cutting or by gluing, but we may equivalently think of this as the condition that adjacent pages always evolve by a one-handle addition, but in either direction: either S_i is obtained from S_{i-1} by a one-handle addition or else S_i is obtained from S_{i+1} by a one-handle addition. One-handles associated to negative hyperbolic points are those already described in Definition 3.1 as "adds", while positive hyperbolic points correspond to adding a handle as the page index decreases. We will call a foliated open book *sorted* if a one-handle, after being added with respect to some direction (i.e., increasing or decreasing indices), persists for all subsequent pages with respect to that direction. See Figure 3.2.

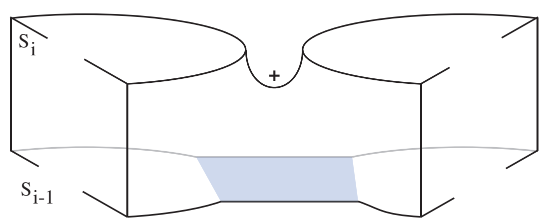

FIGURE 3.2. Here S_{i-1} is obtained from S_i by adding the shaded one-handle. The sorted condition requires that this handle persist for all S_j with $0 \leq j < i$. Note that the binding has been blown up as $B \times I$ for ease of viewing.

Recall that the elliptic points $E = A_i \cap B$ partition as $E = E_+ \coprod E_-$, where each interval is oriented from a point $e_+ \in E_+$ to a point $e_- \in E_-$. We impose the following conventions on the cutting and gluing arcs that govern how the pages evolve:

- If $S_i \to S_{i+1}$ cuts S_i along a properly embedded arc, the endpoints of the arc lie near the e_+ end of the intervals of A_i. We decorate S_i and all prior pages with a copy of the cutting arc and label these arcs as γ_i^+. If S_j is decorated with multiple cutting arcs near the same point e_+, the indices decrease with the orientation of A_j.
- If $S_i \to S_{i+1}$ adds a one-handle to S_i, the points of the attaching sphere separate any γ^+ endpoints from the e^- on the intervals of A_i. We decorate S_{i+1} and all subsequent pages with a copy of the cocore of the attached one-handle and label these arcs as γ_i^-. If S_j is decorated with multiple cocores near the same point e_-, the indices decrease with the orientation of A_j.

If we take the perspective that gluing is simply cutting in with the order of the indices reversed, then the second bullet point can be phrased in identical language to the first. Figure 3.3 illustrates these conventions in an example.

Definition 3.3. A foliated open book is *sorted* if the arcs $\gamma^- \cup \gamma^+$ are mutually disjoint on all the pages where they appear. We denote a sorted foliated open book by $(\{S_i\}_{i=0}^{2k}, h, \{\gamma_i^\pm\})$.

A foliated open book which is sorted has a ghost page: a minimal surface formed by cutting along all of the arcs which may not actually coincide with any S_i in the foliated open book. Although this is not precise statement, it may be helpful in understanding the following notation-heavy definition.

Suppose $(\{S_i\}_{i=0}^{2k}, h, \{\gamma_i^\pm\})$ is a sorted foliated open book for foliated contact three-manifold $(M, \xi, \mathcal{F})$. On each page S_i, let P_i be the complement of a "cornered" neighborhood of $A_i \cup (\bigcup_{i<j} \gamma_j^+) \cup (\bigcup_{i \geq j} \gamma_j^-)$, with corners at E. This P_i is the ghost page and exists as subsurface of each S_i. The copies of P_i may be identified via the flow of a vector field transverse to the pages, and we denote the composition of these identifications from $P_0 \subset S_0$ onto $P_{2k} \subset S_{2k}$ by ι.

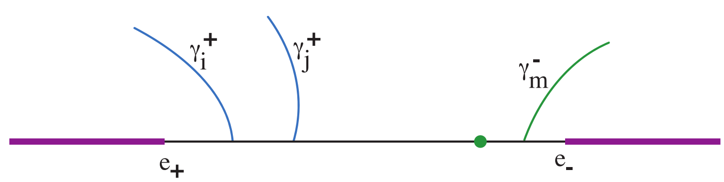

FIGURE 3.3. An indicative interval of A_n. Here $i > j \geq n > m$. The arcs γ_i^+ and γ_j^+ show arcs that will be cut along on higher-index pages. The bold dot indicates where a one-handle could be attached on some later page, while the arc γ_m^- is the cocore of a handle already attached.

3.3. **Sorting by stabilization.** In this subsection we examine the operation of positive stabilization on a foliated open book and show how it can be used to render a non-sorted foliated open book sorted. The idea is straightforward: each stabilization adds a one-handle to every page of the foliated open book by taking a connect sum with a foliated open book for the standard tight S^3. Since the number of sorting arcs $\gamma^\pm$ is controlled by the foliation, and hence unchanged by stabilization, repeating the process sufficiently many times gives the sorting arcs enough room in the enlarged page to avoid each other. Of course, the arcs that guide the stabilization must be chosen carefully, and we explain how to do this below. The formal proof that this is always possible may be found in [LV20].

As shown in [LV20], stabilization may be understood as a concrete example of gluing two foliated open books. Choose an arc $(\gamma, \partial\gamma)$ embedded in a fixed page (S_t, B) of a foliated open book. A regular neighborhood of this arc may be chosen so that its boundary is a sphere whose signed singular foliation has two hyperbolic singularities. Choosing such neighborhoods in two separate foliated open books yields manifolds with matching foliated boundaries. Shift the t coordinate by $\frac{1}{2}$ and the two may be glued to construct a foliated open book for the connect sum of the two original manifolds, with pages formed as the Murasugi sum of the pages of the original foliated open books. If one of the manifolds was an open book with annular pages supporting the tight contact structure on S^3, then the contactomorphism type of the manifold is unchanged and we say that the foliated open book has been *positively stabilized*.

The description above applies with minor modification to all versions of open books, but a distinguishing feature of foliated open books is the non-homegeneity of the pages. An arc on S_t may not persist to some later page $S_{t'}$, or $S_{t'}$ may have a non-trivial mapping class group even though S_t was a collection of disks. This highlights that there are two choices to be made when defining a stabilization of a foliated open book: which page, and which arc?

With a goal of removing intersections of the form $\gamma_i^+ \cap \gamma_j^-$, choose a regular page between the hyperbolic points h_i^+ and h_j^-. We will stabilize along an arc in this page so that as γ_i^+ rises up through the manifold, the subinterval that would collide with the descending γ_j^- picks of the monodromy of the foliated open book for S^3 and instead undergoes a Dehn twist around the core of the annular Murasugi summand of the page.

Example 3.4. Here we illustrate the process described above in a concrete example. Figure 3.4 shows the pages of a foliated open book for an overtwisted ball studied in [LV21], with the letters labeling elliptic points.

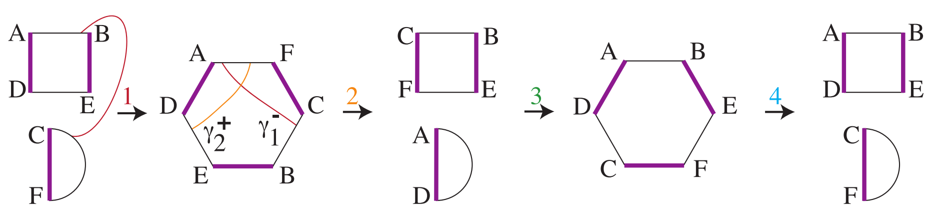

FIGURE 3.4. A foliated open book for an overtwisted ball. The intersection $\gamma_2^+ \cap \gamma_1^-$ on page S_1 dictates a specific stabilization. The result of this stabilization is shown in Figure 3.5.

Since each component of each page is a disk, there is a unique (up to isotopy) way to identify successive pages. The first hyperbolic singularity is negative and corresponds to adding a one-handle to S_0 as shown; on the second page (S_1), the cocore of the one-handle is recorded as an arc γ_1^-. However, the second hyperbolic singularity is positive and corresponds to cutting the second page along the arc labelled γ_2^+ to get the third page. Because γ_1^- and γ_2^+ intersect, we choose a copy of S_1 and stabilize along an arc that crossed γ_2^+ and γ_1^- once. The result is shown in Figure 3.5. One can think of γ_1^- as undergoing a right-handed twist as it ascends, or γ_2^+ as undergoing a left-handed twist as it descends, and since the two curves now avoid each other, we may proceed with increasing t until γ_3^- and γ_4^+ intersect on the new S_3 page.

To remove the intersection $\gamma_3^- \cap \gamma_4^+$, we analogously stabilize along an arc intersecting each of these curves once. Finally, a sorted foliated open book is seen in Figure 3.6. Since the gluing map is inherited from the original foliated open book, it remains translation in the page as drawn.

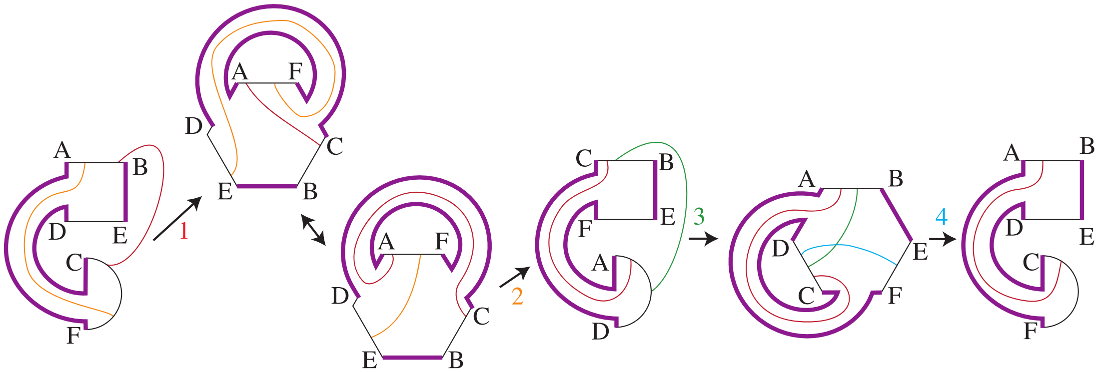

FIGURE 3.5. The stabilization of the foliated open book from Figure 3.4. Note that γ_3^+ and γ_4^- intersect on the new page S_3, so the foliated open book remains unsorted.

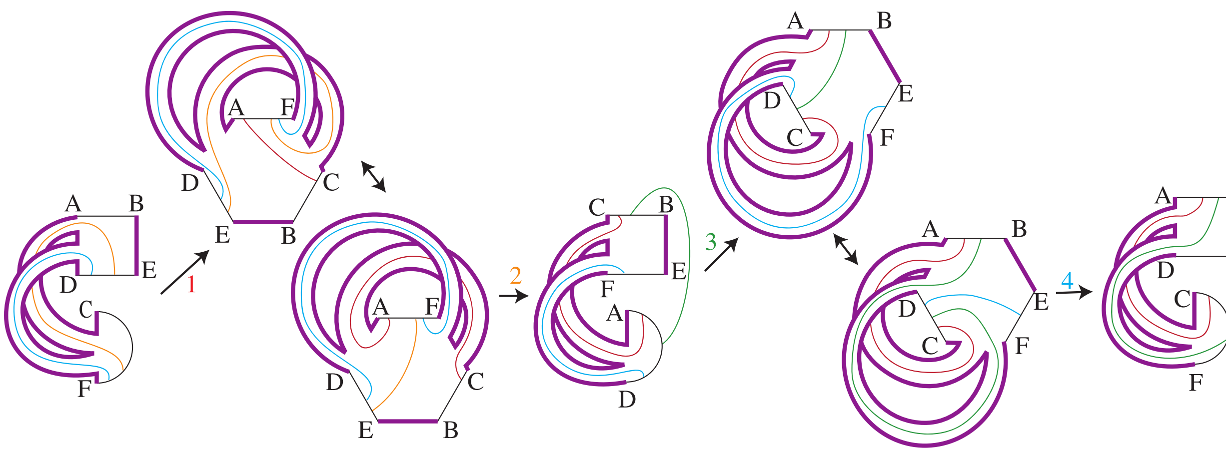

FIGURE 3.6. A sorted foliated open book for a neighborhood of an overtwisted disk, obatined from the foliated open book in Figure 3.4 by a sequence of two stabilizations.

4. THE BORDERED CONTACT INVARIANT

Let $(M, \xi, \mathcal{F})$ be a foliated contact three-manifold. In [AFH+21], a bordered sutured manifold $(M, \Gamma, \mathcal{Z})$ was associated to $(M, \xi, \mathcal{F})$. As explained in Section 2, we can convert the bordered sutured data to multipointed bordered data for a simpler perspective. We describe the parametrization on the boundary of the resulting bordered manifold $(M, \mathcal{Z})$ directly below.

Use the foliation to define a natural parametrization of ∂M via a pointed matched multicircle $\mathcal{Z} = (Z, a, m)$. Recall that the data of the foliation remembers the page index associated to each leaf, and in particular, that there is a distinguished regular leaf A_0. Let $D \subset \partial M$ be a closed neighborhood of A_0, and let $Z = \partial D$. Note that D is a union of n disks, where $2n$ is the number of elliptic points in the foliation. Let δ_i be subarc of the positive (resp. negative) separatrix for h_i^+ (resp. h_i^-) that lies in $\partial M \setminus (\text{int} D)$. Define $a \subset Z$ to be the set of points that are the boundaries of δ_i and let m be the matching induced on the points in a by δ_i. For each component of Z, mark a basepoint with a smallest possible $(0, 2\pi)$-coordinate. See Figure 4.1. It is easy to check that $\mathcal{Z} = (Z, a, m)$ together with the embedding of the arcs δ_i parametrizes ∂M.

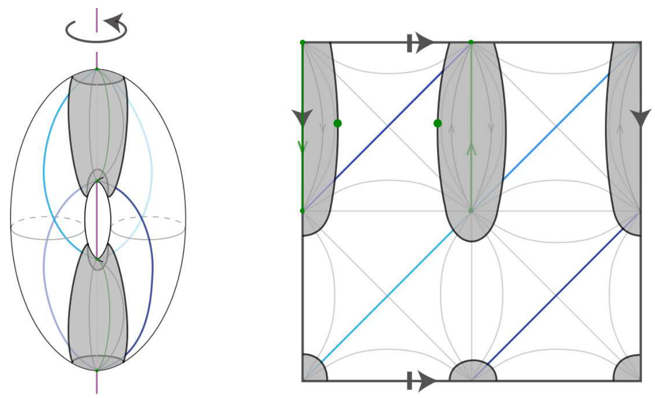

FIGURE 4.1. Left: The bordered three-manifold associated to the foliated solid torus from Figure 3.1.

Let $(\{S_i\}_{i=0}^{2k}, h, \{\gamma_i^\pm\})$ be an abstract sorted foliated open book for a foliated contact three-manifold $(M, \xi, \mathcal{F})$. The sortedness condition ensures that the first page of the open book, together with its (indexed) γ_i^+ arcs; the last page together with its (indexed) γ_j^- arcs; and the monodromy h fully describe the manifold. In fact, the union of the first and last page naturally describes a (cornered) handlebody decomposition for M. Using the data of $(\{S_i\}, h, \{\gamma_i^\pm\})$, we describe a multipointed bordered Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z})$ for this handlebody decomposition, along with a preferred generator. We outline the construction below; cf. [AFH+21, Section 3].

Let g_i be the genus of S_i and let n_i be the number of boundary components of S_i. Recall that the boundary of the cornered surface S_i is $B \cup A_i$, where B is a union of circles and arcs, and A_i is a union of intervals only.

We let $\Sigma = S_0 \cup_B -S_0$. In order to distinguish the two copies, we will write

$$\Sigma = S_\epsilon \cup_B -S_0,$$

but we emphasize that S_ϵ can be identified with S_0. The surface Σ has genus $2g_0 + n_0 - 1$ and $|A_0|$ boundary components.

For $i \in H_-$, consider the S_{2k} copies of the sorting arcs γ_i^-, and let $\beta_i^- = -h(\gamma_i^-)$ on $-S_0$. For $i \in H_+$, consider the S_ϵ copies of the sorting arcs γ_i^+. The endpoints of γ_i^+ lie near the E_+ end of intervals of A_ϵ. Isotope the arcs $\{\gamma_i^+\}$ (simultaneously, to preserve disjointness) near the endpoints along $-\partial\Sigma$ until they all lie in $I_+ \subset A_0$; the isotopy stops after crossing E_+ and before encountering $\cup_{j \in H_-} - h(\gamma_j^-) \subset -S_0$. Call the resulting arcs β_i^+. Define a set of arcs $\boldsymbol{\beta}^a = \{\beta_1^a, \ldots, \beta_{2k}^a\}$ by

$$\beta_i^a = \begin{cases} \beta_i^+ & \text{if } i \in H_+, \\ \beta_i^- & \text{if } i \in H_-. \end{cases}$$

As in [AFH+21], we use the notation β_i^a or $\beta_i^\pm$ if $i \in H_\pm$ interchangeably.

Let $\boldsymbol{b} = \{b_1, \ldots, b_{2g_0+n_0+|A_0|-k-2}\}$ be a set of cutting arcs for $P_\epsilon \subset S_\epsilon$ disjoint from $\boldsymbol{\beta}^a$ and with endpoints on B, so that each connected component of $S_\epsilon \backslash (\boldsymbol{b} \cup \boldsymbol{\beta}^a)$ is a disk with exactly one interval of A_ϵ on its boundary. (In [AFH+21], we show this can always be achieved.) In other words, $\boldsymbol{b}$ is a basis for $H_1(P_\varepsilon, B)$. Recalling the identification $S_\epsilon = S_0$, we may push $b_i \subset S_0$ through M to lie on S_0 again and define

$$\beta_i = b_i \cup -h \circ \iota(b_i) \subset S_\epsilon \cup_B -S_0,$$

where ι is the identification of P_0 with P_{2k} from Section 3.2. Write $\boldsymbol{\beta}^c = \{\beta_1, \ldots, \beta_{2g_0+n_0+|A_0|-k-2}\}$.

For each cutting arc $b_i \in \boldsymbol{b}$ on S_ϵ, let a_i be an isotopic curve formed by pushing the endpoints negatively along the boundary so that a_i and b_i intersect once transversely. Similarly, for each arc $b_i^+ := S_\epsilon \cap \beta_j^+$, let $\tilde{a}_j$ be an isotopic curve formed by pushing the endpoints negatively along the boundary so that $\tilde{a}_j$ and b_i^+ (and equivalently $\tilde{a}_j$ and β_j^+) intersect once transversely. We "double" each of these arcs to form the α-circles which define the handlebody $S_0 \times [0, \epsilon]$. Namely, define

$$\alpha_i = a_i \cup -a_i \subset S_\epsilon \cup_B -S_0$$
$$\widetilde{\alpha}_j = \tilde{a}_j \cup -\tilde{a}_j \subset S_\epsilon \cup_B -S_0,$$

and write $\boldsymbol{\alpha}^c = \{\widetilde{\alpha}_i\}_{i \in H_+} \cup \{\alpha_1, \ldots, \alpha_{2g_0+n_0+|A_0|-k-2}\}$. Place a basepoint on each interval of $A_\epsilon \subset S_\epsilon \subset \Sigma$. Write

$$\mathbf{z} = \{z_1, \ldots, z_{|A_\epsilon|}\}$$

for the set of basepoints.

We say that a mutipointed bordered Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z})$ constructed as above is *adapted* to the sorted abstract foliated open book $(\{S_i\}, h, \{\gamma_i^\pm\})$ and to the corresponding foliated contact three-manifold $(M, \xi, \mathcal{F})$.

Let $\mathcal{H}$ be a mutipointed bordered Heegaard diagram adapted to $(\{S_i\}, h, \{\gamma_i^\pm\})$. In [AFH$^+$21], we show that any such diagram is admissible. (In fact, in [AFH$^+$21] neighborhoods of basepoints are drilled out to obtain a bordered sutured diagram for a certain bordered sutured manifold naturally associated to $(M, \xi, \mathcal{F})$, but we suppress this discussion here.) Using the notation introduced above, define

$$\mathbf{x} = \{x_1, \ldots, x_{2g_0+n_0+|A_0|-k-2}\} \cup \{x_i^+ \mid i \in H_+\}$$

to be the set of unique intersection points

$$x_i = a_i \cap b_i \in S_\epsilon \subset \Sigma$$
$$x_i^+ = \tilde{a}_i \cap b_i^+ \in S_\epsilon \quad \text{if } i \in H_+.$$

We will use $\mathbf{x}$ to define two contact invariants in bordered sutured Floer homology.

By Proposition [AFH$^+$21, Proposition 3.4] and [Zar10, Section 3.4], the diagram $\overline{\mathcal{H}} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \overline{\mathcal{Z}})$ obtained by exchanging the roles of the two sets of curves and formally replacing the arc diagram $\mathcal{Z}$ of β-type (which is to say, parametrized by arcs which are part of the second set of curves) with the identical arc diagram $\overline{\mathcal{Z}}$ of α-type (parametrized by arcs which are part of the first set of curves) is a multipointed bordered diagram for $(-M, \overline{\mathcal{Z}})$. Write $\overline{\mathcal{Z}} = (Z, a, m)$. We have the following proposition.

Proposition 4.1 (cf. [AFH$^+$21, Proposition 3.5]). *The above* $\mathbf{x}$ *gives a well defined generator*

$$\mathbf{x}_D := \mathbf{x} \in \widetilde{CFD}(\overline{\mathcal{H}})$$

with $I_D(\mathbf{x}) = I(H_-)$ *and* $\delta^1(\mathbf{x}_D) = 0$, *and a well defined generator*

$$\mathbf{x}_A := \mathbf{x} \in \widetilde{CFA}(\overline{\mathcal{H}})$$

with $I_A(\mathbf{x}_A) = I(H_+)$ *and* $m_{i+1}(\mathbf{x}_A, a(\boldsymbol{\rho}_1), \ldots, a(\boldsymbol{\rho}_i)) = 0$ *for all* $i \geq 0$ *and all sets of Reeb chords* $\boldsymbol{\rho}_j$ *in* (Z, a).

Example 4.2. We illustrate the construction outlined above using the sorted foliated open book in Figure 3.6. For convenience, in Figure 4.2 we display again the pages S_0 and $-S_4$, along with the sorting arcs decorations.

Figure 4.3 shows the associated Heegaard diagram.

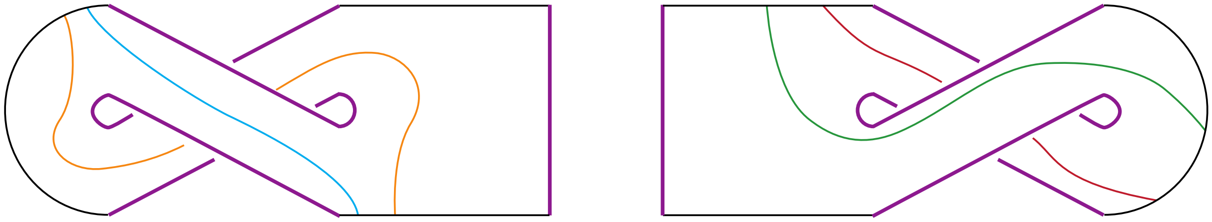

FIGURE 4.2. The first page (to the left) and the mirror of the last page (to the right) of the sorted foliated open book in Figure 3.6.

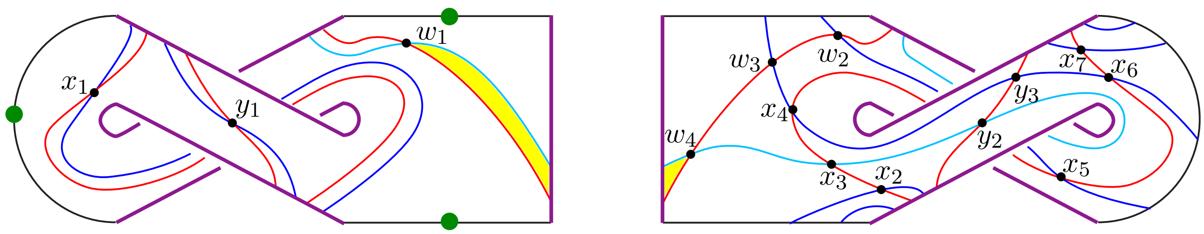

FIGURE 4.3. The Heegaard diagram for the sorted foliated open book in Figure 3.6. The monodromy h is the identity, so the images $\beta_i^- = -h(\gamma_i^-)$ are simply the sorting arcs γ_i^- on $-S_0$. Intersection points are labelled differently from the above definition, for convenience. The contact generator $\mathbf{x}$ is the triple $\{x_1, y_1, w_1\}$, or $x_1y_1w_1$ for short.

5. VANISHING OF THE CONTACT CLASS FOR OVERTWISTED STRUCTURES – A LOCAL ARGUMENT

In this section, we illustrate the power of invariants compatible with cut-and-paste constructions by providing a local argument that the contact class $c(\xi)$ for closed contact manifolds vanishes if the contact structure is overtwisted.

We begin by showing that the bordered contact invariant vanishes for a neighborhood of an overtwisted disk. Specifically, we consider the foliated open book constructed in [LV21] for a three-ball neighborhood $(B^3, \xi_{\mathrm{OT}}, \mathcal{F}_{\mathrm{OT}})$. In Example 3.4, we stabilized the foliated open book from [LV21] to a sorted one. Example 4.2 then constructed the associated Heegaard diagram $\mathcal{H}$; see Figure 4.3. The generator $x_1y_1w_1 \in \widetilde{CFD}(\overline{\mathcal{H}})$ represents the contact class. We claim that there is a unique holomorphic curve asymptotic to $x_1y_1w_4$ at $-\infty$, and it ends at $x_1y_1w_1$.

Indeed, x_1 and y_1 cannot be starting moving coordinates for a holomorphic curve; the only non-basepointed regions at these intersection points are the thin strips supported on the S_ϵ part of the diagram, but the orientation on these strips is *into* x_1 and y_1. So any holomorphic curve starting from $x_1y_1w_4$ must only have w_4 as a moving coordinate. A curve that hits the boundary of the Heegaard diagram would need to have a moving coordinate on a β-arc. Since w_4 is on a β-circle, all holomorphic curves starting from $x_1y_1w_4$ project to the interior of the diagram. Thus, any such curve with a single moving coordinate projects to an immersed bigon. By counting local coefficients, the yellow bigon from $x_1y_1w_4$ to $x_1y_1w_1$ in Figure 4.3 represents the unique such curve.

Thus, considering $\widetilde{CFD}(\overline{\mathcal{H}})$, we have $\delta^1(x_1y_1w_4) = x_1y_1w_1$. Or, if one prefers to consider $\widetilde{CFA}(\overline{\mathcal{H}})$, we have $m_1(x_1y_1w_4) = x_1y_1w_1$, whereas higher products m_i vanish on $x_1y_1w_4$. It follows that there is a type D (resp. type A) homotopy equivalence from $\widetilde{CFD}(\overline{\mathcal{H}})$ (resp. $\widetilde{CFA}(\overline{\mathcal{H}})$) to an equivalent structure, carrying $x_1y_1w_1$ to zero.

Proof of Corollary 4. Suppose (M, ξ) is a closed overtwisted three-manifold. By Giroux flexibility, (M, ξ) contains an overtwisted disk whose neighborhood is contactomorphic to the contact three-ball $(B^3, \xi_{\mathrm{OT}}, \mathcal{F}_{\mathrm{OT}})$ studied above. Thus, (M, ξ) decomposes as the union of two foliated contact three-manifolds, one of which is $(B^3, \xi_{\mathrm{OT}}, \mathcal{F}_{\mathrm{OT}})$. By the above computation, Theorem 3, and functoriality for $\boxtimes$, it follows that $c(\xi) = 0$. □

REFERENCES

[AFH+21] Akram Alishahi, Viktória Földvári, Kristen Hendricks, Joan E. Licata, Ina Petkova, and Vera Vértesi. Bordered floer homology and contact structures, 2021.

[Ghi05] Paolo Ghiggini. Strongly fillable contact 3-manifolds without stein fillings. *Geom. Topol.*, 9(3):1677–1687, 2005.

[Ghi08] Paolo Ghiggini. Knot Floer homology detects genus-one fibred knots. *Amer. J. Math.*, 130(5):1151–1169, 2008.

[HKM08] Ko Honda, William H. Kazez, and Gordana Matić. Contact structures, sutured Floer homology and TQFT. arXiv:0807.2431, 2008.

[HKM09a] Ko Honda, William H. Kazez, and Gordana Matić. The contact invariant in sutured Floer homology. *Invent. Math.*, 176(3):637–676, 2009.

[HKM09b] Ko Honda, William H. Kazez, and Gordana Matić. On the contact class in Heegaard Floer homology. *Journal of Differential Geometry*, 83(2):289–311, October 2009.

[LOT18] Robert Lipshitz, Peter S. Ozsvath, and Dylan P. Thurston. Bordered Heegaard Floer homology. *Mem. Amer. Math. Soc.*, 254(1216):viii+279, 2018.

[LV20] Joan E. Licata and Vera Vértesi. Foliated open books. arXiv:2002.01752, 2020.

[LV21] Joan E. Licata and Vera Vértesi. A User's Guide to Morse Foliated Open Books. citation in progress, 2021.

[Ni07] Yi Ni. Knot Floer homology detects fibred knots. *Invent. Math.*, 170(3):577–608, 2007.

[OS04] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and genus bounds. *Geom. Topol.*, 8:311–334, 2004.

[OS05] Peter Ozsváth and Zoltán Szabó. Heegaard Floer homology and contact structures. *Duke Math. J.*, 129(1):39–61, 2005.

[OS08] Peter Ozsváth and Zoltán Szabó. Holomorphic disks, link invariants and the multi-variable Alexander polynomial. *Algebr. Geom. Topol.*, 8(2):615–692, 2008.

[Zar09] Rumen Zarev. Bordered Floer homology for sutured manifolds. arXiv:0908.1106, 2009.

[Zar10] Rumen Zarev. Joining and gluing sutured Floer homology. arXiv:1010.3496, 2010.

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