Knots and contact structures

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Thanks for P. Massot and S. Schönenberger for some of the pictures

Thermodynamics

E	internal energy
T	temperature
S	entropy
P	pressure
V	volume

First Law of Thermodynamics

$$dE = \delta Q - \delta W$$

($Q = \text{processed heat}$, $W = \text{work on its surroundings}$) for a reversible process $\delta Q = TdS$, $\delta W = PdV$

$$dE = TdS - PdV$$

Since E, S and V are thermodynamical functions of a state, the above is true non-reversible processes too.

Thermodynamics – geometric setup

Define the 1–form:

$$\alpha = dE - TdS + PdV$$

States of the gas are on integrals of $\ker \alpha$.

How many independent variables are there?

= What is the maximal dimension of an integral submanifold?

$$\alpha \wedge (d\alpha)^2 = dE \wedge dT \wedge dS \wedge dP \wedge dV$$

$$(d\alpha = -dT \wedge dS + dP \wedge dV \neq 0)$$

$\Rightarrow$ Max dimensional integral manifolds are 2 dimensional.

Deduce state equations for ...

- ...ideal gases;
- ...van der Waals gases.

Contact structures – definition

Contact structure

on a $(2n+1)$–manifold M is a "maximally nonintegrable" hyperplane distribution ξ in the tangent space of M.

Locally: $\xi = \ker\alpha$ $\quad$ $(\alpha \in \Omega_1(M))$
"maximally nonintegrable" $\Leftrightarrow$ $\alpha \wedge (d\alpha)^n \neq 0$

Standard contact structure

On $\mathbb{R}^{2n+1} = \{(x_1, \dots, x_n, y_1, \dots, y_n, z)\}$ $\xi_{\mathrm{st}} = \ker\alpha$

$$\alpha_{\mathrm{st}} = dz + \sum_{i=0}^{n} x_i dy_i$$

α is contact: $(d\alpha = \sum_{i=0}^{n} dx_i \wedge dy_i)$

$$\alpha \wedge (d\alpha)^n = 2^n dz \wedge dx_1 \wedge dy_1 \wedge \dots \wedge \wedge dx_n \wedge dy_n \neq 0$$

Darboux's theorem

Every contact manifold locally looks like $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$.

Contact structures – origin

- thermodynamics;
- odd dimensional counterparts of symplectic manifolds;
- classical mechanics, contact element (Sophus Lie, Elie Cartan, Darboux);
- Hamiltonian dynamics;
- geometric optics, wave propagation (Huygens, Hamilton, Jacobi);
- Natural boundaries of symplectic manifolds.

Contact structures – applications

- (Eliashberg) New proof for Cerf's Theorem: $\mathrm{Diff}(S^3)/\mathrm{Diff}(D^4) = 0$;
- (Akbulut–Gompf) Topological description of Stein domains;
- (Ozsváth–Szabó) HF detects the genus of a knot;
- (Ghiggini, Juhász, Ni) HFK detects fibered knots;
- (Kronheimer–Mrowka) First step to the Poincaré conjecture: Every nontrivial knot in S^3 has property P;
- (Kronheimer–Mrowka) Knots are determined by their complement;
- (Kronheimer–Mrowka, Ozsváth–Szabó) The unknot, trefoil and the figure eight knot are determined by their surgery.

Contact structures on 3–manifolds

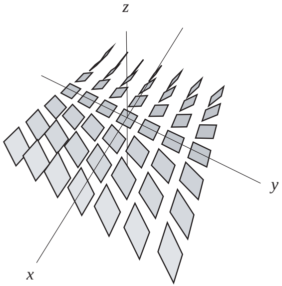
Contact structure (n=1)

on a 3–manifold Y is a "maximally nonintegrable" plane distribution ξ in the tangent space of Y.

"maximally nonintegrable" $\Leftrightarrow$ it rotates (positively) along any curve tangent to ξ

Standard contact structure on $\mathbb{R}^3$

$\xi = \ker(dz + x\,dy)$
$= \langle \frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - \frac{\partial}{\partial y} \rangle$

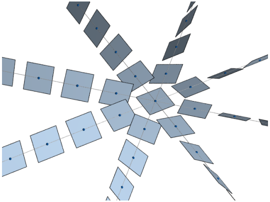


Darboux's theorem

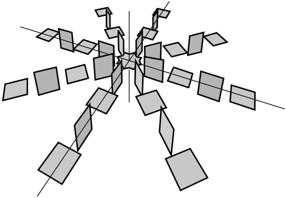
Contact structures locally look like $(\mathbb{R}^3, \xi_{\mathrm{st}})$.

Examples

$$\xi_{\mathrm{sym}} = dz + r^2 d\vartheta = \langle \tfrac{\partial}{\partial r}, r^2\tfrac{\partial}{\partial z} - \tfrac{\partial}{\partial \vartheta} \rangle$$



$$\xi_{\mathrm{OT}} = \ker(\cos r\,dz + r\sin r\,d\vartheta) = \langle \tfrac{\partial}{\partial r}, r\sin r\tfrac{\partial}{\partial z} - \cos r\tfrac{\partial}{\partial \vartheta} \rangle$$

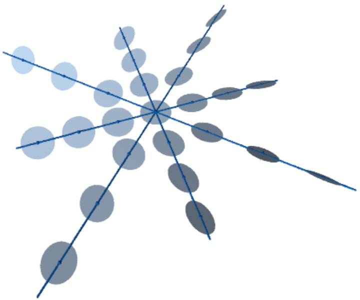


Equivalence of contact structures

Contact isotopy

Two contact structures are isotopic if one can be deformed to the other with an isotopy of the underlying space. $\exists \Psi_t : Y \to Y$, with $\Psi_0 = id$ and $(\Psi_1)_*(\xi_0) = \xi_1$

ξ_{sym} and ξ_{st} are isotopic:



Classification of contact structures

Do all 3–manifolds admit contact structures?

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Yes (Martinet)

several proof using different technics:

- (Martinet, 1971) surgery along transverse knots;
- (Thurston–Winkelnkemper, 1975) open books;
- (Gonzalo, 1978) branched cover;
- (Ding–Geiges–Stipsicz) surgery along Legendrian knots.

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How many contact structures does a 3–manifold admit?

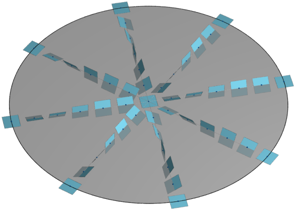
∞

Lutz twist: Once a contact structure is found we can modify it in the neighborhood of an embedded torus.

Tight vs. overtwisted contact structures

overtwisted disc

$D \hookrightarrow Y$ such that D is tangent to ξ



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$$\begin{tikzcd}
 & \textit{overtwisted } (\supseteq \text{overtwisted disc}) \\
\xi \arrow[ur] \arrow[dr] & \\
 & \textit{tight } (\text{not overtwisted})
\end{tikzcd}$$

Theorem (Eliashberg):

$\{$overtwisted ctct structures$\}$/isotopy $\leftrightarrow$ $\{$2–plane fields$\}$/homotopy

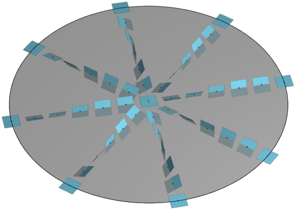
$\Rightarrow$ tight contact structures are "interesting"

geometric meaning: boundaries of complex/symplectic 4–manifolds

Tight vs. overtwisted contact structures

overtwisted disc

$D \hookrightarrow Y$ such that D is tangent to ξ along ∂D



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ξ → *overtwisted* ($\supseteq$ overtwisted disc)

ξ → *tight* (not overtwisted)

Theorem (Eliashberg):

{overtwisted ctct structures}/isotopy $\leftrightarrow$ {2–plane fields}/homotopy

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- (Etnyre–Honda) $-\Sigma(2,3,5)$, the Poincaré homology sphere with reverse orientation does not admit tight contact structure;
- (Lisca–Stipsicz) there exist ∞ many 3–manifold with no tight contact structure.

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How many tight contact structures does a 3–manifold admit?

Characterization done on:

- (Giroux, Honda) Lens spaces;
- (Honda) circle bundles over surfaces;
- (Ghiggini, Ghiggini–Lisca–Stipsicz, Wu, Massot) some Seifert fibered 3–manifolds.

Methods for classification

How can we prove tightness?

- fillability;
- Legendrian knots.

How can we distinguish contact structures?

- homotopical data;
- contact invariant from HF-homologies (Seifert fibered 3–manifolds);
- embedded surfaces;
  if we know the contact structure on the surface, then it is also known in a neighborhood of the surface

How the contact structure on a surface can be encoded?

  - characteristic foliation;
  - on convex surfaces a multicurve is enough.

- embedded curves.

Knots in contact 3–manifolds

Legendrian knot

is a knot L whose tangents lie in the contact planes:

$$TL \in \xi \qquad \text{or} \qquad \alpha(TL) = 0$$

We have already seen Legendrian knots:

- an oriented plane field is a contact structure, if it "rotates" along Legendrian foliations..
- the boundary of an OT disc is Legendrian.

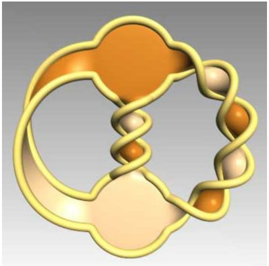
ξ is tangent to D along ∂D $\Leftrightarrow$ ξ does not "twist" as we move along ∂D

Classical Invariants

Thurston-Bennequin number

$$tb(L) = \mathrm{lk}(L, L')$$

where L' is a push off L' of L in the transverse direction;



If Σ a Seifert surface of L (i.e. $\partial\Sigma = L$), then:

$$tb_\Sigma(L) = \mathrm{lk}(L, L') = \#(L \cap \Sigma)$$

Note:

D is OT $\Leftrightarrow tb_D = 0$

Jump back to the proof

Rotation number

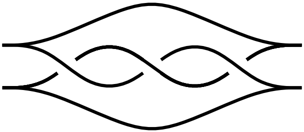
$rot(L)$ is a relative Euler number of ξ on Σ w.r.t. TL

Knots in $(S^3, \xi_{\rm st})$

Recall:

$\xi = \ker(dz + x dy)$

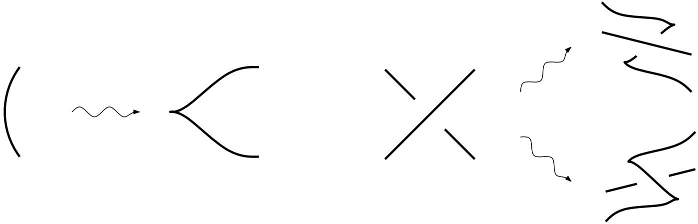
$TK \subset \xi \iff x = -\frac{dz}{dy} \neq \infty$



Claim:

Any knot can be put in Legendrian position.

Proof:



$\square$

Knots in (S^3, ξ_{st})

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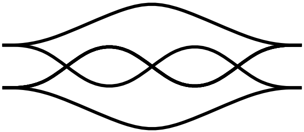

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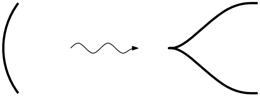
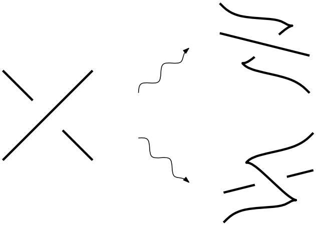




$\square$

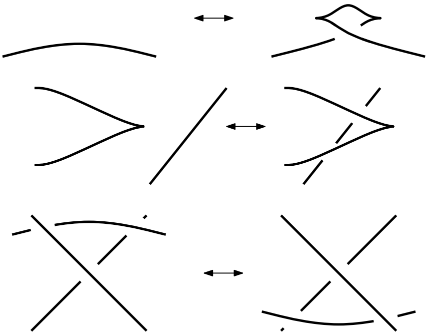
Legendrian isotopy

Legendrian isotopy

Isotopy through Legendrian knots.

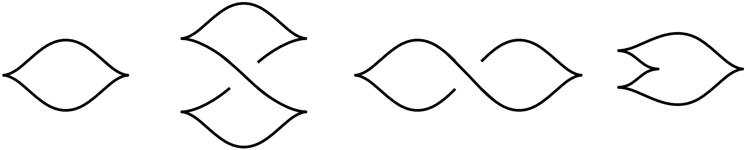
Legendrian Reidemeister moves

Legendrian isotopic knots are related by the following moves:



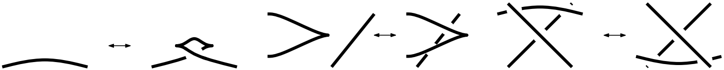
Are they Legendrian isotopic?

Legendrian unknots



Which ones are Legendrian isotopic?

Remember:



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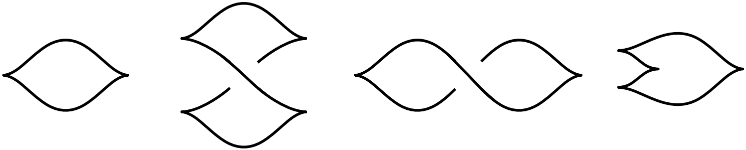

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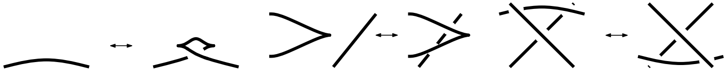

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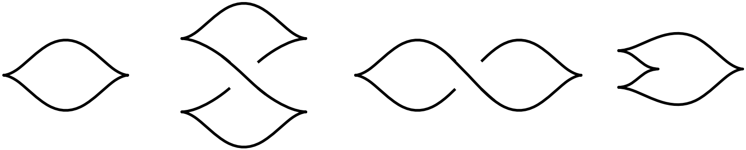

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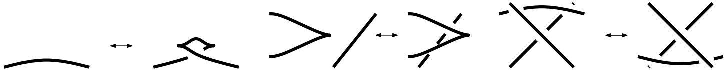
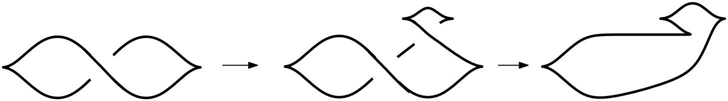
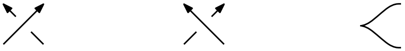


$A \cong B$ $\surd$ and $C \cong D$:



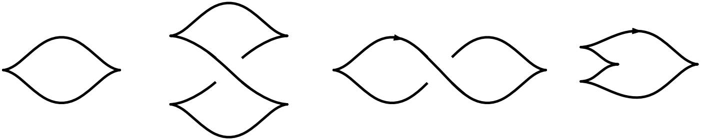
Classical invariants in (S^3, ξ_{st})

Thurston-Bennequin invariant:



rotation number:

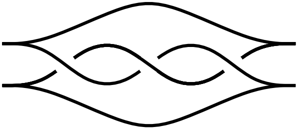
for the Legendrian unknots:



The two definition agree in $(S^3, \xi_{\mathrm{st}} = dz + x\,dy)$

The "new definition"

=3-0-2=1



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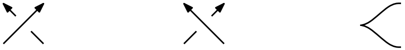

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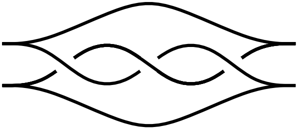
$$tb_\Sigma(L) = \mathrm{lk}(L, L') = \#(L \cap \Sigma)$$



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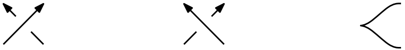

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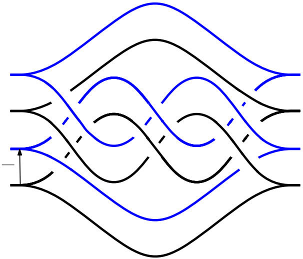

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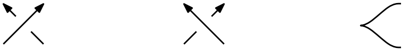

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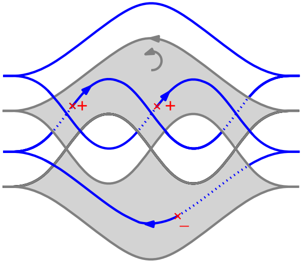
$\frac{\partial}{\partial z}$ is a transverse direction:
$tb = 1 + 1 - 1 = 1$



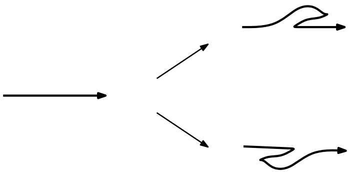
The two definitions agree in general.$\surd$

A new smooth knot invariant

tb can be decreased:

$tb(L_\pm) = tb(L) - 1$

$rot(L_\pm) = rot(L) \pm 1$



Definition:

K is a smooth knot type, then:

$$\overline{tb}(K) = \max\{tb(L) : L \text{ is Legendrian repr. of } K\}$$

Bennequin inequality:

Σ is a (genus g) Seifert surface for a Legendrian knot L, then:

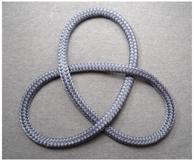
$$tb(L) + |rot(L)| \leq -\chi(\Sigma)$$

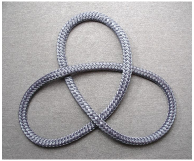
(S^3, ξ_{st}) is tight.

If L is the unknot then $tb(L) + |rot(L)| \leq -\chi(D) = -1$, thus $tb(L) \leq -1$. So $tb(L) \neq 0$ $\square$

$\overline{tb}$ distinguishes mirrors.

Right and left-handed-trefoils



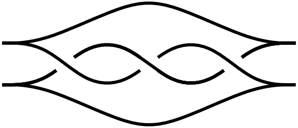


Bounds for the Thurston Bennequin number

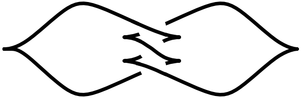
The Seifert surface of the trefoil is a punctured torus, thus

$$tb(L) + |rot(L)| \leq -\chi(\Sigma) = 1$$

the right handed trefoil realizes this bound ($\overline{tb} = 1$):



but the left handed trefoil does not. ($\overline{tb} = -6$)



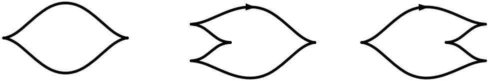
Classification of Legendrian unknots

Legendrian isotopy $\Rightarrow$

- smoothly isotopy;
- *rot* "=";
- *tb* "=".

We have seen:

$tb(L) \leq -1$

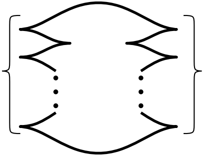


Theorem (Eliashberg-Fraser):

For any pair

$\{(t, r) : t + |r| \leq -1 \ \& \ r \equiv t \mod 2\}$

there is exactly one Legendrian unknot with $tb = t$ and $rot = r$



Proof on the "Algebraic Geometry and Differential Topology Seminar" this Friday

Classification of Legendrian knots

For the unknot we had:
tb "=" & *rot* "=" $\Leftrightarrow$ Legendrian isotopic

Definition:

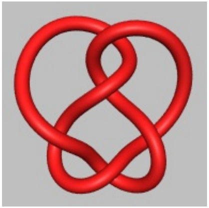
A knot type is called *Legendrian simple* if *tb* and *rot* is enough to classify its Legendrian representations.

Legendrian simple knots:

- (Eliashberg-Fraser) unknot;
- (Etnyre-Honda) torus knots, figure eight knot;
- . . .

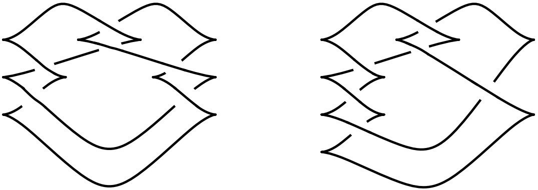
Chekanov's example:

First example for a Legendrian nonsimple knot: the 5_2 knot



Some nonsimple knot types

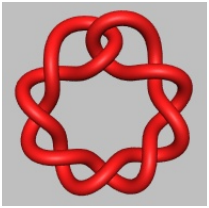
Checkanov's example:



$tb = 1$ and $rot = 0$

Other nonsimple knot types:

- (Epstein–Fuchs–Meyer) twist knots;
- Ng
- Ozsváth–Szabó–Thurston



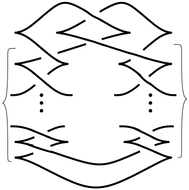
Further classification results

Classification of nonsimple knot types

- (Etnyre–Honda) Can classify Legendrian realizations of $K_1 \# K_2$ in terms of the classification of the Legendrian realizations of L_1 and L_2
- (Etnyre–Honda) $(2,3)$-cable of the $(2,3)$ torus-knot;
- (Etnyre–Ng-V) Classification of Legendrian twist knots (work in progress).

Legendrian Twist knots with maximal tb

- (Chekanov, Epstein–Fuchs–Meyer): $\sim n$ are known to be different
- (Etnyre–Ng-V) There are exactly $\left\lceil \frac{(\frac{(2n+1)}{2}+1)^2}{2} \right\rceil$ different Legendrian representations (work in progress).

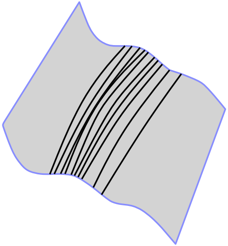


Thanks for your attention!

Equivalent characterizations of contact structures

(Y, ξ) is contact iff locally:

- ξ is totally nonintegrable;
- $\xi = \ker \alpha$, where $\alpha \in \Omega^1(Y)$ and $\alpha \wedge d\alpha \neq 0$;
- ξ rotates (positively) along a Legendrian foliation;



- ξ rotates (positively) along any Legendrian foliation;
- ξ is isotopic to $(\mathbb{R}^3, \xi_{\mathrm{st}})$;
- ξ is isotopic to $(\mathbb{R}^3, \xi_{\mathrm{sym}})$;

▸ Jump back to 3-dimensional contact structures

Property P

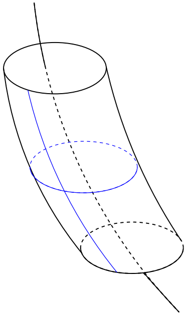
Surgery

cut out a tubular neighborhood of K

glue back along a diffeomorphism $\phi : T^2 \to T^2$

such a map is determined by $\phi(\mu) = p\mu' + q\lambda'$

The surgery is then called $\frac{p}{q}$-surgery



Property P

K has Property P if surgery along K cannot give a counterexample for the Poincaré Conjecture.

Fact (Lickorish, Wallace)

Any 3–manifold can be obtained from S^3 by surgery along a link.

Jump back to the applications

Lutz twist

Lutz twist

We can change ξ along a knot $T \pitchfork \xi$:

ξ is standard along T:

on $\nu(T)$

$$\xi = \ker(\cos(\frac{\pi}{2}r)dt + r\sin(\frac{\pi}{2}r)d\varphi)$$

Change ξ on $\nu(T)$ to:

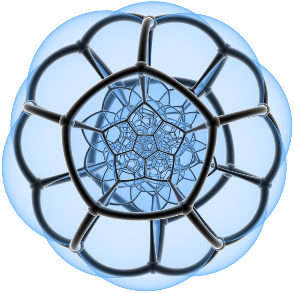
$$\xi' = \ker(\cos(\pi - \frac{3\pi}{2}r)dt + r\sin(\pi - \frac{3\pi}{2}r)d\varphi)$$

Jump back to the classification

Poincaré homology sphere

from the dodecahedron:

Glue each pair of opposite faces of the dodecahedron by using the minimal clockwise twist.



a factor of $SO(3)$

with the rotational symmetries of the dodecahedron (A_5)

Jump back to the applications

Convex surfaces