COPY-PASTE METHODS IN CONTACT TOPOLOGY

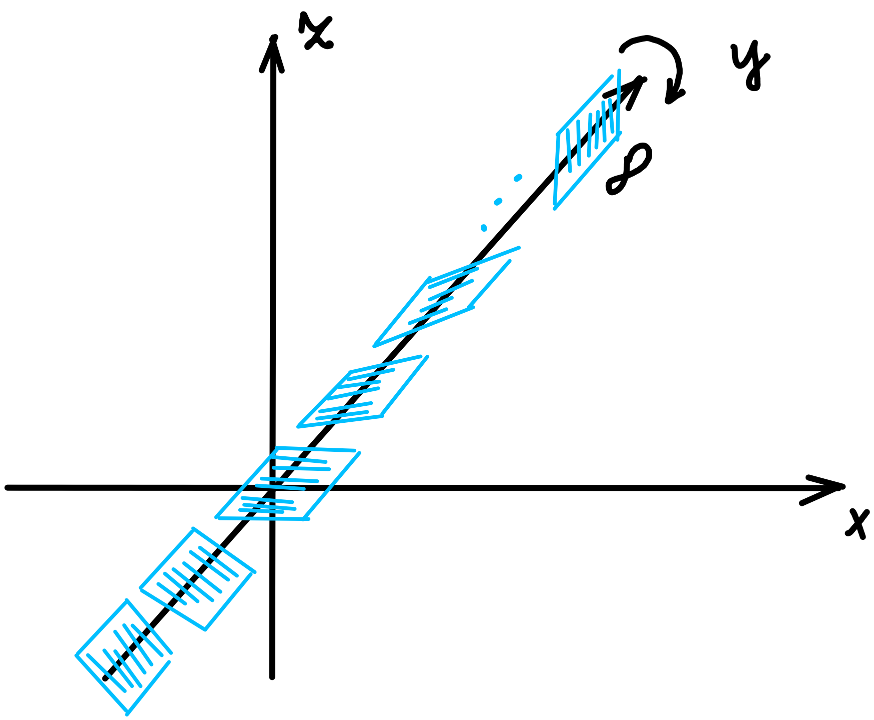
VERA VÉRTESI
UNIVERSITÉ DE NANTES/UCSB

JANUARY 2013

Contact Structures

a totally non-integrable planefield on a 3-manifold

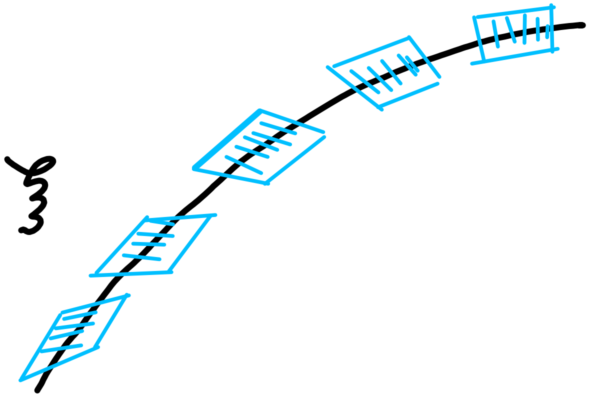
standard contact structure on $\mathbb{R}^3$: $\xi_{st} = \ker(dz - y\,dx) = \left\langle \frac{\partial}{\partial y},\ y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\rangle$



Thm (Darboux) Every contact structure is locally isotopic to ξ_{st}.

Legendrian Knots

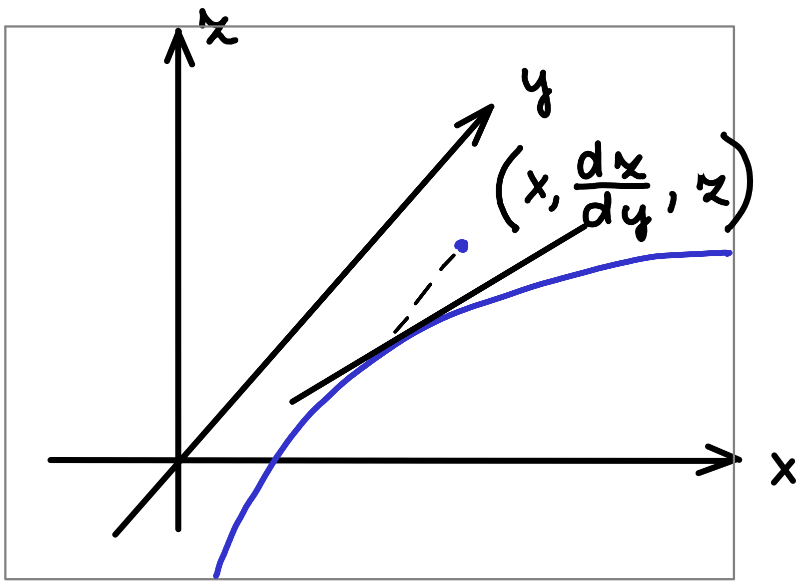
a knot K is **Legendrian** if $TK \in \xi$



Fact: Every knot can be put in Legendrian position

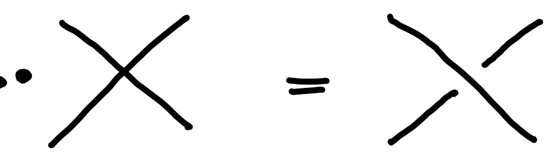
in $(\mathbb{R}^3, \xi_{st})$: $TK \in \xi_{st} = \ker(dz - y\,dx) \iff y = \frac{dz}{dx}$

projection to the (x,z)-plane: **front projection**

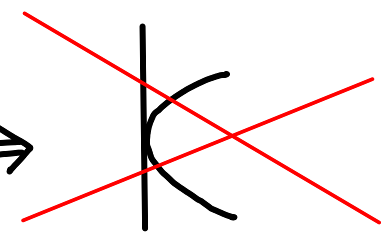


the slope determines the y-coordinate:

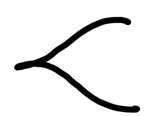
$\Rightarrow$ •



• $\infty \neq y = \frac{dz}{dx}$ $\Rightarrow$



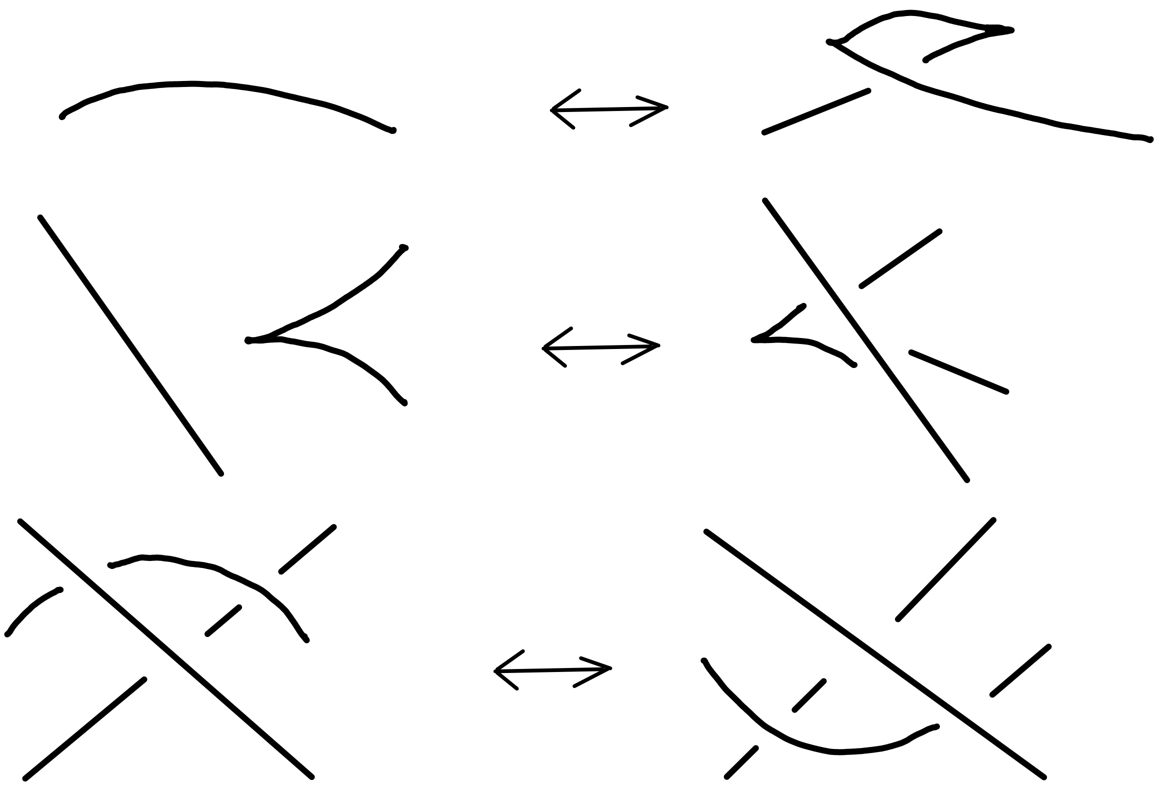
instead:



EQUIVALENCE OF LEGENDRIAN KNOTS

Legendrian isotopy: isotopy through Legendrian knots

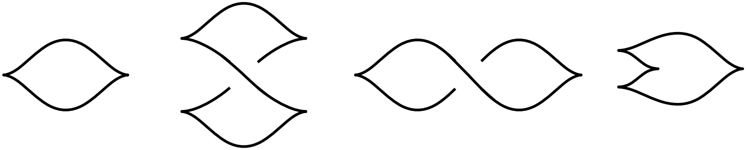
Legendrian Reidemeister moves



Thm two front projections correspond to Legendrian isotopic knots $\Longleftrightarrow$ related by a sequence of Legendrian Reidemeister moves

Are they Legendrian isotopic?

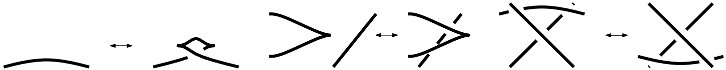
Legendrian unknots



A B C D

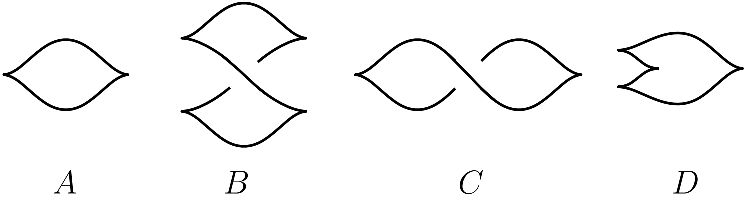
Which ones are Legendrian isotopic?

Remember:



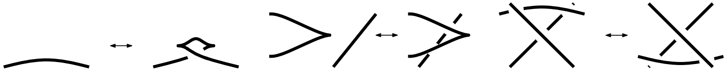
Are they Legendrian isotopic?

Legendrian unknots



Which ones are Legendrian isotopic?

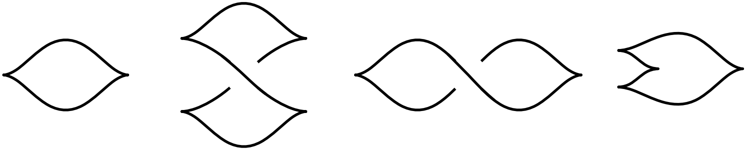
Remember:



$A \cong B\ \surd$

Are they Legendrian isotopic?

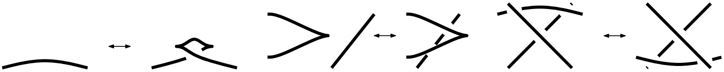
Legendrian unknots



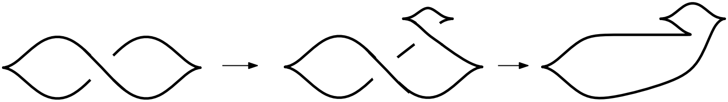
A B C D

Which ones are Legendrian isotopic?

Remember:

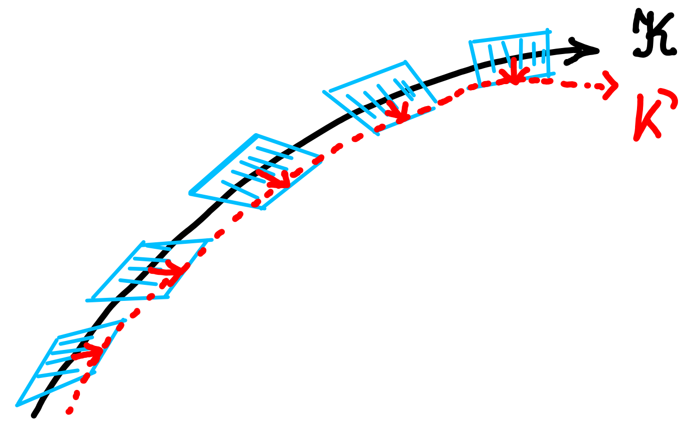


$A \cong B$ $\surd$ and $C \cong D$:



Classical Invariants

Thurston-Bennequin invariant



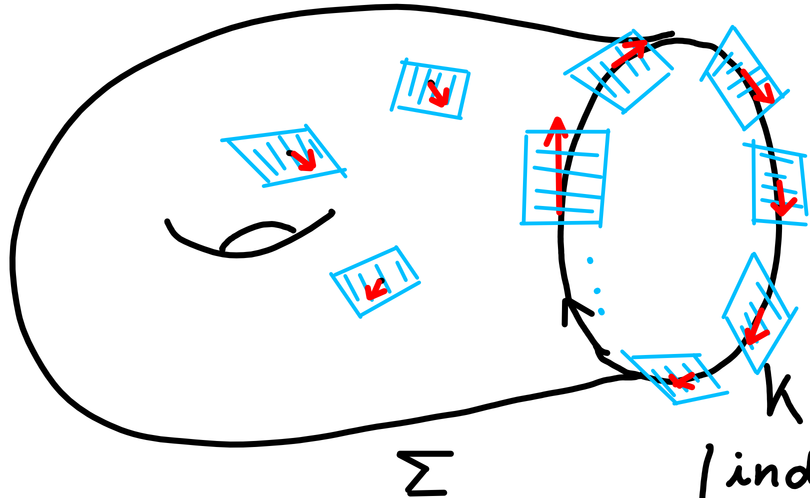
K': push off of $\mathcal{K}$ in ξ

$tb(\mathcal{K}) = lk(\mathcal{K}, K')$ ($\mathcal{K}$ is nullhomologous)

rotation number

Σ a Seifert surface of $\mathcal{K}$

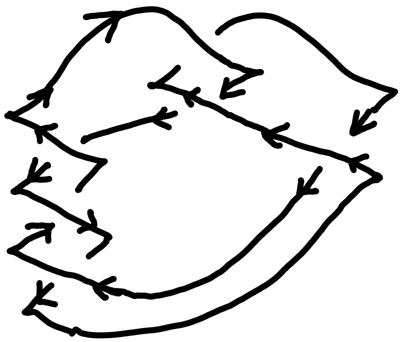
$rot_\Sigma(\mathcal{K})$ is the relative euler number of ξ on Σ w.r.t. $T\mathcal{K}$



s - section of ξ over Σ

$rot_\Sigma(\mathcal{K}) = \langle PD[s^{-1}(0)], [\Sigma] \rangle$ (independent of Σ if $H_2 = 0$)

Classical Invariants of Front Projections in $(\mathbb{R}^3, \xi_3)$



$$tb(K) = w(K) - \frac{1}{2}\#\{\text{cusps}\}$$

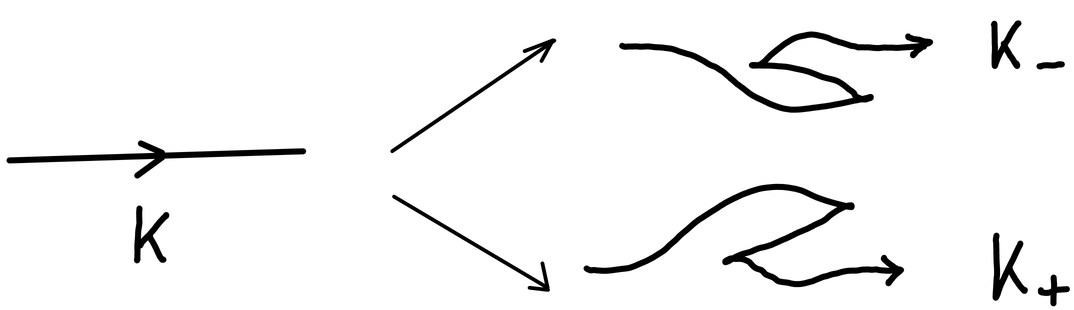
$$6 - \frac{1}{2}10 = 1$$

$$rot(K) = \frac{1}{2}\left(\#\left\{\begin{matrix}\text{downward}\\ \text{cusps}\end{matrix}\right\} - \#\left\{\begin{matrix}\text{upward}\\ \text{cusps}\end{matrix}\right\}\right)$$

$$\frac{1}{2}\left(\quad 5 \quad - \quad 5 \quad\right) = 0$$

Rmk tb can be always decreased :

stabilization



$$tb(K_\pm) = tb(K) - 1$$

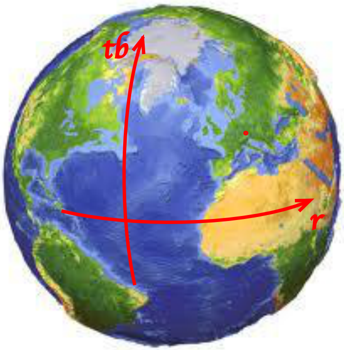
$$rot(K_\pm) = rot(K) \pm 1$$

but ! can not always be increased

$\overline{tb}(K)$ = maximal tb amongst all Legendrian representations

Classification of Legendrian knots

Legendrian isotopy $\Rightarrow$
- smoothly isotopy;
- r "="
- tb "="



K smooth knot

$\mathcal{L}(K)$ the set of Legendrian knots representing K.

$$\begin{array}{lccc} \phi_K : & \mathcal{L}(K) & \to & \mathbb{Z} \times \mathbb{Z} \\ & L & \mapsto & (\mathrm{r}(L), \mathrm{tb}(L)) \end{array}$$

Geography

determine the image of ϕ

Botany

for each point (r, t) in the image determine $\phi^{-1}(r, t)$

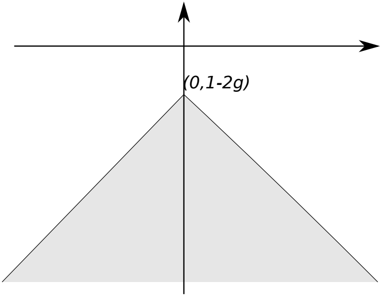
Geography

Bennequin inequality

$tb(L) + |r(L)| \leq 1 - 2g(\Sigma)$

Fact

$tb(L) + |r(L)|$ is always odd.



Geography

Bennequin inequality

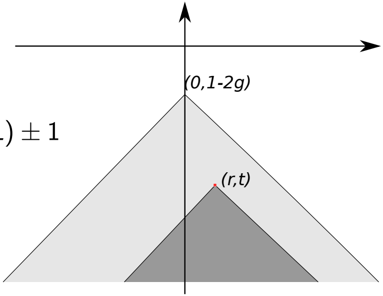
$tb(L) + |r(L)| \leq 1 - 2g(\Sigma)$

Fact

$tb(L) + |r(L)|$ is always odd.

Stabilization

$tb(L_\pm) = tb(L) - 1 \quad r(L_\pm) = r(L) \pm 1$



Geography

Bennequin inequality

$tb(L) + |r(L)| \leq 1 - 2g(\Sigma)$

Fact

$tb(L) + |r(L)|$ is always odd.

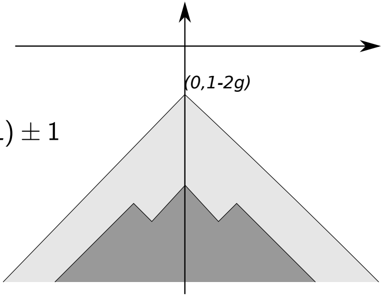
Stabilization

$tb(L_\pm) = tb(L) - 1 \quad r(L_\pm) = r(L) \pm 1$

reflecting the front projection

$$\begin{array}{rccc} f: & \mathbb{R}^3 & \to & \mathbb{R}^3 \\ & (x,y,z) & \mapsto & (x,-y,-z) \end{array}$$

preserves ξ_{st}, and isotopic to $\mathrm{id}_{\mathbb{R}^3}$
the image of ϕ is symmetric about the r-axis.



LEGENDRIAN CLASSIFICATION OF KNOTS

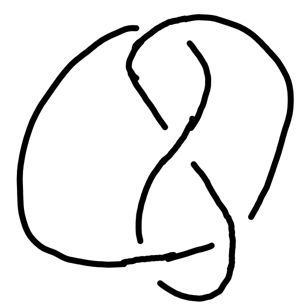
Q. How different representations of a given knot type can be distinguished?

- Is tb & rot enough?

if YES then the knot type is Legendrian simple

Legendrian simple knot types

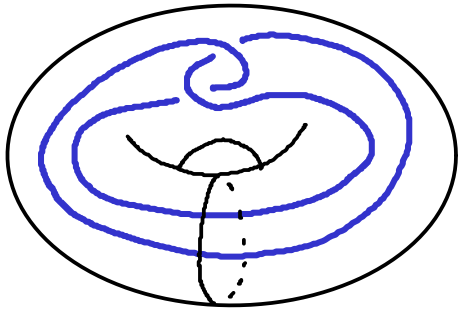
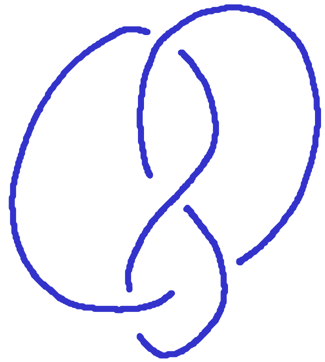
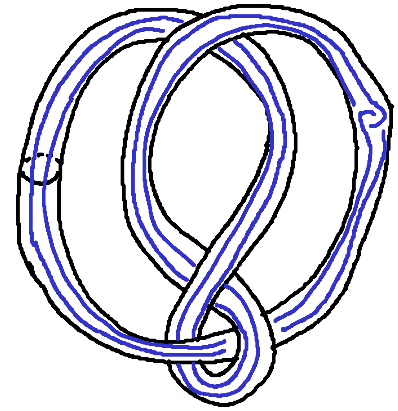
- unknot (Eliashberg - Fraser)
- torus knots, figure eight knot



non Legendrian simple knot types

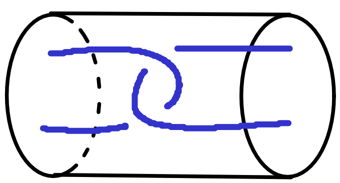
- Chekanov (5_2)
- Epstein - Fuchs - Meyer
- Ng
- Ozsváth - Szabó - Thurston

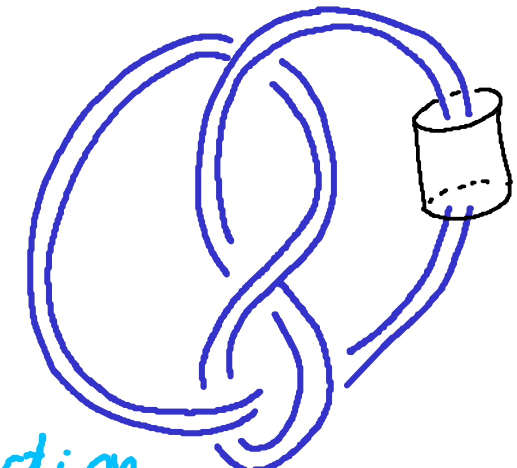
Satellite Construction

 &  $\rightsquigarrow$ 

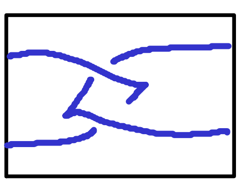
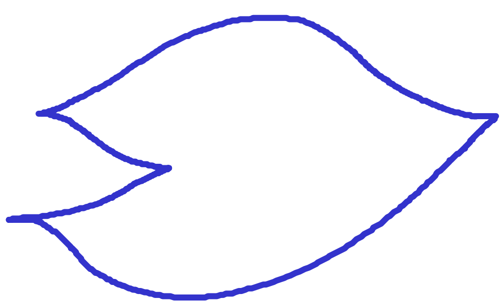
pattern — companion — satellite

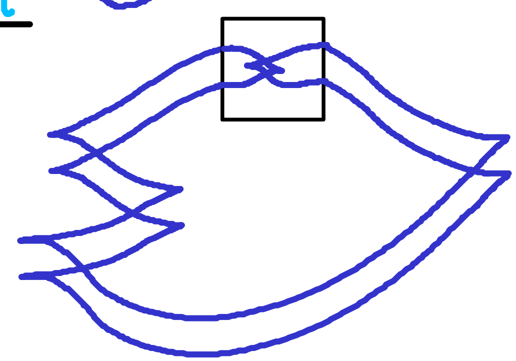
$pt \times S^1 \longmapsto$ Seifert framing

or 

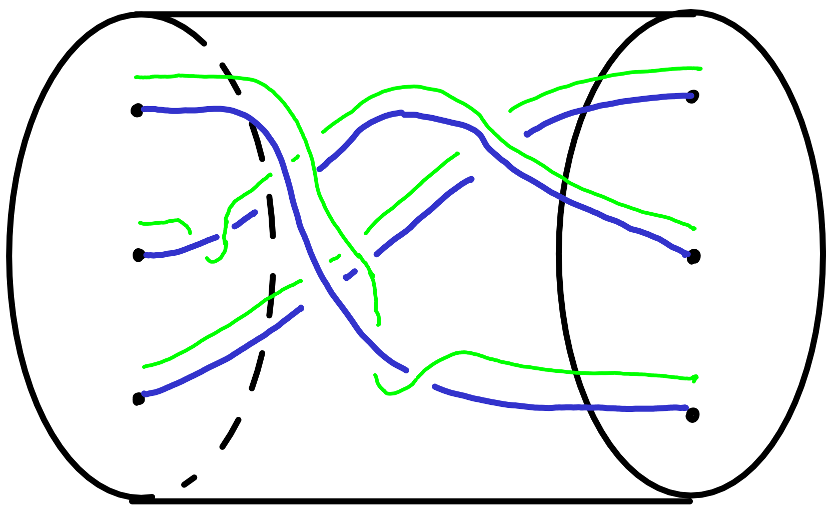


Legendrian satelite construction

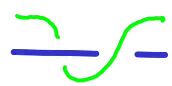
 & 

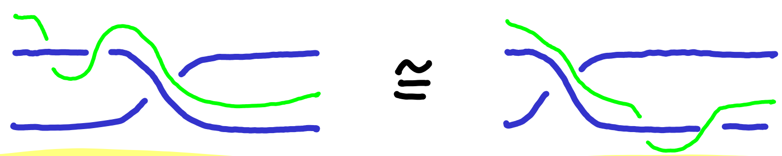


FRAMED BRAIDS



braids with a framing on each strand

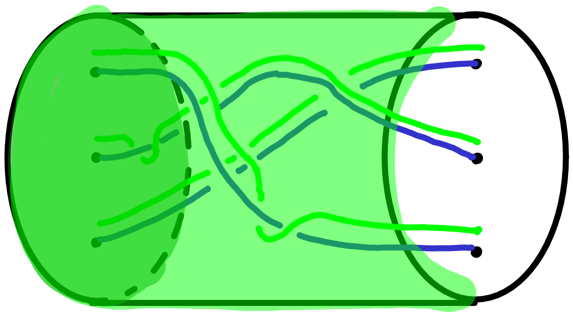
new generators:  τ_i $\langle \tau_i \rangle = \mathbb{Z}^n$

$B_n \circlearrowleft \mathbb{Z}^n$: 

$$\sigma_i \tau_i \sigma_i^{-1} = \begin{cases} \tau_{i-1} & \text{if } j = i-1 \,; \\ \tau_{i+1} & \text{if } j = i \,; \\ \tau_i & \text{otherwise} \,. \end{cases}$$

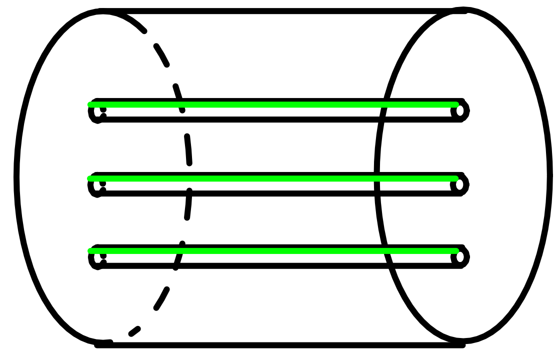
So the framed braid group is: $F_n = B_n \ltimes \mathbb{Z}^n$

The Mapping Class Group of a Punctured Disc



- leave out a small neighborhood of each strand, s.t. the framing is on the boundary

- by fixing the green part straighten everything

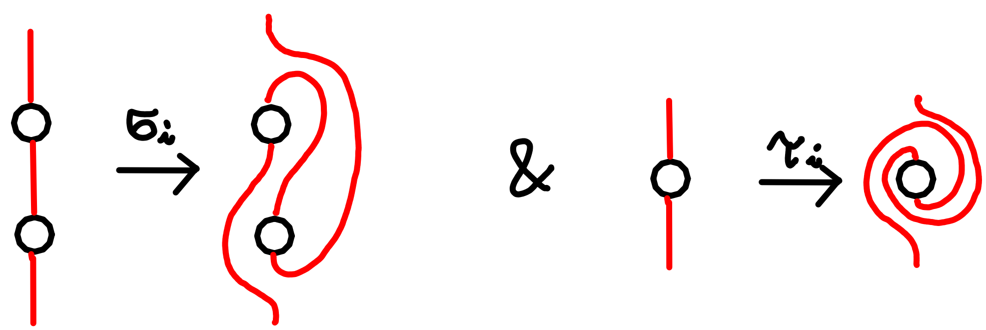


$\rightsquigarrow$ a diffeomorphism $\varphi : \left(D^2 - \nu(n \text{ pts}), \partial(\nu(n \text{ pts}))\right) \circlearrowleft$

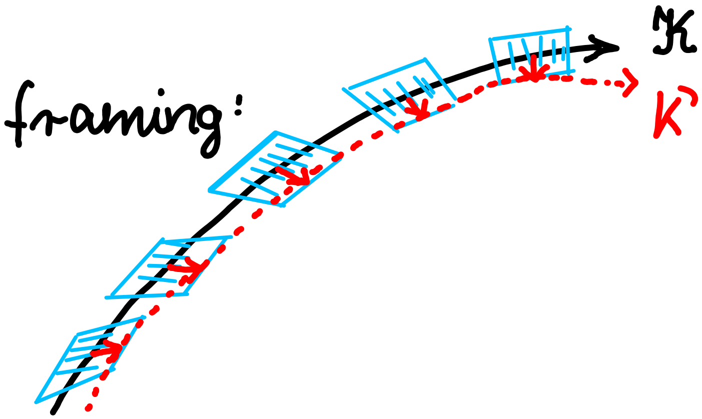
$$\updownarrow$$

F_n

the generators :



LEGENDRIAN REPRESENTATIONS OF BRAIDS

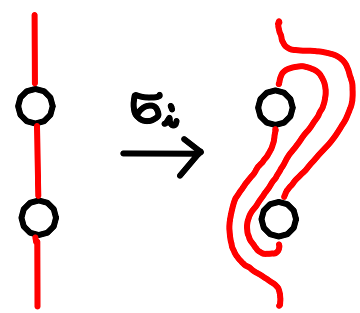


Remember: A Legendrian arc has a natural framing: the Thurston - Bennequin framing

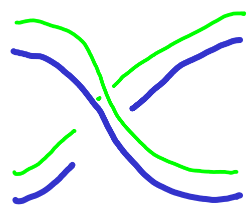
→ the set of Legendrian braids

$$L_n \subseteq F_n$$

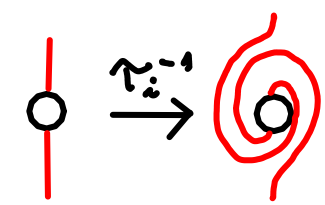
Question: Which framed braids can be represented by Legendrian braids?



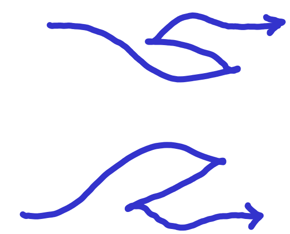
is



so ✓ &

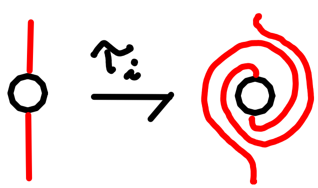


is

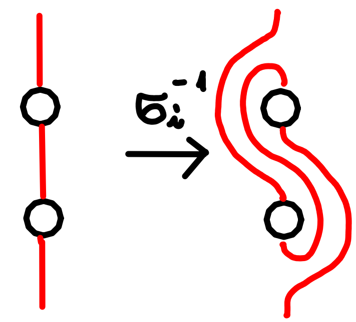


✓

but!



or

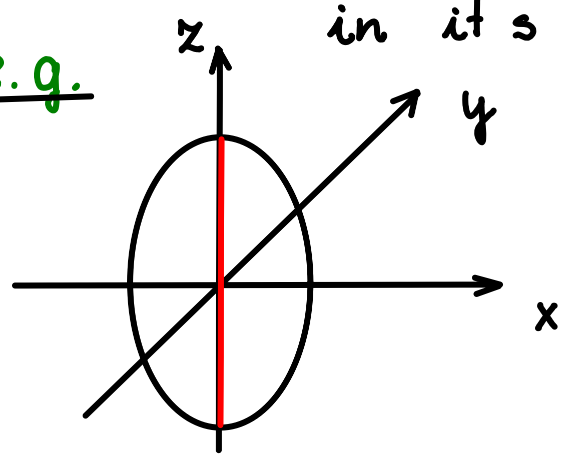


can not be represented

Convex Surfaces

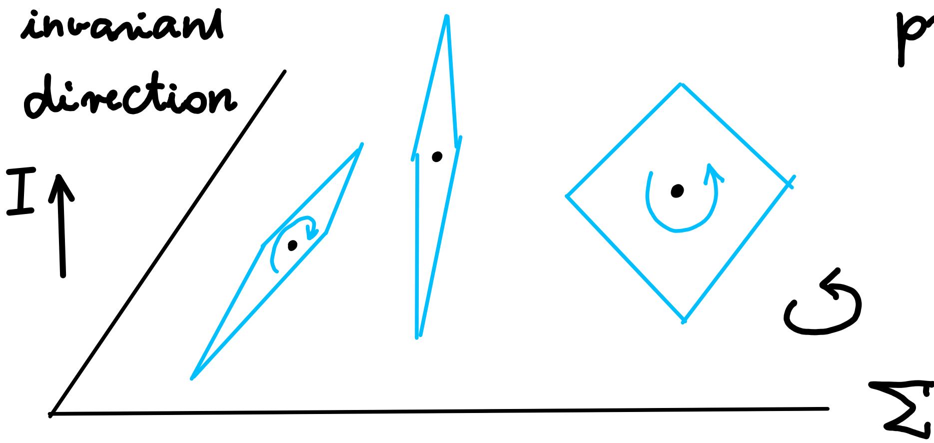
$\Sigma \hookrightarrow (Y, \xi)$ is convex if the contact structure is I-invariant in it's neighborhood

e.g.



$D = \{y^2 + z^2 \leq 1\} \times \{x_0\}$

is invariant in the $\xrightarrow{\frac{\partial}{\partial x}}$ direction



project ξ_x to $T\Sigma$

- at some pts is not onto
  $\parallel$
  1- dimensional submanifold of Σ

  Γ_z = dividing curve

- if it is onto $\begin{cases} \to \text{the orientation of } \xi_x \text{ and of } \Sigma \text{ agree } \Sigma_+ \\ \to \text{the orientation of } \xi_x \text{ and of } \Sigma \text{ disagree } \Sigma_- \end{cases}$

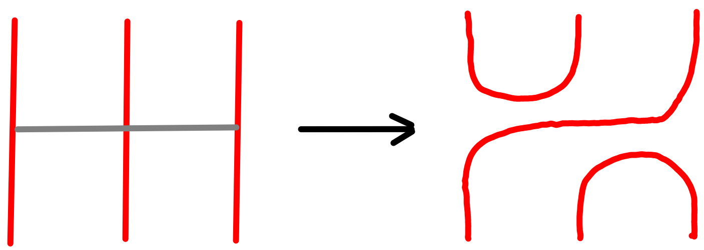
$\rightsquigarrow \quad \Sigma - \Gamma_z = \Sigma_+ \cup^* \Sigma_-$

Convex Surface Theory

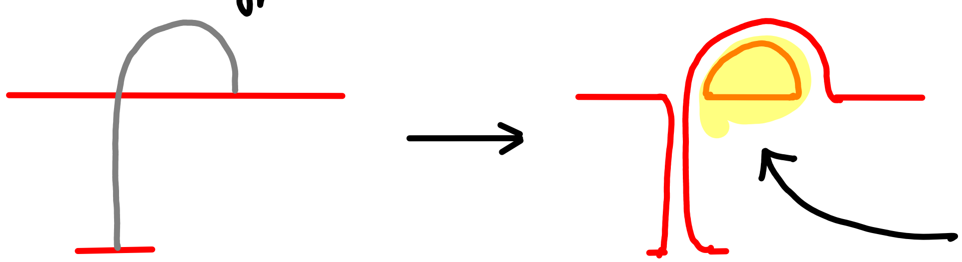
while isotoping Σ in a contact manifold

- if Σ remains convex at all times $\rightarrow \Gamma_\Sigma$ does not changes
- if Σ fails to be convex then Γ_Σ changes by a <u>bypass attachment</u>:

this is a local operation, Γ_Σ is unchanged on other parts



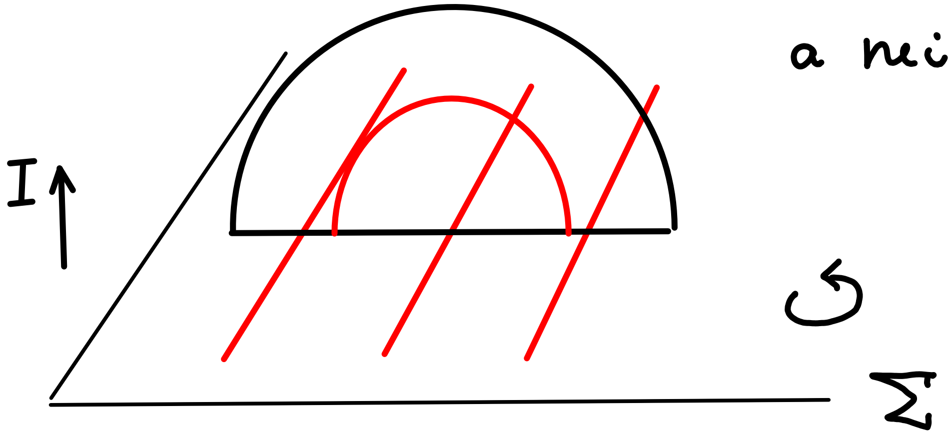
<u>Rmk</u> some bypasses cannot occur in tight contact structures



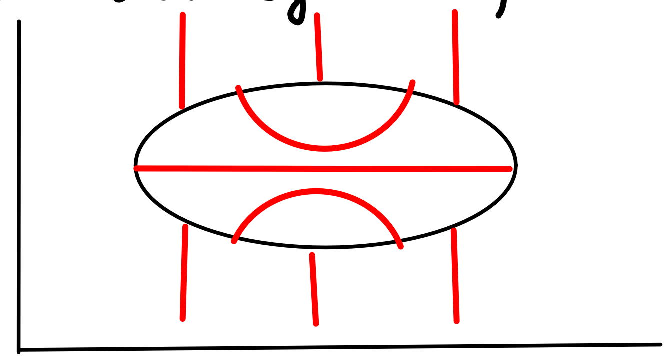
a trivial curve

TIGHT CONTACT STRUCTURES ON $\Sigma \times I$

• a bypass defines a contact structure on $\Sigma \times I$:



a neighborhood of this from above:



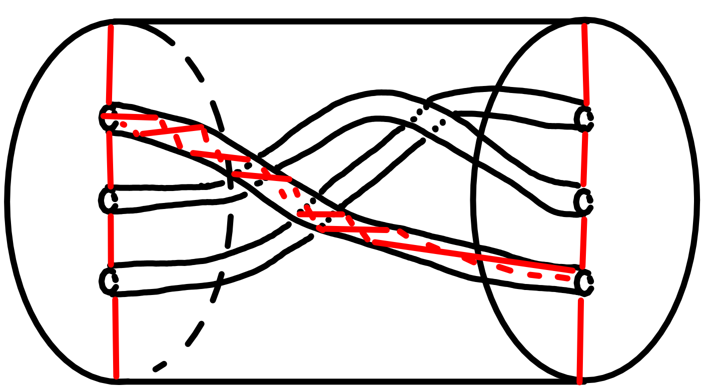
• a contact structure on $\Sigma \times I$ is built up from bypasses

Back to Legendrian Braids

Fact • A Legendrian arc has a standard neighborhood with convex boundary having a 2 component dividing curve each of which representing the Thurston-Bennequin framing

$$\frac{\left\{\begin{array}{l}\text{Legendrian representations}\\ \text{of } A \text{ with given T-B framing}\\ \text{in } (D^3, \xi_{st})\end{array}\right\}}{\text{Legendrian isotopy}} \overset{1:1}{\longleftrightarrow} \frac{\left\{\begin{array}{l}\text{tight contact structures on}\\ D^3 - \nu(A) \text{ with the convex bdry}\\ \text{given by the T-B framing}\end{array}\right\}}{\text{isotopy}}$$

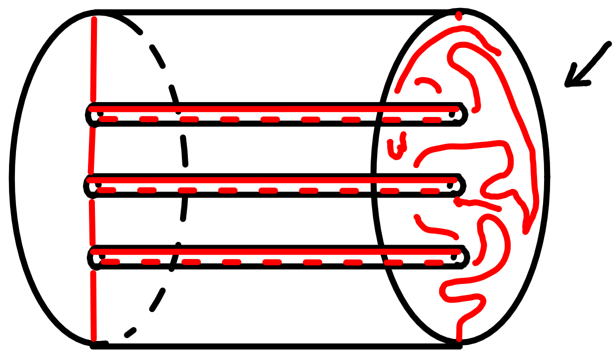
Given a Legendrian braid, take out its standard nbhd:



need to understand isotopy classes of tight contact structures with this boundary

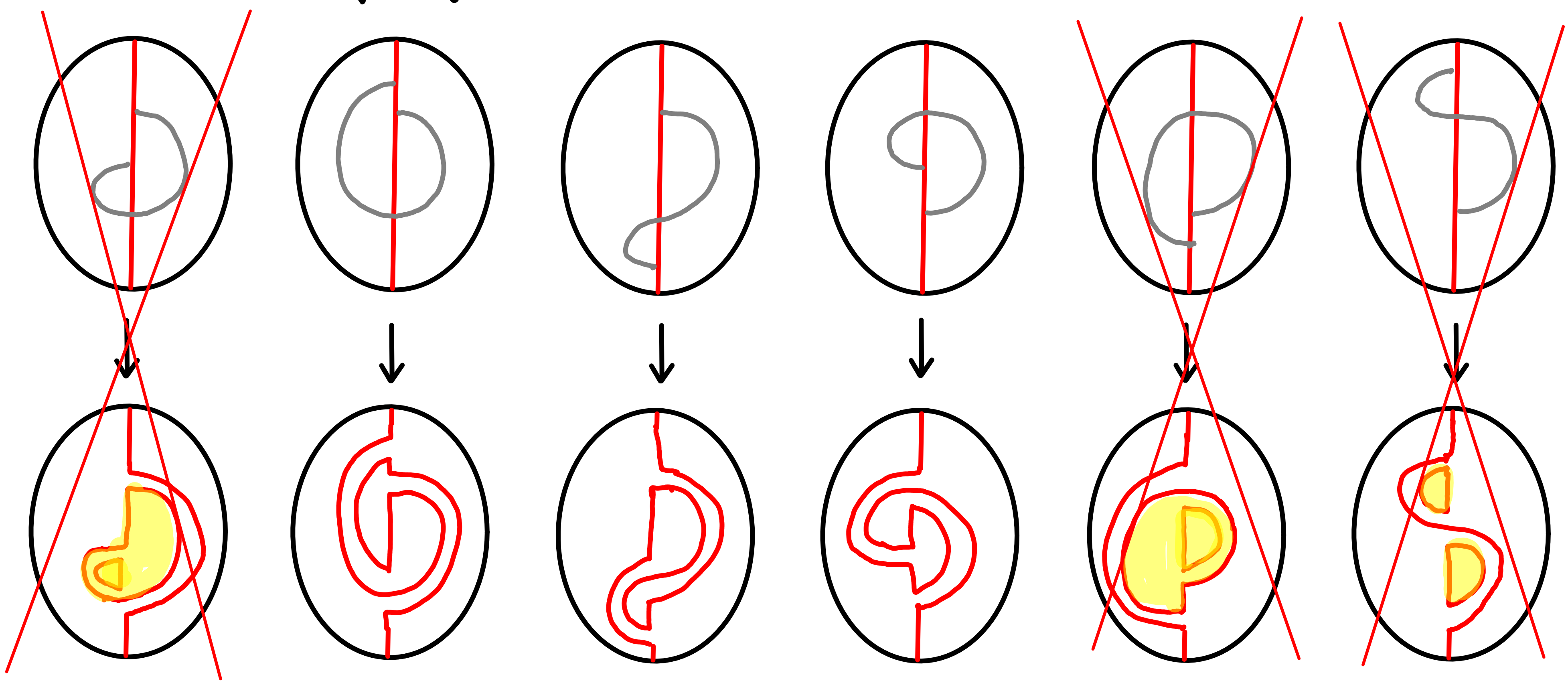
Generators of the Monoid of Legendrian Braids

Straighten everything out!



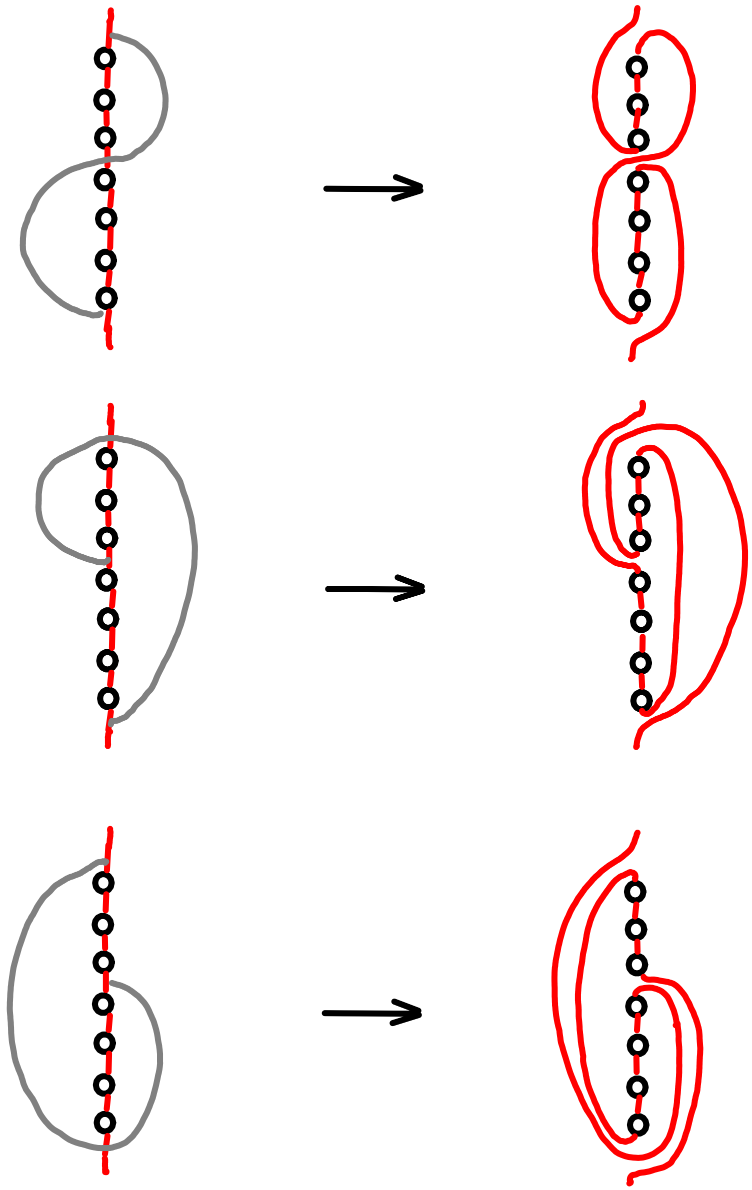
all framing-info is encoded in the dividing curve

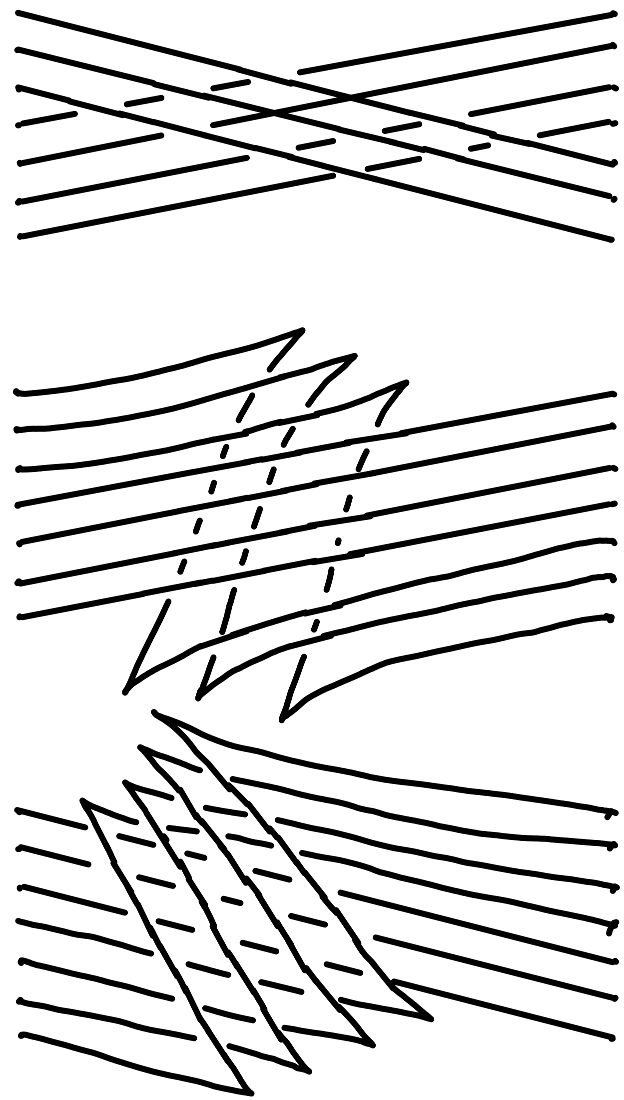
- the contact structure is built up from bypasses.
- What kind of bypasses can occur?



Generators of the Monoid of Legendrian Braids - ctd

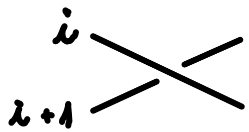
<u>bypass attachment</u> <u>Legendrian front</u>



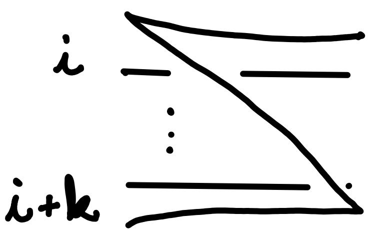


THE GENERATORS

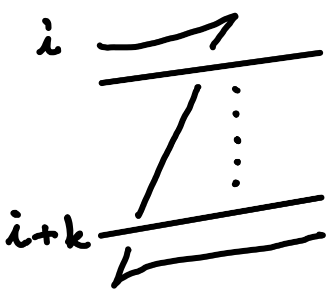
Legendrian front — **framed braid word**



$\sigma_i := X_i$



$\tau_{i+k}^{-1}\,\sigma_{i+k-1}^{-1}\cdots\sigma_i^{-1} := S_{i,i+k}$



$\tau_i^{-1}\,\sigma_i^{-1}\cdots\,\sigma_{i+k}^{-1} := Z_{i,i+k}$

Question: What are the relations?

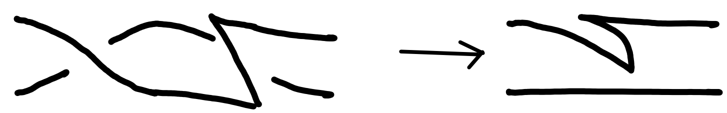
2-Braids

Thm (Etnyre-V) Legendrian 2-braids are generated by:

$x \qquad s \qquad z \qquad St_i^- \qquad St_i^+$

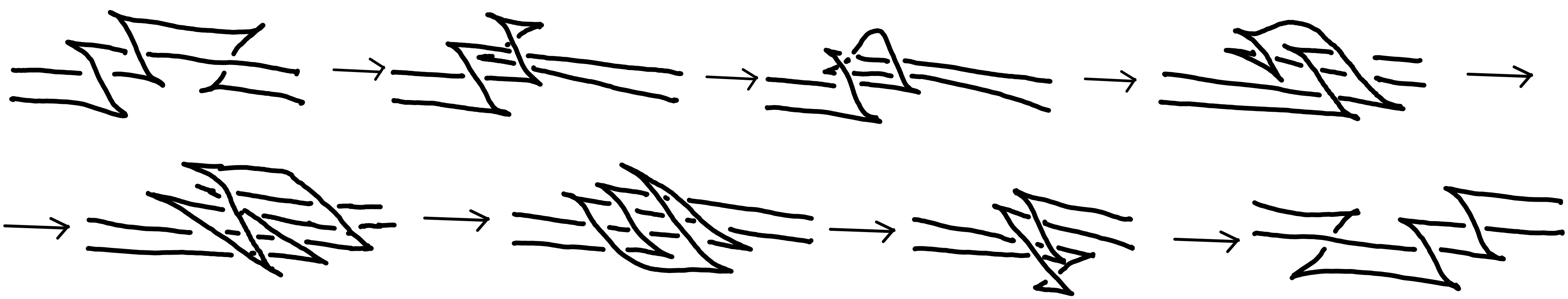
w/ relations: • $St_i^{\pm} \begin{matrix} x \\ s \\ z \end{matrix} = \begin{matrix} x \\ s \\ z \end{matrix} St_{i+1}^{\pm}$

• $xs = St_1^-$



similarly: $xz = St_2^+$, $sx = St_2^-$, $zx = St_1^+$

• $ssz = zss$

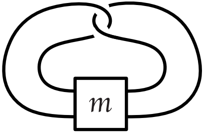


& similarly $zzs = szz$

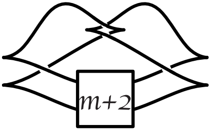
Classification of Legendrian twist knots

(Etnyre-Ng-V)

twist knots



Legendrian representations of twist knots:



in the box:

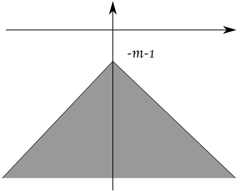
if $m>0$ *if* $m<0$

Classification of Legendrian twist knots

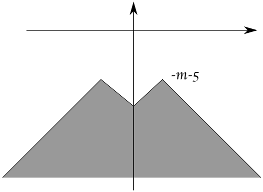
(Etnyre-Ng-V)

- m positive $\Rightarrow$ K is Legendrian simple

m odd

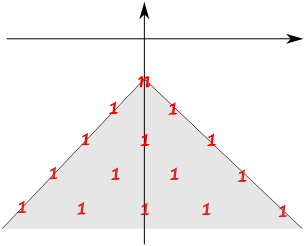


m even

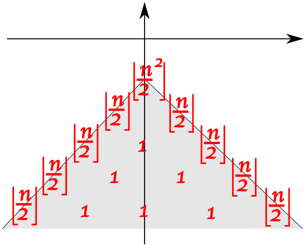


- m negative

m odd



m even



Classification of other knots

"Thm" (Etnyre - V): $\mathcal{K}$ Legendrian simple + Extra condition
$\Rightarrow$ positive Whitehead doubbles with companion $\mathcal{K}$ are Legendrian simple
$\Rightarrow$ positive braid satelites with companion $\mathcal{K}$ have unique Legendrian representatives with maximal Thurston - Bennequin number

Extra condition ensures that all Legendrian representatives of the satellite lie inside the standard contact neighborhood of the Legendrian representative of $\mathcal{K}$ w/ maximal Thurston - Bennequin number

E.g. ! unknot does not satisfy Extra condition !
figure eight knot
torus knots
iterated torus knots

Thanks
for your
Attention!