A SELF-PAIRING THEOREM FOR TANGLE FLOER HOMOLOGY

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Abstract. We show that for a tangle T with $-\partial^0 T \cong \partial^1 T$ the Hochschild homology of the tangle Floer homology $\widetilde{CT}(T)$ is the link Floer homology of the closure $T' = T/(-\partial^0 T \sim \partial^1 T)$ of the tangle, linked with the tangle axis.

1. introduction

Tangle Floer homology is an invariant of tangles in 3-manifolds with boundary S^2 or $S^2 \coprod S^2$, or in closed 3-manifolds, which takes the form of a differential graded module, bimodule, or a chain complex, respectively [5]. It behaves well under gluing and recovers knot Floer homology. Before we state the main results, we recall some definitions from [5] and make some new ones.

Definition 1.1. An *n-marked sphere* $\mathcal{S}$ is a sphere S^2 with n oriented points $t_1, \ldots, t_n$ on its equator $S^1 \subset S^2$ numbered respecting the orientation of S^1.

Definition 1.2. A *marked* (m,n)*-tangle* $\mathcal{T}$ in an oriented 3-manifold Y with two boundary components $\partial^0 Y \cong S^2$ and $\partial^1 Y \cong S^2$ is a properly embedded 1–manifold T with $(-\partial^0 Y, -(\partial^0 Y \cap \partial T))$ identified with an m-marked sphere and $(\partial^1 Y, \partial^1 Y \cap \partial T)$ identified with an n-marked sphere (via orientation-preserving diffeomorphisms). We denote $-(\partial^0 Y \cap \partial T)$ and $\partial^1 Y \cap \partial T$ along with the ordering information by $-\partial^0 \mathcal{T}$ and $\partial^1 \mathcal{T}$.

Definition 1.3. A *strongly marked* (m,n)*-tangle* $(Y, \mathcal{T}, \gamma)$ is a marked (m,n)-tangle $(Y, \mathcal{T})$, along with a framed arc γ connecting $\partial^0 Y$ to $\partial^1 Y$ in the complement of $\mathcal{T}$ such that γ and its framing λ_γ (viewed as a push off of γ) have ends on the equators of the two marked spheres, and we see $-\partial^0 \mathcal{T}, -\partial^0 \gamma, -\partial^0 \lambda_\gamma$ and $\partial^1 \mathcal{T}, \partial^1 \gamma, \partial^1 \lambda_\gamma$ in this order along each equator. See Figure 1.

As a special case, an (m,n)-tangle in $\mathbb{R}^2 \times I$ is a cobordism (contained in $[1,\infty) \times \mathbb{R} \times I$) from $\{1,\ldots,m\} \times \{0\} \times \{0\}$ to $\{1,\ldots,n\} \times \{0\} \times \{1\}$. A tangle in $\mathbb{R}^2 \times I$ can be thought of as a strongly marked tangle, by compactifying $\mathbb{R}^2 \times I$ to $S^2 \times I$, taking the images of $\mathbb{R} \times \{0\} \times \{0\}$ and $\mathbb{R} \times \{0\} \times \{1\}$ to be the equators of the marked spheres, and setting $(\gamma, \lambda_\gamma) := (\{(-1,0)\} \times I, \{(0,0)\} \times I)$.

We turn our attention to strongly marked tangles $(Y, \mathcal{T}, \gamma)$ with $-\partial^0 \mathcal{T} \cong \partial^1 \mathcal{T}$.

Definition 1.4. A strongly marked tangle $(Y, \mathcal{T}, \gamma)$ is called *closable* if $-\partial^0 \mathcal{T} \cong \partial^1 \mathcal{T}$. Given a closable tangle $(Y, \mathcal{T}, \gamma)$, we can "glue it to itself" to form its *closure* (T', Y', γ') by identifying the two boundary components of Y, $-\partial^0 Y$ and $\partial^1 Y$, with the same marked sphere. The *surgered closure* of $(Y, \mathcal{T}, \gamma)$ is the pair $(T_0, Y_{0(\gamma)})$, where the link T_0 is the union of T' and the negatively oriented meridian μ_γ of γ in the 0-surgery $Y_{0(\gamma)}$ of Y' along the framed knot γ'. We call μ_γ the *tangle axis* of the tangle $\mathcal{T}$. See Figure 1.

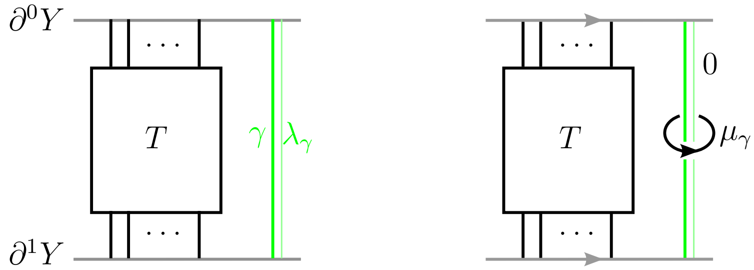

FIGURE 1. Left: A strongly marked tangle (T, Y, γ). Right: The surgered closure $(T_0, Y_0(\gamma))$ of the tangle (T, Y, γ).

When Y is $S^2 \times I$ and (γ, λ_γ) is a product as above, then $Y_0(\gamma) \cong S^3$ and T_0 is the link formed by the closure of $T \subset \mathbb{R}^2 \times I \subset S^3$ and an unknot that is the boundary of a disk containing $-\partial^0 T \sim \partial^1 T$, see Figure 2. For example, for a braid $B \in \mathbb{R}^2 \times I$, the tangle axis is precisely the braid axis.

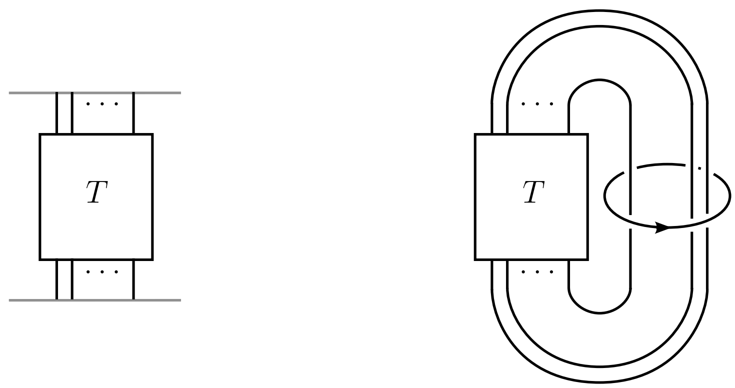

FIGURE 2. Left: A tangle $T \subset \mathbb{R}^2 \times I$. Right: The corresponding link $T_0 \subset S^3$.

For a tangle $(Y, \mathcal{T}, \gamma)$, the tangle Floer homology $\widetilde{CT}(Y, \mathcal{T}, \gamma)$ is a left-right DA bimodule over $(\mathcal{A}(-\partial^0\mathcal{T}), \mathcal{A}(\partial^1\mathcal{T}))$, where $\mathcal{A}(-\partial^0\mathcal{T})$ and $\mathcal{A}(\partial^1\mathcal{T})$ are differential graded algebras associated to $-\partial^0\mathcal{T}$ and $\partial^1\mathcal{T}$, respectively, see [5]. For a closable tangle, these two algebras are the same, and one can take the Hochschild homology of the bimodule, see [1, Section 2.3.5]. We show that this Hochschild homology is the knot Floer homology of the surgered closure of the tangle.

Theorem 1.1. *Let $(Y, \mathcal{T}, \gamma)$ be a closable strongly marked tangle. Then*

$$HH(\widetilde{CT}(Y, \mathcal{T}, \gamma)) \cong \widehat{HFK}(T_0, Y_{0(\gamma)}).$$

Note that so far gradings for tangle Floer homology have only been defined when the underlying manifold is $S^2 \times I$ or B^3, so Theorem 1.1 only claims an ungraded isomorphism.

For the special case of a tangle T in $\mathbb{R}^2 \times I$, we state a graded version. In this case, $HH(\widetilde{CT}(T))$ inherits the Maslov and Alexander gradings M and A from $\widetilde{CT}(T)$, and also carries a *strands* grading S, see Section 3. Let l be the number of components of T_0, and label the components $L_0 = \mu_\gamma, L_1, \ldots, L_{l-1}$. The link Floer homology of T_0, $\widetilde{HFL}(T_0)$, is multigraded, with Maslov grading M in $\mathbb{Z} + \frac{l-1}{2}$ and Alexander multigrading $(A_0, \ldots, A_{l-1})$

in $(\frac{1}{2}\mathbb{Z})^l$, with each $\frac{1}{2}\mathbb{Z}$ factor corresponding to a component of the link [4]. Let $\widetilde{HFL}(T_0, 0)$ be $\widetilde{HFL}(T_0)$ with multigrading collapsed to a trigrading by M, A_0, and $A' := A_1 + \cdots A_{l-1} = A - A_0$ (here $A = A_0 + \cdots + A_{l-1}$ is the Alexander grading on $\widetilde{HFK}(T_0)$).

Theorem 1.2. *Let T be a tangle in $\mathbb{R}^2 \times I$ with $-\partial^0 T \cong \partial^1 T$, and let T_0 be its surgered closure. Then there is an isomorphism*

$$HH(\widetilde{CT}(T)) \cong \widetilde{HFL}(T_0, 0)$$

which respects the trigrading in the following sense. If the isomorphism maps a homogeneous element $\mathbf{x} \in HH(\widetilde{CT}(T))$ to an element $\mathbf{y} \in \widetilde{HFL}(T_0)$, then $\mathbf{y}$ is homogeneous and

$$\begin{aligned} M(\mathbf{y}) &= M(\mathbf{x}) + S(\mathbf{x}) - a - 1 \\ A'(\mathbf{y}) &= A(\mathbf{x}) - S(\mathbf{x}) + \frac{l}{2} + n - a - 1 \\ A_0(\mathbf{y}) &= S(\mathbf{x}) - \frac{n+1}{2}, \end{aligned}$$

where $n = |\partial^1 T|$, and a is the number of positively oriented points in $\partial^1 T$.

Remark 1.3. The authors are in the process of upgrading the invariants in [5] to have Alexander multigradings, corresponding to different components of the tangle. The arguments in this paper automatically imply that the isomorphism from Theorem 1.2 respects the multigrading, with appropriate additive constants.

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2. Algebra review

Let A be a unital differential graded algebra over a ground ring $\mathbf{k}$, where $\mathbf{k}$ is a direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. The unit gives a preferred map $\iota : \mathbf{k} \to A$. We assume that A is *augmented*, i.e. there is a map $\epsilon : A \to \mathbf{k}$ such that $\epsilon(1) = 1$, $\epsilon(ab) = \epsilon(a)\epsilon(b)$, and $\epsilon(\partial a) = 0$. The *augmentation ideal* $\ker \epsilon$ is denoted by A_+.

A *type DA bimodule over* (A, A) is a graded $\mathbf{k}$-bimodule M, together with degree 0, $\mathbf{k}$-linear maps

$$\delta^1_{1+j} : M \otimes A[1]^{\otimes j} \to A \otimes M[1],$$

satisfying a certain compatibility condition, see [1, Definition 2.2.42].

A *DA* bimodule is *bounded* if the structure maps behave in a certain nice way, see [1, Definition 2.2.45]. We will not recall the complete definition of boundedness here, but we point out that the structures arising from nice Heegaard diagrams are bounded, and moreover the only nonzero structure maps in that case are δ^1_1 and δ^1_2. We will call a *DA* bimodule *nice* if it is bounded and $\delta^1_i = 0$ for all $i > 2$.

Given a bounded type *DA* bimodule N over (A, A), one can define a chain complex $(N^\circ, \widetilde{\partial})$ whose homology agrees with the Hochschild homology of the $\mathcal{A}_\infty$-bimodule $A \boxtimes N$ corresponding to N, see [1, Section 2.3.5]. The vector space N°, called the the *cyclicization* of N, is the quotient $N/[N, \mathbf{k}]$, where $[N, \mathbf{k}]$ is the submodule of N generated by elements $xk - kx$, for $x \in N$ and $k \in \mathbf{k}$. The differential $\widetilde{\partial}$ is easy to describe when N is nice. We recall the construction in this special case below.

Define a cyclic rotation map $R : (A \otimes N)^\circ \to (N \otimes A_+)^\circ$ by

$$R(a \otimes x) = x \otimes [(\mathrm{id} - \iota \circ \epsilon)(a)]$$

The map $\epsilon \otimes \mathrm{id} : A \otimes N \to \mathbf{k} \otimes N = N$ descends to a map $(A \otimes N)^\circ \to N^\circ$, which we will also denote ϵ. We denote the cyclicizations of $\delta_1^1 : N \to A \otimes N$ and $\delta_2^1 : N \otimes A_+ \to A \otimes N$ by δ_1^1 and δ_2^1 as well. Finally, $\widetilde{\partial}$ is defined as

$$\widetilde{\partial} = \epsilon \circ \delta_1^1 + \epsilon \circ \delta_2^1 \circ R \circ \delta_1^1.$$

Given a tangle $(Y, \mathcal{T}, \gamma)$, one can represent it by a multipointed bordered Heegaard diagram $\mathcal{H}$ with two boundary components $\mathcal{H}_0$ and $\mathcal{H}_1$, see [5, Section 8.2]. To $-\mathcal{H}_0$ and $\mathcal{H}_1$ one associates differential algebras $\mathcal{A}(-\mathcal{H}_0)$ and $\mathcal{A}(\mathcal{H}_1)$, and to $\mathcal{H}$ a *DA* bimodule $\widetilde{CT}(\mathcal{H})$ over $\mathcal{A}(-\mathcal{H}_0)$ and $\mathcal{A}(\mathcal{H}_1)$. The structure maps on the bimodule are obtained by counting certain holomorphic curves in $\mathcal{H} \times I^2$. See [5, Sections 7.2 and 10.3] For a tangle in $\mathbb{R}^2 \times I$, the bimodule can also be defined in terms of sequences of strand diagrams corresponding to a decomposition of the tangle into elementary pieces, see [5, Sections 3 and 5.2]. We do not recall the two constructions here, but refer the reader to [5].

3. Proof via special diagrams

We prove both theorems via nice diagrams.

Proof of Theorem 1.1. Let $(Y, \mathcal{T}, \gamma)$ be a closable strongly marked (n, n)-tangle, and let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbb{X}, \mathbb{O}, \mathbf{z})$ be a nice bordered Heegaard diagram for $(Y, \mathcal{T}, \gamma)$, which exists by [5, Proposition 12.1]. Glue $\mathcal{H}$ to itself by identifying $-\partial^0\mathcal{H}$ and $\partial^1\mathcal{H}$, and call the result $\mathcal{H}'$ (note this is not a valid Heegaard diagram). Recall that $\mathbf{z} = \{\mathbf{z}_1, \mathbf{z}_2\}$ is a set of two arcs in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ with boundary on $\partial\Sigma \setminus \boldsymbol{\alpha}$, oriented from the left to the right boundary, and let $\mathbf{z}_1'$ and $\mathbf{z}_2'$ be the resulting closed curves in $\mathcal{H}'$. Surger $\mathcal{H}'$ along $\mathbf{z}_1'$ and $\mathbf{z}_2'$, and place 4 basepoints in the 4 resulting regions: X_1, O_1, X_2, and O_2 in the region whose boundary contains a^0_{n+1},a^0_1, a^0_{2n+2}, and a^0_{n+2}, respectively. The result is a diagram $\mathcal{H}^\circ = (\Sigma^\circ, \boldsymbol{\alpha}^\circ, \boldsymbol{\beta}^\circ, \mathbb{X}^\circ, \mathbb{O}^\circ)$, see Figure 3. Now

The reader can verify that $\mathcal{H}^\circ$ represents $(T_0, Y_{0(\gamma)})$. Observe that the generators of $\mathcal{H}'$, or equivalently the generators of $\mathcal{H}^\circ$, correspond to generators $\mathbf{x}$ of $\mathcal{H}$ with $\bar{o}^0(\mathbf{x}) = o^1(\mathbf{x})$.

Denote the algebra $\mathcal{A}(\partial^1\mathcal{H}) \cong \mathcal{A}(-\partial^0\mathcal{H})$ by $\mathcal{A}$, and its ring of idempotents by $\mathbf{k}$. Recall that $\mathcal{A}$ has a basis over $\mathbb{F}_2$ consisting of strand diagrams [5, Section 7]. We define the augmentation map $\epsilon : \mathcal{A} \to \mathbf{k}$ on this basis explicitly: it is the identity on generators in $\mathbf{k} \subset \mathcal{A}$ and zero on generators $a \notin \mathbf{k} \subset \mathcal{A}$. The structure maps on the *DA* bimodule $\widetilde{CT}(\mathcal{H})$ count the following types of domains (see [5, Sections 10 and 12]):

(1) empty provincial rectangles and bigons. These contribute to δ_1^1, with image in $\mathbf{k} \otimes \widetilde{CT}(\mathcal{H}) \subset \mathcal{A} \otimes \widetilde{CT}(\mathcal{H})$.
(2) empty rectangles that intersect $\partial^0\mathcal{H}$ (the left boundary of $\mathcal{H}$). These contribute to δ_1^1, with image in $\mathcal{A}_+ \otimes \widetilde{CT}(\mathcal{H}) \subset \mathcal{A} \otimes \widetilde{CT}(\mathcal{H})$.
(3) sets of empty rectangles, each of which intersects $\partial^1\mathcal{H}$ (the right boundary of $\mathcal{H}$). These comprise δ_2^1, whose image is entirely contained in $\mathbf{k} \otimes \widetilde{CT}(\mathcal{H})$.

The differential on the Hochschild complex $(\widetilde{CT}(\mathcal{H})^\circ, \widetilde{\partial})$ then counts the following domains on $\mathcal{H}$. The map $\epsilon \circ \delta_1^1$ counts provincial rectangles and bigons, and then forgets the idempotent

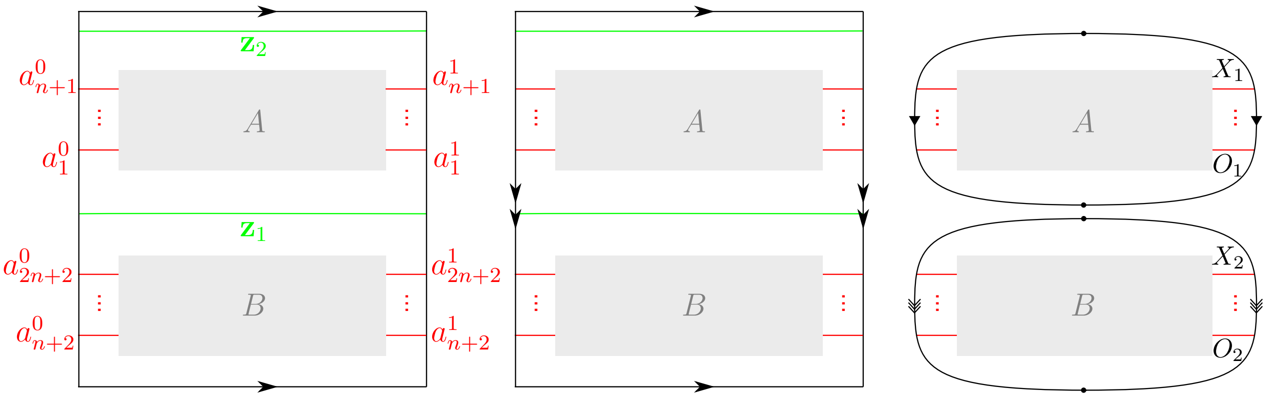

FIGURE 3. Left: A Heegaard diagram $\mathcal{H}$ for a closable tangle $(Y, \mathcal{T}, \gamma)$. The left edge is $\partial^0\mathcal{H}$ and the right edge is $\partial^1\mathcal{H}$. Middle: The diagram $\mathcal{H}'$ obtained by self-gluing $\mathcal{H}$. Right: The Heegaard diagram $\mathcal{H}^\circ$ obtained by surgery on the two green curves on $\mathcal{H}'$.

component of the output. These are exactly the empty rectangles and bigons in $\mathcal{H}^\circ$ that do not cross $-\partial^0\mathcal{H} = \partial^1\mathcal{H}$. The map $\epsilon \circ \delta^1_2 \circ R \circ \delta^1_1$ counts rectangles as follows. Since the rotation map R is zero on elements $e \otimes x$ with $e \in \mathbf{k}$, only the part of δ^1_1 that counts domains of Type (2) contributes. Thus, the image of $R \circ \delta^1_1$ is generated by elements of form $y \otimes a$, where $a \in \mathcal{A}_+$ is a generator with only one moving strand. Thus, the part of δ^1_2 that contributes to $\epsilon \circ \delta^1_2 \circ R \circ \delta^1_1$ counts individual empty rectangles that intersect $\partial^1\mathcal{H}$. To sum up, $\epsilon \circ \delta^1_2 \circ R \circ \delta^1_1$ counts pairs of empty rectangles, one with an edge on $\partial^0\mathcal{H}$ and one with an edge on $\partial^1\mathcal{H}$, which glue up to a rectangle after the identification $-\partial^0\mathcal{H} \sim \partial^1\mathcal{H}$. These are exactly the empty rectangles in $\mathcal{H}^\circ$ that cross $-\partial^0\mathcal{H} \sim \partial^1\mathcal{H}$.

Thus, $(\widetilde{CT}(\mathcal{H})^\circ, \widetilde{\partial}) \cong \widetilde{CFK}(\mathcal{H}^\circ)$. □

Proof of Theorem 1.2. We already discussed the isomorphism in the proof of Theorem 1.1. It remains to identify the gradings.

Let $\mathcal{H}$ be a Heegaard diagram for T obtained by plumbing annular bordered grid diagrams, as in [5, Section 4]. By gluing on a diagram for the straight strands $\partial^1 T \times I$ if necessary, we may assume that $\mathcal{H}$ has even genus, which we denote by $2g$ (this makes for an easier gradings argument). See the top diagram in Figure 4. We modify $\mathcal{H}$ to a diagram $\mathcal{H}^\circ$ for T_0, as in the proof of Theorem 1.1. See the bottom diagram in Figure 4.

Call the part of $\mathcal{H}^\circ$ away from the four regions resulting from the surgery on $\mathbf{z}'_i$ the *nice* part of $\mathcal{H}^\circ$ (this is the part that is the plumbing of grid diagrams). As in Figure 3, we can draw $\mathcal{H}^\circ$ on the plane, as the union of two $2g$-punctured disks with certain identifications of the boundary, see Figure 7. As seen on Figure 7, we refer to the top/bottom disk as the *top/bottom half* of $\mathcal{H}^\circ$, respectively.

Denote the set of generators of $\mathcal{H}$ by $\mathfrak{S}(\mathcal{H})$, and the subset of generators with i occupied α-arcs on the right by $\mathfrak{S}_i(\mathcal{H})$. Denote the subsets of $\mathfrak{S}(\mathcal{H})$ and $\mathfrak{S}_i(\mathcal{H})$ that correspond to generators of $\mathcal{H}^\circ$ by $\mathfrak{S}(\mathcal{H})^\circ$ and $\mathfrak{S}_i(\mathcal{H})^\circ$, and the corresponding sets of generators of $\mathcal{H}^\circ$ by $\mathfrak{S}(\mathcal{H}^\circ)$ and $\mathfrak{S}_i(\mathcal{H}^\circ)$, respectively. For a generator $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, denote the corresponding generator in $\mathfrak{S}(\mathcal{H}^\circ)$ by $\mathbf{x}^\circ$. Define a *strands* grading on generators by $S(\mathbf{x}^\circ_i) = i$ for $\mathbf{x}^\circ_i \in \mathfrak{S}_i(\mathcal{H}^\circ)$.

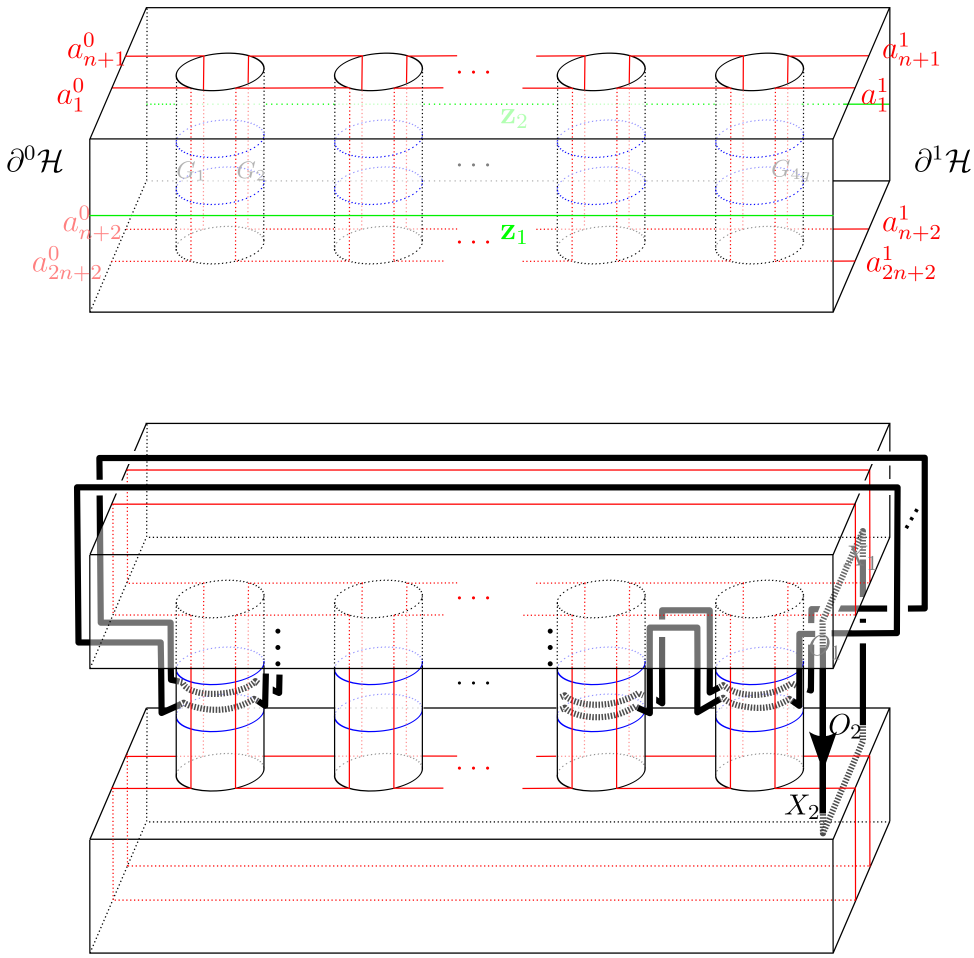

FIGURE 4. Top: A diagram for T in $S^2 \times I$. There are two grids on each vertical annulus, Xs and Os omitted for simplicity. Bottom: The corresponding diagram for T_0 in S^3, along with T_0 in bold black.

Since $\mathcal{H}^\circ$ is a diagram for S^3, any two generators are connected by a domain. Let $\mathbf{x}^\circ, \mathbf{y}^\circ \in \mathfrak{S}(\mathcal{H}^\circ)$, and let $B \in \pi_2(\mathbf{x}^\circ, \mathbf{y}^\circ)$. By adding regions of $\Sigma^\circ \setminus \boldsymbol{\alpha}^\circ$, we can assume that the domain of B is contained entirely in the top half of $\mathcal{H}^\circ$ and has zero multiplicity in the lowest region (the one containing O_1) of the top half of $\mathcal{H}^\circ$. The oriented boundary of B splits into two pieces, $\partial^\alpha B \subset \boldsymbol{\alpha}^\circ$ and $\partial^\beta B \subset \boldsymbol{\beta}^\circ$. The piece $\partial^\alpha B$ is the union of arcs in $\boldsymbol{\alpha}^\circ$ such that $\partial(\partial^\alpha B) = \mathbf{y}^\circ - \mathbf{x}^\circ$. Let $\alpha_1, \dots, \alpha_{n+1}$ be the α-circles in $\mathcal{H}^\circ$ resulting from the gluing of the α-arcs in $\mathcal{H}$, labelled so that $a_i^0 \in \alpha_i$, and let $x_i = \mathbf{x}^\circ \cap \alpha_i$, $y_i = \mathbf{y}^\circ \cap \alpha_i$. Below, we turn our attention to the oriented arcs $c_i := \partial^\alpha B \cap \alpha_i$, and to each c_i we associate a number $t_i \in \{-1, 0, 1\}$. Since B is contained in the top half of $\mathcal{H}^\circ$, there are three possibilities for each c_i:

- c_i is contained in the rightmost/leftmost grid if and only if x_i and y_i are (in this case, define $t_i = 0$);
- c_i covers both the rightmost and the leftmost grid, and is oriented to the right, as seen on Figure 5, if and only if x_i is in the rightmost grid and y_i is in the leftmost grid (in this case, define $t_i = 1$);

- c_i covers both the rightmost and the leftmost grid, and is oriented to the left, as seen on Figure 5, if and only if x_i is in the leftmost grid and y_i is in the rightmost grid (in this case, define $t_i = -1$).

For $1 \le i \le n$, let R_i be the region of $\Sigma^\circ \setminus (\boldsymbol{\alpha}^\circ \cup \boldsymbol{\beta}^\circ)$ containing the image of the interval (a_i^0, a_{i+1}^0) in Σ°, and let R_{n+1} be the topmost region of the top half of $\mathcal{H}^\circ$. See Figure 5.

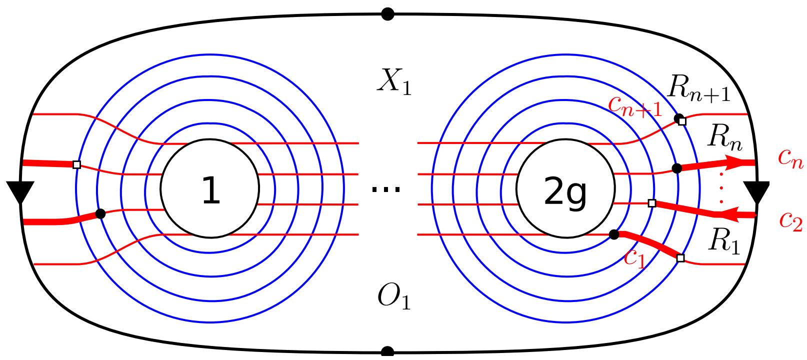

FIGURE 5. The top half of $\mathcal{H}^\circ$, along with the arcs c_i in thick red (c_{n+1} is just a point).

It is not hard to see that the multiplicity of B at each R_i is $t_1 + \dots + t_i$, and that the multiplicity of B at R_{n+1} is zero if and only if the generators $\mathbf{x}$ and $\mathbf{y}$ of $\mathcal{H}$ corresponding to $\mathbf{x}^\circ$ and $\mathbf{y}^\circ$ occupy the same number of arcs on the right.

Suppose $\mathbf{x}^\circ, \mathbf{y}^\circ \in \mathfrak{S}_i(\mathcal{H}^\circ)$. By the above, $\mathbf{x}^\circ$ and $\mathbf{y}^\circ$ are connected by a domain B contained entirely in the nice part of $\mathcal{H}^\circ$, i.e. $n_{X_1}(B) = n_{X_2}(B) = n_{O_1}(B) = n_{O_2}(B) = 0$, so $A_0(\mathbf{x}) = A_0(\mathbf{y})$. Then B is the result of self-gluing a domain B' in $\mathcal{H}$. Note that the left and right multiplicities of B' match up, i.e. if $p_0 \in \partial^0\mathcal{H}$ and $p_1 \in \partial^1\mathcal{H}$ are points that are identified in $\mathcal{H}^\circ$, then $m(\partial^0 B', p_0) = -m(\partial^1 B', p_1)$. Recall the definitions of the sets $S^i_{\mathbb{O}}, S^i_{\mathbb{X}}, S^i_{\mathbf{x}}, S^i_{\mathbf{y}}$ and of the gradings of domains from [5, Section 11.2]. For each of the four types of sets, $p_0 \in \bar{S}^0_\bullet$ if and only if $p_1 \in S^1_\bullet$. Further, $e(B) = e(B')$, and $n_p(B) = n_p(B')$ for any point p. Then

$$\begin{aligned} M(B') =& -e(B') - n_{\mathbf{x}}(B') - n_{\mathbf{y}}(B') + \frac{1}{2} m([\partial^\partial B'], \bar{S}^0_{\mathbf{x}} + \bar{S}^0_{\mathbf{y}} + S^1_{\mathbf{x}} + S^1_{\mathbf{y}}) \\ & - m([\partial^\partial B'], \bar{S}^0_{\mathbb{O}} + S^1_{\mathbb{O}}) + 2n_{\mathbb{O}}(B') = -e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) + 2n_{\mathbb{O}}(B) = M(B) \\ A(B') =& \frac{1}{2} m([\partial^\partial B'], \bar{S}^0_{\mathbb{X}} - \bar{S}^0_{\mathbb{O}} + S^1_{\mathbb{X}} - S^1_{\mathbb{O}}) + n_{\mathbb{O}}(B') - n_{\mathbb{X}}(B') = n_{\mathbb{O}}(B) - n_{\mathbb{X}}(B) = A(B). \end{aligned}$$

Thus, the relative (M, A) gradings are the same in $\widetilde{CT}(\mathcal{H})$ as in $\widehat{CFK}(\mathcal{H}^\circ)$, and the relative A_0 grading is zero, i.e.

$$\begin{aligned} M(\mathbf{x}^\circ) - M(\mathbf{y}^\circ) &= M(\mathbf{x}) - M(\mathbf{y}) \\ A(\mathbf{x}^\circ) - A(\mathbf{y}^\circ) &= A(\mathbf{x}) - A(\mathbf{y}) \\ A_0(\mathbf{x}^\circ) - A_0(\mathbf{y}^\circ) &= 0. \end{aligned}$$

Next, we compare the gradings of generators with distinct numbers of occupied arcs on the right. The plumbing of $4g$ grid diagrams for $\mathcal{H}$ corresponds to a sequence of $4g$ shadows $\mathcal{P}_1, \dots, \mathcal{P}_{4g}$, see [5, Sections 3 and 4] and Figure 6. Suppose $\mathfrak{S}_i(\mathcal{H}^\circ) \neq \emptyset$ and $\mathfrak{S}_{i+j}(\mathcal{H}^\circ) \neq \emptyset$ for some i, j, and let $\mathbf{x}_i \in \mathfrak{S}_i(\mathcal{H})^\circ, \mathbf{x}_{i+j} \in \mathfrak{S}_{i+j}(\mathcal{H})^\circ$. The generator $\mathbf{x}_{i+j}$ has j more strands than $\mathbf{x}_i$ in each even-indexed shadow $\mathcal{P}_{2t}$, so in particular pick one strand in each $\mathcal{P}_{2t}$, and let p_{2t-1} and p_{2t} be the endpoints of the strand in $\partial^0\mathcal{P}_{2t}$ and $\partial^1\mathcal{P}_{2t}$, respectively. Replacing these $2g$ strands with the strands from p_{2t} to p_{2t+1} produces a generator $\mathbf{x}_{i+j-1} \in \mathfrak{S}_{i+j-1}(\mathcal{H})^\circ$. Repeating this procedure shows that $\mathfrak{S}_k(\mathcal{H})^\circ \neq \emptyset$ for every $i+1 \leq k \leq i+j-1$ as well.

Then it suffices to choose two generators $\mathbf{x}_i \in \mathfrak{S}_i(\mathcal{H})^\circ$ and $\mathbf{x}_{i+1} \in \mathfrak{S}_{i+1}(\mathcal{H})^\circ$ for each i such that $\mathfrak{S}_i(\mathcal{H})^\circ \neq \emptyset$ and $\mathfrak{S}_{i+1}(\mathcal{H})^\circ \neq \emptyset$, and understand the relative (M, A, A_0) grading for the corresponding generators $\mathbf{x}_i^\circ, \mathbf{x}_{i+1}^\circ$. Let $\mathbf{x}_{i+1} \in \mathfrak{S}_{i+1}(\mathcal{H})^\circ$ and let $p_0, \ldots, p_{4g} \simeq p_0$ be as above. Modify $\mathbf{x}_{i+1}$ as follows. First exchange each p_t with the topmost point in $\partial^0\mathcal{P}_t$ (along with any associated strands), so that now $p_0, \ldots, p_{4g} \simeq p_0$ are the topmost points of the shadows. Then, if the strands on $\mathcal{P}_{2t}$ ending at p_{2t-1} and p_{2t} are distinct, resolve their crossing so that now there is a strand connecting p_{2t-1} and p_{2t}. Let $\mathbf{x}_i$ be the generator obtained from $\mathbf{x}_{i+1}$ by replacing these strands with the strands from p_{2t} to p_{2t+1}, as above. Now $\mathbf{x}_i \in \mathfrak{S}_i(\mathcal{H})^\circ$ and $\mathbf{x}_{i+1} \in \mathfrak{S}_{i+1}(\mathcal{H})^\circ$ agree almost everywhere, except that $\mathbf{x}_{i+1}$ contains the strand at the very top of each even-indexed shadow, and $\mathbf{x}_i$ contains the strand at the very top of each odd-indexed shadow. The Maslov and Alexander gradings on strand generators are defined by counting various intersections of strands, see [5, Section 3.4], and one sees that $M(\mathbf{x}_{i+1}) = M(\mathbf{x}_i)$ and $A(\mathbf{x}_{i+1}) = A(\mathbf{x}_i)$.

Switching back to Heegaard diagrams, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ differing in the above way is equivalent to saying that the region R_{n+1} connects $\mathbf{x}_{i+1}^\circ$ to $\mathbf{x}_i^\circ$. Since $e(R_{n+1}) = 1 - g$, $n_{\mathbf{x}_{i+1}^\circ}(R_{n+1}) = g/2 = n_{\mathbf{x}_i^\circ}(R_{n+1})$, $n_{X_1}(R_{n+1}) = 1$ and $n_p(R_{n+1}) = 0$ for any other $p \in \mathbb{X} \cup \mathbb{O}$, we see that

$$\begin{aligned} M(\mathbf{x}_{i+1}^\circ) - M(\mathbf{x}_i^\circ) &= 1 \\ A(\mathbf{x}_{i+1}^\circ) - A(\mathbf{x}_i^\circ) &= 0 \\ A_0(\mathbf{x}_{i+1}^\circ) - A_0(\mathbf{x}_i^\circ) &= 1. \end{aligned}$$

So for $i < j$ and arbitrary $\mathbf{x}_i^\circ \in \mathfrak{S}_i(\mathcal{H}^\circ)$, $\mathbf{x}_j^\circ \in \mathfrak{S}_j(\mathcal{H}^\circ)$, we have

$$M(\mathbf{x}_j^\circ) - M(\mathbf{x}_i^\circ) = M(\mathbf{x}_j) - M(\mathbf{x}_i) + j - i \tag{1}$$

$$A(\mathbf{x}_j^\circ) - A(\mathbf{x}_i^\circ) = A(\mathbf{x}_j) - A(\mathbf{x}_i) \tag{2}$$

$$A_0(\mathbf{x}_j^\circ) - A_0(\mathbf{x}_i^\circ) = j - i. \tag{3}$$

The argument that the isomorphism respects the absolute gradings is analogous to the one from [5, Section 6]. With the $4g$ grids arranged as in Figure 6, indexed $G_1, \ldots, G_{4g}$ from left to right, let $\mathbf{x}_{\mathbb{O}}^\circ$ be the generator formed by the bottom-left corner x_j of each O_j in G_{4i} and G_{4i+1}, the top-right corner x_j of each O_j in G_{4i+2} and G_{4i+3}, the very top-right corner x'_{4i+1} of each grid G_{4i+1}, and the bottom-left corner x'_{4i+3} of each grid G_{4i+3}. Define $\mathbf{x}_{\mathbb{X}}^\circ$ analogously, by replacing O_j with X_j in the above definition.

Denote the β circle containing each x_i or x'_i by β_i or β'_i, respectively. Form a set of circles $\boldsymbol{\gamma}$ by performing handleslides (which are allowed to cross $\mathbb{X}$ but not $\mathbb{O}$) of all β_i and perturbations of all β'_i, as in Figure 7. We look at the holomorphic triangle map associated to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbb{O})$, see [2, 3]. Let k be the number of Os in $\mathcal{H}$ (so the number of Os in $\mathcal{H}^\circ$ is $k+2$). Observe that $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbb{O})$ is a diagram for $(\#^{k+2} S^1 \times S^2)$, and let Θ be the top-dimensional generator. Let $\mathbf{y}$ be the generator of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbb{O})$ nearest to $\mathbf{x}_{\mathbb{O}}^\circ$. There is a holomorphic triangle that maps $\mathbf{x}_{\mathbb{O}}^\circ \otimes \Theta$ to $\mathbf{y}$, so $M(\mathbf{x}_{\mathbb{O}}^\circ) = M(\mathbf{y})$.

Observe that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbb{O})$ is a diagram for S^3 with $k+2$ basepoints, so, as a group graded by the Maslov grading, we have $\widehat{HF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbb{O}) \cong H_{*+k+1}(T^{k+1})$. The diagram has 2^{k+1}

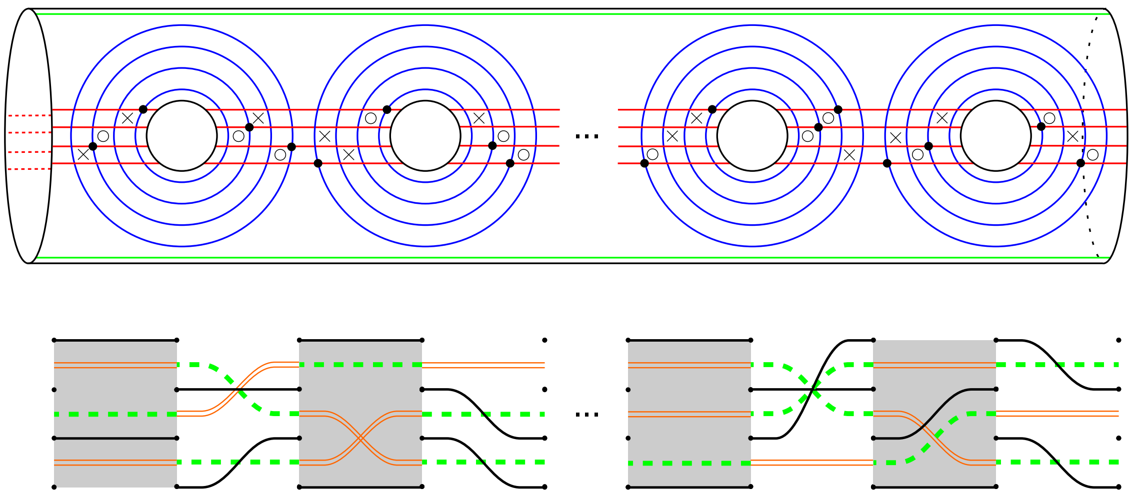

FIGURE 6. Top: A Heegaard diagram for T coming from a plumbing of grids and the "bottom-most" generator $\mathbf{x}_{\mathbb{O}}$. Bottom: The corresponding sequence of shadows for T, and the strands diagram for $\mathbf{x}_{\mathbb{O}}$.

generators, so they are a basis for the homology. Let $\mathbf{y}'$ be the generator obtained from $\mathbf{y}$ by replacing the intersection of γ'_{4i+1} and the topmost α of G_{4i+1} with the intersection of γ'_{4i+1} and the bottommost α of G_{4i+2}, and the intersection of γ'_{4i+3} and the bottommost α of G_{4i+3} with the intersection of γ'_{4i+3} and the topmost α of G_{4i+4}. There are k disjoint bigons going into $\mathbf{y}'$, so $M(\mathbf{y}') \leq -k$. The shaded $4g$-gon on Figure 7 from $\mathbf{y}'$ to $\mathbf{y}$ shows that $M(\mathbf{y}') - M(\mathbf{y}) = 1$. Thus, $M(\mathbf{y}) = -k-1$, so $M(\mathbf{x}^\circ_{\mathbb{O}}) = -k-1$. By Equation 1, for an arbitrary generator $\mathbf{x}^\circ_i$ we have

$$M(\mathbf{x}^\circ_i) = M(\mathbf{x}_i) + i - a - 1.$$

Similarly, for the z-normalized (or, $\mathbb{X}$-normalized, to match the notation in this paper) grading N, we have $N(\mathbf{x}^\circ_{\mathbb{X}}) = -k-1$. Since $N = M - 2A - (k+2-l)$, we get

$$A(\mathbf{x}^\circ_{\mathbb{X}}) = \frac{1}{2}(M(\mathbf{x}^\circ_{\mathbb{X}}) - N(\mathbf{x}^\circ_{\mathbb{X}}) - (k+2-l)) = \frac{1}{2}(M(\mathbf{x}^\circ_{\mathbb{X}}) + l - 1).$$

On the other hand, we can compute $M(\mathbf{x}_{\mathbb{O}})$ and $A(\mathbf{x}_{\mathbb{O}})$ using the definition from [5, Section 3.4]. The computation is analogous to the one from [5, Section 6], and we see that $M(\mathbf{x}_{\mathbb{O}}) = -k$, $N(\mathbf{x}_{\mathbb{X}}) = -k$, $A(\mathbf{x}_{\mathbb{X}}) = \frac{1}{2}M(\mathbf{x}_{\mathbb{X}})$. Since $M(\mathbf{x}^\circ_{\mathbb{X}}) = M(\mathbf{x}_{\mathbb{X}}) + n - 2a - 1$, we get

$$A(\mathbf{x}^\circ_{\mathbb{X}}) = A(\mathbf{x}_{\mathbb{X}}) + \frac{1}{2}(n - 2a + l - 2).$$

The Alexander multigrading on a generator $\mathbf{x}^\circ$ can be described by the relative Spin^c structure $\mathfrak{s}(\mathbf{x}^\circ) \in \underline{\mathrm{Spin}}^c(S^3, L)$, see [4]. In the case when the link is in S^3, one can think of A_i by looking at the projection of a Seifert surface for L_i onto $\mathcal{H}^\circ$. Specifically, for a generator $\mathbf{x}^\circ_i \in \mathfrak{S}_i(\mathcal{H}^\circ)$, we can compute its A_0 grading in the following way. Connect X_1 to O_1 and X_2 to O_2 away from $\boldsymbol{\beta}^\circ$, and O_1 to X_2 and O_2 to X_1 away from $\boldsymbol{\alpha}^\circ$ to obtain a curve C on $\mathcal{H}^\circ$ representing L_0, so that C is negative the boundary of a disk D that is a neighborhood of the rightmost grid (in general C may be immersed but not necessarily embedded). Then

$$A_0(\mathbf{x}^\circ_i) = \frac{1}{2}(e(D) + 2n_{\mathbf{x}^\circ_i}(D) - n_{\mathbb{X}}(D) - n_{\mathbb{O}}(D)) = i - \frac{n+1}{2}.$$

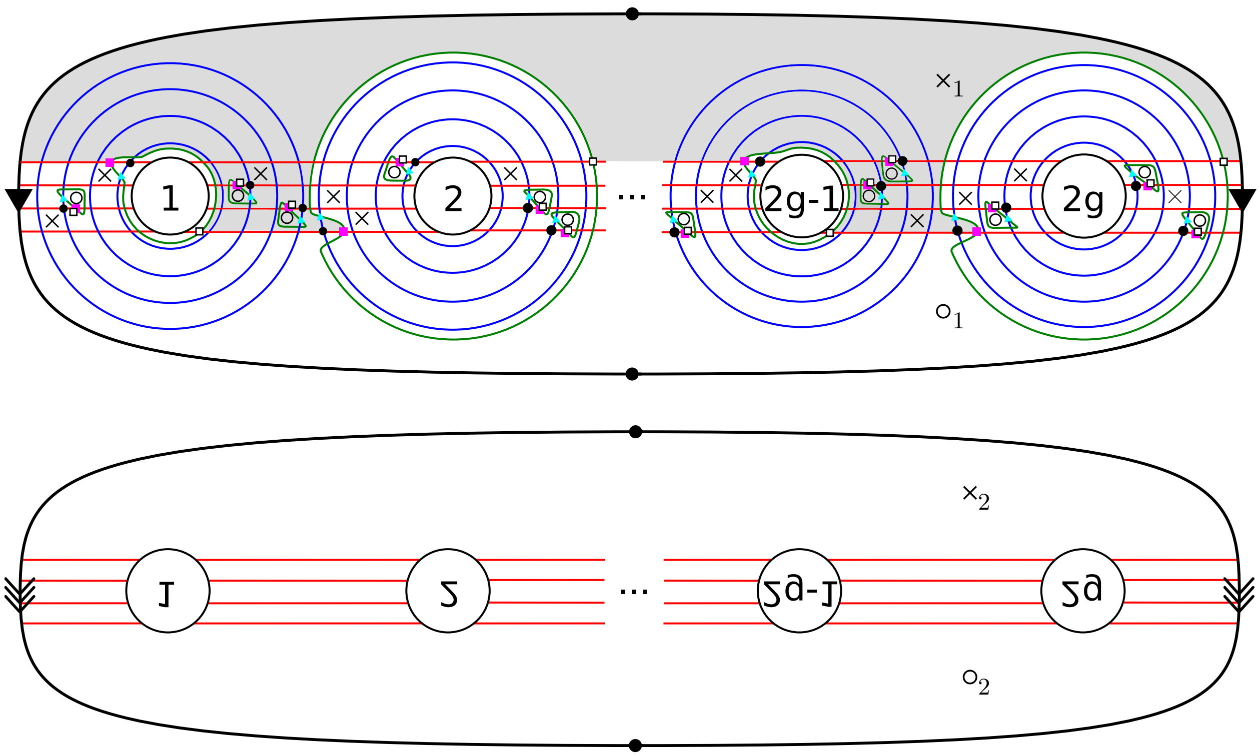

FIGURE 7. The Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbb{O})$, with the original $\mathbb{X}$ markings left in. The black dots form the generator $\mathbf{x}_{\mathbb{O}}$, the purple squares form $\mathbf{y}$, the white squares from $\mathbf{y}'$, and the cyan triangles form Θ.

It follows that

$$A'(\mathbf{x}_i^\circ) = A(\mathbf{x}_i) - S(\mathbf{x}_i) + \frac{l}{2} + n - a - 1$$

This completes the identification of gradings. $\square$

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