Fundamental equations of Mie theory

DBW

Abstract

Rainbows are fascinating optical phenomena and the mathematics and physics behind them is vast and rich. With ray optics much aspects of rainbows can be understood, but for a deeper understanding Mie theory is needed. Mie theory provides a framework to discuss the scattering of light from small sphere-like particles. To give those that are interested an easier way to an understanding of the basic equations of Mie theory, this note was written.

1 Introduction

Ray optics provide a good framework to discuss several optical phenomena, including the primary and secondary arcs of rainbows. Some other aspects of rainbows – the interference fringes for example – might be already well discussed with wave front analysis. To generally solve the problem and discuss some less known properties of rainbows and glory phenomena, a thorough discussion of Maxwell equations is needed. This is precisely what Mie theory does. Mie theory can then be applied to discuss scattering of light on small water droplets, which causes rainbows and glory phenomena, but also other atmospheric optical phenomena such as scattering of light from dust.

Mie theory provides a set of basic equations for the scattered electromagnetic field, which can easily be implemented on a computer. Simulations might then provide good insights in how light is scattered from small spheres and how the scattering depends on the color of light, i.e., on the frequency.

These notes were written as a result of trying to understand how the basic equations of Mie theory come can be derived. Many different conventions for special functions are used throughout the literature, but even the Maxwell equations can be written in different unit systems. In order to have one set of valid equations and knowing which conventions are used, the author wanted to verify each step in deriving the basic Mie equations.

The basic problem is the following: A plane wave of electromagnetic radiation in air (or vacuum) hits on a small dielectric sphere of radius a, e.g. made of water. We want to find the scattered radiation. To treat the problem, we use a coordinate system with its origin at the center of the droplet. By spherical symmetry we use spherical coordinates, with origin at the center of the water droplet:

$$x = r\cos(\varphi)\sin(\vartheta)\,, \quad y = r\sin(\varphi)\sin(\vartheta)\,, \quad z = r\cos(\vartheta)\,. \tag{1}$$

The unit basis vectors in cartesian coordinates are denoted $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$; the unit basis vectors in spherical coordinates are denoted $\mathbf{e}_r$, $\mathbf{e}_\vartheta$ and $\mathbf{e}_\varphi$.

To tackle the basic problem, a solution to the Maxwell equations needs to be found. Finding a solution that describes radiation can be made more managable by introducing some vector spherical harmonics, that is, vector fields satisfying Helmholtz equation. They are the vector analogues of the spherical harmonic functions Y_l^m that arise in many areas, such as quantum mechanics, potential problems in electromagnetism, 3D computer graphics, electronic configurations, magnetic fields on planets, cosmic microwave background radiation, ... The vector spherical harmonics are used in many interesting fields such as fluid dynamics, electrodynamics magneto-hydrdynamics. For example the magnetic fields in stars have an impact on their development; the magnetic field can well be described by vector spherical harmonics. Care has to be taken in directly using vector spherical harmonics, since several different definitions exist. The definitions are to be adapted to the application of interest. In this note we define some vector spherical harmonics that are adapted to electromagnetic radiation of some well-defined frequency.

The note is organized as follows: In section 2 we give the basic equations that are to be solved, that is, we give the Maxwell equations. In section 3 we give some standard mathematic machinery that enables us to bring the Maxwell equations into a managable form and reduce them to second-order differential equations in one variabel. In section 4 we then introduce the vector spherical harmonics and relate them to the Maxwell equations. Up to this point no explicit expansions in vector spherical harmonics has been made. In order to do so, more properties of spherical Bessel functions and associated Legendre functions are needed, which will be given in 5. In section 6 we give an expansion of the incoming plane waves in terms of vector spherical harmonics, and in section 7 we use the interface conditions at the boundary of the water droplet to obtain expansion in vector spherical harmonics for the electromagnetic field inside the droplet and for outgoing electromagnetic field; these expansions are the fundamental equations of Mie theory. In section 8 we discuss some various directions one can take with these fundamental equations and a few other aspects. Finally, in section 9 the reader finds a list with the commonly used symbols in this text; we however do not give coefficients of expansions in this glossary as they are used only within one section.

Historically, the solutions that we try to present in this note, were first derived by Gustav Mie in work BEITRÄGE ZUR OPTIK TRÜBER MEDIEN, SPEZIELL KOLLOIDALER METALLÖSUNGEN that appeared in Annalen der Physik, 330 (3). Half a century later, Hendrik

C. van de Hulst wrote a book[1] also discussing Mie theory. A more recent treatment is written by Craig F. Bohren and Donald R. Huffmann[2], again half a century later. In this note, we can merely present some details of only a small part of the derivation of the basic equations of Mie theory. We hope we can encourage the reader to fill in the details, find out, what more is known about the subject, and consult the original literature and other historic works on the subject.

We hope to have made no calculation errors or typos. If anyone finds some mistakes, please write an email to westradennis at gmail dot com.

2 Maxwell equations

We use the macroscopic version of Maxwell equations

$$\begin{aligned} &\nabla\cdot\mathbf{D}=\rho\,,\quad \nabla\cdot\mathbf{B}=0\\ &\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{B}\,,\quad \nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial}{\partial t}\mathbf{D} \end{aligned} \tag{2}$$

together with the constitutive relations $\mathbf{D}=\epsilon\mathbf{E}$ and $\mathbf{B}=\mu\mathbf{H}$. For the scattering of light on water droplets, we can assume $\rho=0$ and $\mathbf{J}=0$.

We wish to find solutions for the following problem: An incoming plane wave of the form $\mathbf{E}=\vec{e}_x\mathcal{E}e^{ikz-i\omega t}$ hits a water droplet; calculate the scattered radiation. By superimposing such plane waves one can get an understanding of the more daily life problem of sunlight hitting raindrops in the atmosphere.

We choose as variables E and H and elimenate the time dependence by writing $\mathbf{E}(x,y,z,t)=\mathbf{E}(x,y,z)e^{-i\omega t}$ and $\mathbf{H}(x,y,z,t)=\mathbf{H}(x,y,z)e^{-i\omega t}$. The material properties are inside μ and ϵ, which we assume to take the value μ_2 and ϵ_2 inside a sphere of radius a – the droplet – and the value μ_1 and ϵ_1 outside this sphere – in the air. The equations that are to be solved then become

$$\begin{aligned} &\nabla\cdot\mathbf{E}=0\,,\quad \nabla\cdot\mathbf{H}=0\\ &\nabla\times\mathbf{E}=i\omega\mu\mathbf{H}\,,\quad \nabla\times\mathbf{H}=-i\omega\epsilon\mathbf{E}\,. \end{aligned} \tag{3}$$

Combining these equations one finds that $\mathbf{E}$ and $\mathbf{H}$ have to satisfy the vector Helmholtz equations

$$(\triangle+k^2)\mathbf{E}=(\triangle+k^2)\mathbf{H}=0 \tag{4}$$

where $k^2=\epsilon\mu\omega^2$ and $\triangle=\nabla\cdot\nabla$.

[1] 1957: Van de Hulst, Light scattering by small particles. New York: John Wiley and Sons. ISBN 9780486139753

[2] 2010: Bohren and Huffmann, Absorption and scattering of light by small particles. New York: Wiley-Interscience. ISBN 3-527-40664-6.

3 Mathematical preamble

In this sectin we will repeat some standard mathematical machinery, which the experienced reader might want to skip. We will discuss the differential operators curl, grad and div in spherical coordinates and indicate how solutions to the (scalar) Helmholtz equation are found using separation of variables. In order not to obscure the main line of reasoning, which involves determining the scattering of light from a water droplet, we postpone a deeper discussion of the spherical Bessel functions and associated Legendre functions to section 5.

If f is a differentiable function, the gradient of f is given in spherical coordinates by

$$\mathrm{grad} f = \nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\mathbf{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial \varphi}\mathbf{e}_\varphi \,. \tag{5}$$

The divergence of a differentiable vector field $\mathbf{X} = X^r\mathbf{e}_r + X^\vartheta\mathbf{e}_\vartheta + X^\varphi\mathbf{e}_\varphi$ is given in spherical coordinates by

$$\mathrm{div}\,\mathbf{X} = \nabla\cdot\mathbf{X} = \frac{1}{r^2}\partial_r(r^2X^r) + \frac{1}{r\sin\vartheta}\partial_\vartheta(\sin(\vartheta)X^\vartheta) + \frac{1}{r\sin\vartheta}\partial_\varphi X^\varphi \,. \tag{6}$$

The curl of a differentiable vector field $\mathbf{X}$ is a vector field $\nabla\times\mathbf{X}$ whose components are given in spherical coordinates by

$$\begin{aligned}
(\nabla\times\mathbf{X})^r &= \frac{1}{r\sin\vartheta}\Big(\partial_\vartheta(\sin(\vartheta)X^\varphi) - \partial_\varphi X^\vartheta\Big)\\
(\nabla\times\mathbf{X})^\vartheta &= \frac{1}{r\sin\vartheta}\partial_\varphi X^r - \frac{1}{r}\partial_r(rX^\varphi)\\
(\nabla\times\mathbf{X})^\varphi &= \frac{1}{r}\Big(\partial_r(rX^\vartheta) - \partial_\vartheta X^r\Big)
\end{aligned} \tag{7}$$

The Laplacian $\triangle f = \nabla^2 f = \mathrm{div}\,\mathrm{grad} f$ of a smooth function f is then found to be

$$\triangle f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\vartheta}\partial_\vartheta(\sin(\vartheta)\partial_\vartheta f) + \frac{1}{r^2\sin^2\vartheta}\partial_\varphi^2 f \,. \tag{8}$$

To solve the Helmholtz equation $(\triangle + k^2)f$ for a smooth function we use separation of variables and set $f(r,\vartheta,\varphi) = R(r)\Theta(\vartheta)\Phi(\varphi)$ and plug this into the Helmholtz equation in spherical coordinates and find

$$\frac{1}{r^2R}\partial_r(r^2\partial_r R) + \frac{1}{r^2\sin(\vartheta)\Theta}\partial_\vartheta(\sin(\vartheta)\partial_\vartheta\Theta) + \frac{1}{r^2\sin^2(\vartheta)\Phi}\partial_\varphi^2\Phi + k^2 = 0\,. \tag{9}$$

If we multiply this equation with $r^2 \sin^2(\vartheta)$ we see that one term, involving $\partial_\varphi^2 \Phi$, depends on φ and on no other variables. The other terms do not depend on φ. Hence there must be a (perhaps complex) constant, which we already write as $-m^2$, such that

$$\partial_\varphi^2 \Phi = -m^2 \Phi \tag{10}$$

and

$$\frac{1}{r^2 R}\partial_r(r^2 \partial_r R) + \frac{1}{r^2 \sin(\vartheta)\Theta}\partial_\vartheta(\sin(\vartheta)\partial_\vartheta \Theta) - \frac{m^2}{r^2 \sin^2(\vartheta)} + k^2 = 0\,. \tag{11}$$

Now we can perform a similar trick and find there must be some constant C, such that

$$\partial_r(r^2 \partial_r R) + (k^2 r^2 - C)R = 0$$

and

$$\frac{1}{\sin(\vartheta)}\partial_\vartheta(\sin(\vartheta)\partial_\vartheta \Theta) - \frac{m^2 \Theta}{\sin^2(\vartheta)} + C\Theta = 0\,. \tag{12}$$

The Φ-equation is now easily solved for: we can take $\Phi(\varphi) = e^{\pm i m \varphi}$. Indeed, only if the separation constant is of the form $-m^2$ we have periodic solutions. For the constant C some more work is required, but can be found in most mathematical literature[3]. The constant C needs to be of the form $l(l+1)$ for some nonnegative integer l and m can take values ranging from $-l$ to $+l$.

For the radial equation we put $x = kr$ and find the equation

$$(x^2 R')' + (x^2 - l(l+1))R = 0\,, \tag{13}$$

whose solutions are the spherical Bessel functions $w_l(x)$, which come in two forms: the regular one, written $j_l(x)$, and the irregular one, written $y_l(x)$.

For the angular equation we put $\xi = \cos\vartheta$ and find that Θ satisfies the equation

$$((1-\xi^2)\Theta')' + \Big(l(l+1) - \frac{m^2}{1-\xi^2}\Big)\Theta = 0\,, \tag{14}$$

which has as regular solutions the associated Legendre functions P_{lm}.

Any solution$f$ to the Helmholtz equation that is at most irregular at the origin can be expanded in terms of the functions $e^{im\varphi}P_{lm}(\cos\vartheta)j_l(kr)$ and $e^{im\varphi}P_{lm}(\cos\vartheta)y_l(kr)$. Since the special functions, the associated Legendre functions and the spherical Bessel functions, play a prominent role in the sequel, we discuss some more properties of these in section 5.

We often encounter sets of orthogonal functions on some compact space, in particular on $[-1;1]$, $[0,2\pi]$ and on the sphere S^2. With orthogonal we will mean orthogonal with

[3] See for example: Jon Mathews and Robert Lee Walker, *Mathematical methods of physics*, Benjamin 1973.

respect to an inner product of the form $\langle f|g\rangle = \int f^* g$, that is, with respect to integration over the compact space. Since the spaces under consideration are compact, we will only find countable sets of orthogonal functions, in which case the use of the Kronecker delta $\delta_{\alpha,\beta}$ is ubiquitous: $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha,\beta} = 0$ if $\alpha \neq \beta$. With this notation, a set of orthogonal functions is characterized by

$$\langle f_\alpha | f_\beta \rangle = \int f_\alpha^* f_\beta = \mathcal{N}_\alpha \delta_{\alpha,\beta} \,, \tag{15}$$

and if $\mathcal{N}_\alpha = 1$, the set is an orthonormal set.

4 Vector spherical harmonics

We first consider the construction of a solution to the vector Helmholtz equation on the basis of a function f satisfying the Helmholtz equation $(\triangle + k^2)f = 0$. The basic idea is to associate to f two kinds of vector fields, which we denote by $\mathbf{L}$ and $\mathbf{K}$. We write $\mathbf{r}$ for the radial vector field;

$$\mathbf{r} = (x, y, z) = r\mathbf{e}_r \,. \tag{16}$$

Then we define

$$\mathbf{L} = \nabla \times (\mathbf{r}f) = \nabla f \times \mathbf{r} \,, \qquad \mathbf{K} = \frac{1}{k}\nabla \times \mathbf{L} \,. \tag{17}$$

It is then easily checked that we have

$$(\triangle + k^2)\mathbf{L} = (\triangle + k^2)\mathbf{K} = 0 \,, \quad \nabla \cdot \mathbf{L} = \nabla \cdot \mathbf{K} = 0 \,, \tag{18}$$

and $\nabla \times \mathbf{L} = k\mathbf{K}$ and $\nabla \times \mathbf{K} = k\mathbf{L}$. These identities are precisely those that the Maxwell equations dictate for $\mathbf{E}$ and $\mathbf{H}$.

Any complex function f satisfying the Helmholtz equation can be expanded in terms of the basis functions $P_{lm}(\cos\vartheta) j_l(kr) e^{im\varphi}$ and $P_{lm}(\cos\vartheta) y_l(kr) e^{im\varphi}$, where P_{lm} are the associated Legendre functions, j_l and y_l the spherical Bessel functions. In the following section 5 we discuss these functions in more detail. The labels l and m take the following values: $l = 0, 1, 2, 3, \ldots$ and for a given l the label m takes the integer values from $-l$ to $+l$. We will however choose to split $e^{im\varphi}$ in its real and imaginary part so that m takes nonnegative values: $0 \leq m \leq l$.

For the moment it is irrelevant whether we take j_l, y_l or even $h_l = j_l + iy_l$ for the spherical Bessel function, and therefore we write w_l for a spherical Bessel function, be it j_l, y_l or h_l. Later, when we get to write down the expansions, we need to make a choice whether $w_l = j_l$ or $w_l = y_l$ or $w_l = h_l$. The argument of the spherical Bessel functions will be of the form kr, where k is the wave vector, which depends on the medium. Later we will often simply write $x = kr$, as long as no confusion is possible.

Then we define

$$\begin{aligned}\mathbf{L}_{lm} &= \nabla \times \Big(\mathbf{r} P_{lm}(\cos\vartheta) w_l(kr) \cos(m\varphi) \Big) \,, \\ \mathbf{L}'_{lm} &= \nabla \times \Big(\mathbf{r} P_{lm}(\cos\vartheta) w_l(kr) \sin(m\varphi) \Big) \,,\end{aligned} \tag{19}$$

and

$$\mathbf{K}_{lm} = \frac{1}{k} \nabla \times \mathbf{L}_{lm} \,, \quad \mathbf{K}'_{lm} = \frac{1}{k} \nabla \times \mathbf{L}'_{lm} \,. \tag{20}$$

Using the expression (7) we find

$$\begin{aligned}\mathbf{L}_{lm} &= -\frac{mP_{lm}(\cos\vartheta)}{\sin\vartheta} \sin(m\varphi) w_l(kr) \mathbf{e}_\vartheta - \frac{\mathrm{d}P_{lm}(\cos\vartheta)}{\mathrm{d}\vartheta} \cos(m\varphi) w_l(kr) \mathbf{e}_\varphi \\ \mathbf{L}'_{lm} &= \frac{mP_{lm}(\cos\vartheta)}{\sin\vartheta} \cos(m\varphi) w_l(kr) \mathbf{e}_\vartheta - \frac{\mathrm{d}P_{lm}(\cos\theta)}{\mathrm{d}\vartheta} \sin(m\varphi) w_l(kr) \mathbf{e}_\varphi \\ \mathbf{K}_{lm} &= \frac{w_l(kr)}{kr} l(l+1) P_{lm}(\cos\vartheta) \cos(m\varphi) \mathbf{e}_r + \frac{1}{kr} \frac{\mathrm{d}}{\mathrm{d}r} (r w_l(kr)) \frac{\mathrm{d}P_{lm}(\cos\vartheta)}{\mathrm{d}\vartheta} \cos(m\varphi) \mathbf{e}_\vartheta \\ &\quad - \frac{1}{kr} \frac{\mathrm{d}}{\mathrm{d}r} (r w_l(kr)) \frac{mP_{lm}(\cos\vartheta)}{\sin\vartheta} \sin(m\varphi) \mathbf{e}_\varphi \\ \mathbf{K}'_{lm} &= \frac{w_l(kr)}{kr} l(l+1) P_{lm}(\cos\vartheta) \sin(m\varphi) \mathbf{e}_r + \frac{1}{kr} \frac{\mathrm{d}}{\mathrm{d}r} (r w_l(kr)) \frac{\mathrm{d}P_{lm}(\cos\vartheta)}{\mathrm{d}\vartheta} \sin(m\varphi) \mathbf{e}_\vartheta \\ &\quad + \frac{1}{kr} \frac{\mathrm{d}}{\mathrm{d}r} (r w_l(kr)) \frac{mP_{lm}(\cos\vartheta)}{\sin\vartheta} \cos(m\varphi) \mathbf{e}_\varphi\end{aligned} \tag{21}$$

5 Properties of special functions

In this section we discuss some properties of sperical Bessel functions and associated Legendre functions.

We define the Legendre polynomials by

$$P_l(\xi) = \frac{1}{2^l l!} \Big(\frac{\mathrm{d}}{\mathrm{d}\xi} \Big) (\xi^2 - 1)^l \,. \tag{22}$$

The Legendre polynomials are the regular solutions to the differential equation

$$((1-\xi^2) y')' + l(l+1) y = (1-\xi^2) y'' - 2\xi y' + l(l+1) y = 0 \,. \tag{23}$$

The Legendre polynomials are orthogonal:

$$\int_{-1}^{1} P_l(\xi) P_{l'}(\xi) \mathrm{d}\xi = \frac{2\delta_{l,l'}}{2l+1} \,, \tag{24}$$

which is proved in many textbooks, but one proof can also be found in my notes on associated Legendre functions[4].

The Legendre polynomials P_0 up to P_l span the space of polynomials up to degree l and thus ξ^n can be expanded by the Legendre polynomials P_0 up to P_n and thus if $n < l$ we must have $\int_{-1}^{1} \xi^n P_l(\xi)\mathrm{d}\xi = 0$. Another interesting integral is

$$\int_{-1}^{1} \xi^l P_l(\xi)\mathrm{d}\xi = \frac{2^{l+1}(l!)^2}{(2l+1)!}\,, \tag{25}$$

which can be proved by partial integration using the definition (22) and the integral identity

$$\int_{-1}^{1} (1-\xi^2)^l \mathrm{d}\xi = 2^{2l+1}\frac{l!l!}{(2l+1)!}\,. \tag{26}$$

The Legendre polynomials satisfy the differential relation

$$P'_{l+1}(\xi) - P'_{l-1}(\xi) = (2l+1)P_l(\xi) \tag{27}$$

and the recurrence relation

$$(2l+1)\xi P_l(\xi) = (l+1)P_{l+1}(\xi) + lP_{l-1}(\xi)\,. \tag{28}$$

We define the associated Legendre functions by

$$P_{lm}(\xi) = \frac{(-1)^m}{2^l l!}(1-\xi^2)^{m/2}\Big(\frac{\mathrm{d}}{\mathrm{d}\xi}\Big)^{l+m}(\xi^2-1)\,. \tag{29}$$

Clearly we have $P_{l0} = P_l$ and for $m \geq 0$ we have

$$P_{lm}(\xi) = (-1)^m(1-\xi^2)^{m/2}\Big(\frac{\mathrm{d}}{\mathrm{d}\xi}\Big)^m P_l(\xi)\,, \qquad m \geq 0\,. \tag{30}$$

One can show that $P_{l,-m}$ and P_{lm} are proportional. The associated Legendre polynomials are the regular solutions of the differential equation

$$((1-\xi^2)y')' + \Big(l(l+1) - \frac{m^2}{1-\xi^2}\Big)y = (1-\xi^2)y'' - 2\xi y' + \Big(l(l+1) - \frac{m^2}{1-\xi^2}\Big)y = 0\,. \tag{31}$$

Using this differential equation one can show using partial integrations that for fixed m one has

$$\int_{-1}^{1} P_{lm}(\xi)P_{l'm}(\xi)\mathrm{d}\xi = \frac{2\delta_{l,l'}}{2l+1}\frac{(l+m)!}{(l-m)!}\,. \tag{32}$$

[4] See the PDF file behind the link
`https://www.mat.univie.ac.at/~westra/associatedlegendrefunctions.pdf`

The functions $P_{lm}(\cos\vartheta)\cos(m\varphi)$ and $P_{lm}(\cos\vartheta)\sin(m\varphi)$ make up a set of orthogonal functions on the sphere S^2. In fact, any continuous function on the sphere can be expanded in terms of these basis functions. By orthogonality, such an expansion is unique. If f and g are two functions on the sphere, we write $\langle f|g\rangle$ for the inner product on the sphere

$$\langle f|g\rangle = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\vartheta \mathrm{d}\vartheta\, f^* g\,, \tag{33}$$

where f^* denotes the complex conjugate of f. It is often convenient to switch from ϑ to the integration variable $\xi = \cos\vartheta$ and then $\int_0^\pi \sin\vartheta \mathrm{d}\vartheta = \int_{-1}^{1} \mathrm{d}\xi$.

The spherical Bessel functions w_l come in two disguises, j_l and y_l, which are defined by

$$j_l(x) = (-x)^l \Big(\frac{\mathrm{d}}{x\mathrm{d}x}\Big)^l \frac{\sin(x)}{x}\,, \qquad y_l(x) = -(-x)^l \Big(\frac{\mathrm{d}}{x\mathrm{d}x}\Big)^l \frac{\cos(x)}{x}\,. \tag{34}$$

The spherical Bessel functions are solutions to the differential equation

$$x^2 y'' + 2xy' + (x^2 - l(l+1))y = 0\,. \tag{35}$$

Another useful definition is

$$h_l(x) = j_l(x) + i y_l(x)\,, \tag{36}$$

which behaves for large x as $(-i)^{l+1}\frac{e^{ix}}{x}$, which will be necessary when we get to find good expansions for the outgoing radiation.

The behavior of $j_l(x)$ at the origin is given by

$$j_l(x) = \frac{2^l l!}{(2l+1)!} x^l + O(x^l)\,, \tag{37}$$

which can be proved by using the definition (34) and inserting a power series expansion for $\sin(x)$, a calculation which is eased by putting $u = x^2$.

Two equations that will often be useful and will be used without mentioning is

$$\begin{aligned} w_{n+1}(x) &= -w_n'(x) + \frac{n}{x} w_n(x) \\ w_{l+1}(x) + w_{l-1}(x) &= \frac{2l+1}{x} w_l(x)\,. \end{aligned} \tag{38}$$

We sketch a proof. One first puts $D = \frac{\mathrm{d}}{x\mathrm{d}x}$ and shows that $Dx^r = rx^{r-1}$. Then one brings the definitions (34) into the form $w_l = (-x)^l D^l f$. Pulling one D out of the definition of w_{l+1} almost immediately gives the first identity of eqns.(38). One then also has $w_n = -w_{n-1}' + \frac{n-1}{x} w_{n-1}$; differentiating this equation and using the differential equation (35) one arrives at the second of eqns.(38). As a side remark, using eqns.(38) one can easily define w_{-1}.

6 Plane wave expansions

The incoming plane wave hitting the water droplet is taken to be of the form

$$\mathbf{E}^{in} = \mathbf{e}_x \mathcal{E} e^{ikz - i\omega t}\,. \tag{39}$$

From eqn.(39) the magnetic field $\mathbf{H}^{in}$ can be calculated. The field strength is contained in the constant $\mathcal{E}$, which will appear linearly in all quantities. Hence we could as well put it to 1 and re-insert it later. We will however keep it in this section and absorb it in sone other constants later on. The time-dependent factor $e^{-i\omega t}$ can be ignored in this section; we are only concerned with the dependence on r, ϑ and φ.

We start with a rather well-known result. The following expansion holds:

$$e^{ikr\cos\vartheta} = \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\vartheta) j_l(kr)\,. \tag{40}$$

Since we will exploit this result in obtaining an expansion of $\mathbf{E}^{in}$ in terms of $\mathbf{L}_{lm}$, $\mathbf{L}'_{lm}$, $\mathbf{K}_{lm}$ and $\mathbf{K}'_{lm}$ we give a short indication of a proof eqn.(40).

Since e^{ikz} solves the Helmholtz equation and is regular at the origin, it can be expanded in the basic functions as $e^{ikz} = \sum_{lm} a_{lm} P_{lm}(\cos\vartheta) j_l(kr) e^{im\varphi}$, but since $ikz = ikr\cos\vartheta$, there is no φ-dependence and so only the coefficients for $m = 0$ are nozero. We thus have

$$e^{ikr\cos\vartheta} = \sum_{l=0}^{\infty} a_l P_l(\cos\vartheta) j_l(kr)\,, \tag{41}$$

and our task is to determine a_l. As before we put $\xi = \cos\vartheta$. By the orthogonality relation (24) we can determine the a_l by

$$\frac{2}{2l+1} a_l j_l(kr) = \int_{-1}^{1} e^{ikr\xi} P_l(\xi) \mathrm{d}\xi\,. \tag{42}$$

In order not to have to calculate all integrals $\int_{-1}^{1} \xi^k P_l(\xi)\mathrm{d}\xi$ we use a trick, which consists in considering eqn.(42) around $x = kr = 0$. From eqn.(37) we see that $j_l(x) = \frac{2^l l!}{(2l+1)!} x^l$ plus higher order terms. We thus expand $e^{ix\xi} = \sum_n \frac{i^n x^n}{n!} \xi^n$ and consider the term $\frac{i^l x^l}{l!} \xi^l$ – indeed $\int_{-1}^{1} \xi^n P_l(\xi)\mathrm{d}\xi = 0$ for $n < l$ since the Legendre polynomials up to degree n span the space of polynomials up to degree n, so that P_l is orthogonal to them for $l > n$, and for $n > l$ we obtain the higher order terms, so we can restrict our attention to $n = l$. Using relation (25) then leads to $a_l = i^l(2l+1)$.

Now we come to the main part of this section and expand $\mathbf{E}^{in}$ in terms of the vector harmonics

$$\mathbf{E}^{in} = \sum_{lm} A_{lm}\mathbf{K}_{lm} + A'_{lm}\mathbf{K}'_{lm} + B_{lm}\mathbf{L}_{lm} + B'_{lm}\mathbf{L}'_{lm}\,, \tag{43}$$

where we now choose to take $w_l = j_l$ since $\mathbf{E}^{in}$ is definitively regular at the origin. We now take inner products with $\mathbf{e}_r$, $\mathbf{e}_\vartheta$, $\mathbf{e}_\varphi$ use the relations

$$\mathbf{e}_r \cdot \mathbf{e}_x = \sin\vartheta\cos\varphi\,, \quad \mathbf{e}_\vartheta \cdot \mathbf{e}_x = \cos\vartheta\cos\varphi\,, \quad \mathbf{e}_\varphi \cdot \mathbf{e}_x = -\sin\varphi\,, \tag{44}$$

and see that only terms with $m = 1$ can contribute, which excludes all terms with $l = 0$. Furthermore, noting that $\sin\varphi$ and $\cos\varphi$ are orthogonal, we can restrict to an expansion in terms of $\mathbf{K}_{l1}$ and $\mathbf{L}'_{l1}$. We thus write $\mathbf{E}^{in} = \sum_l A_l\mathbf{K}_{l1} + B_l\mathbf{L}'_{l1}$ and need to solve for

$$\begin{aligned}
\sum_{l=0}^{\infty} i^l(2l+1)j_l(x)P_l(\xi)\sin\vartheta &= \sum_{l=1}^{\infty} \frac{j_l(x)}{x} l(l+1)P_{l1}(\xi)A_l \\
\sum_{l=0}^{\infty} i^l(2l+1)j_l(x)P_l(\xi)\cos\vartheta &= \sum_{l=1}^{\infty} A_l X_l(x)\frac{\mathrm{d}P_{l1}(\xi)}{\mathrm{d}\vartheta} + B_l\frac{P_{l1}(\xi)}{\sin\vartheta}j_l(x) \\
\sum_{l=0}^{\infty} i^l(2l+1)j_l(x)P_l(\xi) &= \sum_{l=1}^{\infty} A_l X_l(x)\frac{P_{l1}(\xi)}{\sin\vartheta} + B_l\frac{\mathrm{d}P_{l1}(\xi)}{\mathrm{d}\vartheta}j_l(x)
\end{aligned} \tag{45}$$

where $X_l(x) = \frac{1}{x}(xj_l(x))'$. To solve these equations we note that the functions

$$\Pi_l^{\pm}(\xi) = \frac{P_{l1}(\xi)}{\sqrt{1-\xi^2}} \pm \sqrt{1-\xi^2}\frac{\mathrm{d}P_{l1}(\xi)}{\mathrm{d}\xi} \tag{46}$$

make up two sets of orthogonal functions and we have

$$\int_{-1}^{1} \Pi_k^{\pm}(\xi)\Pi_l^{\pm}(\xi)\mathrm{d}\xi = \delta_{k,l}\frac{2l^2(l+1)^2}{2l+1}\,. \tag{47}$$

This relation is proved by a partial integration and using that P_{l1} satisfies $((1-\xi^2)P'_{l1})' = \left(\frac{1}{1-\xi^2} - l(l+1)\right)P_{l1}$. If we add the second and the third from eqns.(45) we obtain

$$\sum_{l=0}^{\infty} i^l(2l+1)(\xi+1)P_l(\xi) = \sum_{l=1}(A_l X_l(x) + B_l j_l(x))\Pi_l^-(\xi)\,. \tag{48}$$

We now pursue the following strategy: We express the left-hand side as a linear sum of the P_l and then take the inner product left and right with Π_k^- to find a linear relation between A_k, B_k and the j_l. We thus need to evaluate $\langle P_l|\Pi_k^-\rangle$, to which we now turn.

Usindfg $P_{l1}(\xi) = -\sqrt{1-\xi^2}P'_l(\xi)$ and the differential relation satisfied by P_l we find the following equalities

$$\Pi_l^-(\xi) = (\xi+1)\Big(-P'_l(\xi) + (1-\xi)P''_l(\xi)\Big) = (\xi-1)P'_l(\xi) - l(l+1)P_l(\xi)\,. \tag{49}$$

From eqn.(27) we find

$$P_l' = (2l-1)P_{l-1} + (2l-5)P_{l-3} + \dots \,. \tag{50}$$

Using this and eqn.(28) we find after some algebra that

$$\Pi_l^- = -l^2 P_l + (-1)^l \sum_{j=0}^{l-1} (-1)^j (2j+1) P_j \,. \tag{51}$$

Using the latter we find the relation

$$\langle \sum_l c_l P_l, \Pi_k^- \rangle = -k^2 \frac{2c_k}{2k+1} + \sum_{j=0}^{k-1} (-1)^{k+j} 2c_j \,, \tag{52}$$

for any coefficients c_l. We now use again eqn.(28) and rewrite

$$\sum_{l=0}^{\infty} i^l (2l+1) j_l(x) P_l(\xi)(\xi+1) = \sum_{l=0}^{\infty} \Big(l j_{l-1}(x) + i(2l+1) j_l(x) - (l+1) j_{l+1}(x) \Big) i^{l-1} P_l(\xi) \,. \tag{53}$$

We now call $c_l = \Big(l j_{l-1}(x) + i(2l+1) j_l(x) - (l+1) j_{l+1}(x) \Big) i^{l-1}$ and find that we have to solve

$$-k^2 \frac{2c_k}{2k+1} + \sum_{j=0}^{k-1} (-1)^{k+j} 2c_j = \Big(A_k X_k + B_k j_k \Big) \frac{2k^2(k+1)^2}{2k+1} \,. \tag{54}$$

Investigating c_k and the sums $\sum_{j=0}^{p-1} c_k (-1)^k$ for some low values of p and k one easily convinces him or herself that we have the following relation

$$\sum_{j=0}^{p-1} c_k (-1)^k = -p(-i)^p (j_p - i j_{p-1}) \,. \tag{55}$$

Plugging this into eqn.(54) we find that we have to solve

$$A_k X_k + B_k j_k = \frac{1}{k(k+1)} \Big(i^{k+1}(k+1) j_{k-1} - i^{k+1} k j_{k+1} - i^k (2k+1) j_k \Big) \,. \tag{56}$$

Using the explicit expression $X_l(x) = -\frac{l}{2l+1} j_{l+1}(x) + \frac{l+1}{2l+1} j_{l-1}(x)$ and comparing the coefficients in front of the different spherical Bessel functions, one concludes that a solution to the equation to solve is

$$A_k = \frac{i^{k+1}(2k+1)}{k(k+1)} \,, \quad B_k = -\frac{i^k (2k+1)}{k(k+1)} \,. \tag{57}$$

We will check this solution in two ways, using the two other independent equations from eqns.(45). But before doing so, we put the above result into a useful expansion; defining $E_l = \frac{i^{l+1}(2l+1)}{l(l+1)}$ we write the electric and magnetic field of the incoming radiation as

$$\begin{aligned}\mathbf{E}^{in} &= \mathcal{E}\sum_{l=1}^{\infty} E_l(\mathbf{K}_{l1} + i\mathbf{L}'_{l1}) \\ \mathbf{H}^{in} &= \mathcal{E}\sqrt{\frac{\epsilon}{\mu}}\sum_{l=1}^{\infty} E_l(\mathbf{K}'_{l1} - i\mathbf{L}_{l1})\,.\end{aligned} \tag{58}$$

6.1 Checking with the radial equation

Noting that $P_{l1}(\xi) = -\sqrt{1-\xi^2}P'_l(\xi) = -\sin\vartheta P'_l(\xi)$ and using $\frac{j_l(x)}{x} = \frac{j_{l+1}(x)+j_{l-1}(x)}{2l+1}$ we obtain from the first of eqns.(45) the equation, supressing the arguments,

$$\sum_{l=0}^{\infty} i^l(2l+1)j_lP_l = -\sum_{l=1}^{\infty} j_l\frac{l(l-1)}{2l-1}P'_{l-1}A_{l-1} - \sum_{l=0}^{\infty} j_l\frac{(l+1)(l+2)}{2l+3}P'_{l+1}A_{l+1}\,. \tag{59}$$

Then using again eqn.(27) and taking an ansatz $A_l = \frac{2l+1}{l(l+1)}a_li^l$ we bring eqn.(59) into the form

$$j_0P_0 + \sum_{l=1}^{\infty} i^l j_l(P'_{l+1} - P'_{l-1}) = -\sum_{l=1}^{\infty} j_lP'_{l-1}a_{l-1}i^{l-1} - \sum_{l=1}^{\infty} j_lP'_{l+1}a_{l+1}i^{l+1} - ia_1j_0P_0\,. \tag{60}$$

We see that $a_l = i$ is a solution, corroborating the equation for A_l as found in eqn.(57).

6.2 Checking with Π_l^+

We combine the second and the third of eqns.(45) into

$$\sum_{l=0}^{\infty} i^l(2l+1)P_l(\xi)(1-\xi) = \sum_{l=1}^{\infty}(A_lX_l - B_lj_l)\Pi_l^+\,. \tag{61}$$

Using similar algebraic steps one shows that

$$\sum_{l=0}^{\infty} i^l(2l+1)P_l(\xi)(1-\xi) = \sum_{l=0}^{\infty}\Big(i^l(2l+1)j_l - i^{l-1}lj_l - i^{l+1}(l+1)j_{l+1}\Big)P_l(\xi)\,. \tag{62}$$

We call $c_l = i^l(2l+1)j_l - i^{l-1}lj_l - i^{l+1}(l+1)j_{l+1}$.

We express Π_l^+ as

$$\Pi_l^+(\xi) = -(\xi+1)P_l'(\xi) + l(l+1)P_l(\xi)\,. \tag{63}$$

Using similar steps as before, one shows that

$$(1+\xi)P_l'(\xi) = lP_l + \sum_{j=0}^{l-1}(2j+1)P_j\,. \tag{64}$$

With these, we have

$$\Pi_l^+ = l^2 P_l - \sum_{j=0}^{l-1}(2j+1)P_j\,. \tag{65}$$

Hence we find

$$\sum_{k=0}^{\infty} c_k \langle P_k|\Pi_l^+\rangle = \frac{2l^2 c_l}{2l+1} - \sum_{j=0}^{l-1} 2c_j\,. \tag{66}$$

Either considering the first few lowest values for l or by a direct calculation one easily shows that

$$\sum_{j=0}^{l-1} c_j = i^{l-1} l (j_{l-1} - i j_l)\,. \tag{67}$$

Hence

$$\sum_{k=0}^{\infty} c_k \langle P_k|\Pi_l^+\rangle = \frac{2l^2}{2l+1}c_l - 2i^{l-1}l(j_{l-1} - ij_l) \tag{68}$$

which we rearrange as

$$\sum_{k=0}^{\infty} c_k \langle P_k|\Pi_l^+\rangle = 2i^l l(l+1)j_l - 2i^{l+1}\frac{l^2(l+1)}{2l+1}j_{l+1} - 2i^{l-1}\frac{l(l+1)^2}{2l+1}j_{l-1}\,. \tag{69}$$

On the other hand, we also have $\langle \Pi_l^+|\Pi_k^+\rangle = \delta_{kl}\frac{2l^2(l+1)^2}{2l+1}$ and hence

$$\sum_{k=1}^{\infty}(A_k X_k - B_k j_k)\langle \Pi_l^+|\Pi_k^+\rangle = \Big(-\frac{l}{2l+1}j_{l+1}A_l + \frac{l+1}{2l+1}j_{l-1}A_l - B_l j_l\Big)\frac{2l^2(l+1)^2}{2l+1}\,. \tag{70}$$

Comparing the coefficients in front of the different spherical Bessel functions j_l, j_{l+1} and j_{l-1} we againg find $A_l = \frac{i^{l+1}(2l+1)}{l(l+1)}$ and $B_l = -\frac{i^l(2l+1)}{l(l+1)}$.

7 Implementing the interface conditions

Our problem of determining the scattered electromagnetic field can be tackled as follows: The incoming radiation, given by $\mathbf{E}^{in}$ and $\mathbf{H}^{in}$, causes an electromagnetic field inside the droplet, which will be given its electric field $\mathbf{E}^{d}$ and its magnetic field $\mathbf{H}^{d}$, but also an outgoing electromagnetic field, characterized by $\mathbf{E}^{out}$ and $\mathbf{H}^{out}$. All of these can be expanded in terms of the vector spherical harmonics, where now for each one, a good choice of the spherical Bessel function has to be made, and where the radial dependence given by kr is different in the droplet (in water) from that outside the droplet (in the air). The expansion of the incoming fields is already known. The interface conditions on the surface of the droplet will determine the other expansions; the droplet is taken to be a sphere with radius a.

The interface conditions for a water droplet without surface charges and currents are

$$\begin{aligned}
(\mathbf{D}^{in} + \mathbf{D}^{out}) \cdot \mathbf{e}_r &= \mathbf{D}^{d} \cdot \mathbf{e}_r \quad \text{at} \;\; r = a\,, \\
(\mathbf{B}^{in} + \mathbf{B}^{out}) \cdot \mathbf{e}_r &= \mathbf{B}^{d} \cdot \mathbf{e}_r \quad \text{at} \;\; r = a\,, \\
(\mathbf{E}^{in} + \mathbf{E}^{out}) \times \mathbf{e}_r &= \mathbf{E}^{d} \times \mathbf{e}_r \quad \text{at} \;\; r = a\,, \\
(\mathbf{H}^{in} + \mathbf{H}^{out}) \times \mathbf{e}_r &= \mathbf{H}^{d} \times \mathbf{e}_r \quad \text{at} \;\; r = a\,.
\end{aligned} \tag{71}$$

In order to deal with these interface conditions effectively, we introduce the following functions on the sphere:

$$\begin{aligned}
\Delta^{+}_{lm}(\vartheta,\varphi) &= \frac{mP_{lm}(\cos\vartheta)}{\sin\vartheta}\cos(m\varphi) \\
\Delta^{-}_{lm}(\vartheta,\varphi) &= \frac{mP_{lm}(\cos\vartheta)}{\sin\vartheta}\sin(m\varphi) \\
\Gamma^{+}_{lm}(\vartheta,\varphi) &= \frac{\mathrm{d}P_{lm}(\cos\vartheta)}{\mathrm{d}\vartheta}\cos(m\varphi) \\
\Gamma^{-}_{lm}(\vartheta,\varphi) &= \frac{\mathrm{d}P_{lm}(\cos\vartheta)}{\mathrm{d}\vartheta}\sin(m\varphi)\,,
\end{aligned} \tag{72}$$

which can then be used to rewrite the vector harmonics as

$$\begin{aligned}
\mathbf{L}_{lm}(x) &= \Delta^{+}_{lm} w_l(x)\mathbf{e}_\vartheta - \Gamma^{-}_{lm} w_l(x)\mathbf{e}_\varphi \\
\mathbf{L}'_{lm}(x) &= -\Delta^{-}_{lm} w_l(x)\mathbf{e}_\vartheta - \Gamma^{+}_{lm} w_l(x)\mathbf{e}_\varphi \\
\mathbf{K}_{lm}(x) &= l(l+1)P_{lm}(\cos\vartheta)\sin(m\varphi)\frac{w_l(x)}{x}\mathbf{e}_r + \Gamma^{-}_{lm} X_l(x)\mathbf{e}_\vartheta + \Delta^{+}_{lm} X_l(x)\mathbf{e}_\varphi \\
\mathbf{K}'_{lm}(x) &= l(l+1)P_{lm}(\cos\vartheta)\cos(m\varphi)\frac{w_l(x)}{x}\mathbf{e}_r + \Gamma^{+}_{lm} X_l(x)\mathbf{e}_\vartheta - \Delta^{-}_{lm} X_l(x)\mathbf{e}_\varphi
\end{aligned} \tag{73}$$

where we also abbreviated $X_l(x) = \frac{1}{x}(xw_l(x))'$. Note that we indicated the r-dependence through $x = kr$ explicitly; this is just to facilitate the bookkeeping. The newly defined functions $\Delta^{\pm}_{lm}$ and $\Gamma^{\pm}_{lm}$ satisfy some interesting orthogonality properties with respect to the inner product on the sphere (33), which will be useful to use when dealing with the interface conditions.

The integer m that labels the vector harmonics only takes nonnegative values. With this restriction we have

$$\begin{aligned}
&\int_0^{2\pi} \sin(m\varphi)\sin(m'\varphi)\mathrm{d}\varphi = \pi\delta_{m,m'} \\
&\int_0^{2\pi} \cos(m\varphi)\cos(m'\varphi)\mathrm{d}\varphi = \pi\delta_{m,m'}(1+\delta_{m,0}) \\
&\int_0^{2\pi} \sin(m\varphi)\cos(m'\varphi)\mathrm{d}\varphi = 0\,.
\end{aligned} \tag{74}$$

Using these relations, the orthogonality property eqn.(32) of the associated Legendre functions and the differential equation satisfied by them, one finds the following relations hold:

$$\begin{aligned}
&\langle\Delta^+_{lm}|\Delta^-_{l'm'}\rangle = \langle\Gamma^+_{lm}|\Gamma^-_{l'm'}\rangle = \langle\Delta^+_{lm}|\Gamma^-_{l'm'}\rangle = \langle\Delta^-_{lm}|\Gamma^+_{l'm'}\rangle = 0 \\
&\langle\Delta^{\sigma_1}_{lm}|\Gamma^{\sigma_1}_{l'm'}\rangle + \langle\Gamma^{\sigma_2}_{lm}|\Delta^{\sigma_2}_{l'm'}\rangle = 0(!!!) \\
&\langle\Delta^{\pm}_{lm}|\Delta^{\pm}_{l'm'}\rangle + \langle\Gamma^-_{lm}|\Gamma^-_{l'm'}\rangle = \pi\frac{2l(l+1)}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l,l'}\delta_{m,m'} \\
&\langle\Delta^{\pm}_{lm}|\Delta^{\pm}_{l'm'}\rangle + \langle\Gamma^+_{lm}|\Gamma^+_{l'm'}\rangle = \pi\frac{2l(l+1)}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l,l'}\delta_{m,m'}(1+\delta_{m,0})
\end{aligned} \tag{75}$$

where in the second line all signs $\sigma_i \in \{-,+\}$ for $i = 1, 2$ can be chosen at will. We now introduce the following short notation

$$c_{lm} = \pi\frac{2l(l+1)}{2l+1}\frac{(l+m)!}{(l-m)!}\,, \tag{76}$$

with which we then find the following orthogonality relations with respect to the inner product on the sphere: The nonzero inner products are

$$\begin{aligned}
\langle\mathbf{L}_{lm}(x_1)|\mathbf{L}_{l'm'}(x_2)\rangle &= w_l(x_1)w_l(x_2)\delta_{l,l'}\delta_{m,m'}c_{lm} \\
\langle\mathbf{L}'_{lm}(x_1)|\mathbf{L}'_{l'm'}(x_2)\rangle &= w_l(x_1)w_l(x_2)\delta_{l,l'}\delta_{m,m'}c_{lm}(1+\delta_{m,0}) \\
\langle\mathbf{K}_{lm}(x_1)|\mathbf{K}_{l'm'}(x_2)\rangle &= \Big(\frac{w_l(x_1)w_l(x_2)}{x_1x_2}l(l+1) + X_l(x_1)X_l(x_2)\Big)\delta_{l,l'}\delta_{m,m'}c_{lm} \\
\langle\mathbf{K}'_{lm}(x_1)|\mathbf{K}'_{l'm'}(x_2)\rangle &= \Big(\frac{w_l(x_1)w_l(x_2)}{x_1x_2}l(l+1) + X_l(x_1)X_l(x_2)\Big)\delta_{l,l'}\delta_{m,m'}(1+\delta_{m,0})c_{lm}\,.
\end{aligned} \tag{77}$$

whereas all other inner products are zero;

$$\begin{aligned}
\langle\mathbf{L}_{lm}(x_1)|\mathbf{L}'_{l'm'}(x_2)\rangle = \langle\mathbf{L}_{lm}(x_1)|\mathbf{K}_{l'm'}(x_2)\rangle = \langle\mathbf{L}_{lm}(x_1)|\mathbf{K}'_{l'm'}(x_2)\rangle = 0 \\
\langle\mathbf{L}'_{lm}(x_1)|\mathbf{K}_{l'm'}(x_2)\rangle = \langle\mathbf{L}'_{lm}(x_1)|\mathbf{K}'_{l'm'}(x_2)\rangle = \langle\mathbf{K}_{lm}(x_1)|\mathbf{K}'_{l'm'}(x_2)\rangle = 0\,.
\end{aligned} \tag{78}$$

One has to be careful with interpreting the notation here: In each vector field w_l is at the moment still free to choose, but in expressions such as in eqns.(77) a product $w_l(x_1)w_l(x_2)$ one should remember that $w_l(x_1)$ might be stemming from the *same* function as $w_l(x_2)$, which would be wrong. The product $w_l(x_1)w_l(x_2)$ means the product of perhaps two different functions, one evaluated at x_1, the other at x_2; it could thus be $h_l(x_1)j_l(x_2)$, or even $y_l(x_1)h_l(x_2)$, depending on the choice made for the spherical Bessel function inside the vector harmonic on the left-hand side. We think this ambiguity is less awkward than any other choice of notation.

Since we will first focus on the continuity of $\mathbf{E}\times\mathbf{e}_r$ and $\mathbf{H}\times\mathbf{e}_r$ at the surface of the droplet, the vector products with $\mathbf{e}_r$ are to be considered as well. Using $\mathbf{e}_\vartheta\times\mathbf{e}_r = -\mathbf{e}_\varphi$ and $\mathbf{e}_\varphi\times\mathbf{e}_r$ one can write

$$\begin{aligned}
\mathbf{L}_{lm}\times\mathbf{e}_r &= -\Gamma^-_{lm}w_l(x)\mathbf{e}_\vartheta - \Delta^+_{lm}w_l(x)\mathbf{e}_\varphi \\
\mathbf{L}'_{lm}\times\mathbf{e}_r &= -\Gamma^+_{lm}w_l(x)\mathbf{e}_\vartheta + \Delta^-_{lm}w_l(x)\mathbf{e}_\varphi \\
\mathbf{K}_{lm}\times\mathbf{e}_r &= +\Delta^+_{lm}X_l(x)\mathbf{e}_\vartheta - \Gamma^-_{lm}X_l(x)\mathbf{e}_\varphi \\
\mathbf{K}'_{lm}\times\mathbf{e}_r &= -\Delta^-_{lm}X_l(x)\mathbf{e}_\vartheta - \Gamma^+_{lm}X_l(x)\mathbf{e}_\varphi\,,
\end{aligned} \tag{79}$$

which also satisfy some useful orthogonality properties, which are easily deduced, either directly using eqns.(77) and (78), or by very similar calculations: The only nonvanishing inner products are

$$\begin{aligned}
\langle\mathbf{L}_{lm}(x_1)\times\mathbf{e}_r|\mathbf{L}_{l'm'}(x_2)\times\mathbf{e}_r\rangle &= w_l(x_1)w_l(x_2)\delta_{l,l'}\delta_{m,m'}c_{lm} \\
\langle\mathbf{L}'_{lm}(x_1)\times\mathbf{e}_r|\mathbf{L}'_{l'm'}(x_2)\times\mathbf{e}_r\rangle &= w_l(x_1)w_l(x_2)\delta_{l,l'}\delta_{m,m'}c_{lm}(1+\delta_{m,0}) \\
\langle\mathbf{K}_{lm}(x_1)\times\mathbf{e}_r|\mathbf{K}_{l'm'}(x_2)\times\mathbf{e}_r\rangle &= \delta_{l,l'}\delta_{m,m'}c_{lm}X_l(x_1)X_l(x_2) \\
\langle\mathbf{K}'_{lm}(x_1)\times\mathbf{e}_r|\mathbf{K}'_{l'm'}(x_2)\times\mathbf{e}_r\rangle &= \delta_{l,l'}\delta_{m,m'}(1+\delta_{m,0})c_{lm}X_l(x_1)X_l(x_2)\,.
\end{aligned} \tag{80}$$

Again, one has to be careful to interpret products as $w_l(x_1)w_l(x_2)$, in the equations above. But below, we indicate as an additional argument, which spherical Bessel function has to be taken – a detail, which till now barely mattered, so that we tried to keep the notation as little clumsy as was necessary. We will below for example write $\mathbf{L}_{lm}(h, k_1r)$ to indicate that in the vector field $\mathbf{L}_{lm}$ the spherical Bessel function that was taken is h_l and that its argument is k_1r. Also, we will use an index on the objects X_l, thus $X^j_l(x) = \frac{1}{x}(xj_l(x))'$ for example.

The scattered radiation is given by its values outside the droplet, so it need not even by regular at the origin. But being an outgoing wave, its behavior for large x should be of the form $\frac{e^{ix}}{x}$, which dictates that we use h_l. The radiation inside the droplet has to be regular at the origin, since this is the center of the droplet, and hence we have to take j_l for the vector fields inside the droplet.

With these considerations on the choice of spherical harmonics, we first take a completely general ansatz for the outgoing radiation $\mathbf{E}^{out}$, $\mathbf{H}^{out}$ and the radiation inside the droplet

$\mathbf{E}^d$, $\mathbf{H}^d$:

$$\begin{aligned}
\mathbf{E}^{out} &= \sum_{lm} A_{lm}\mathbf{K}_{lm}(h,k_1r) + A'_{lm}\mathbf{K}'_{lm}(h,k_1r) + B_{lm}\mathbf{L}_{lm}(h,k_1r) + B'_{lm}\mathbf{L}'_{lm}(h,k_1r) \\
\mathbf{H}^{out} &= -im_1\sum_{lm} A_{lm}\mathbf{L}_{lm}(h,k_1r) + A'_{lm}\mathbf{L}'_{lm}(h,k_1r) + B_{lm}\mathbf{K}_{lm}(h,k_1r) + B'_{lm}\mathbf{K}'_{lm}(h,k_1r) \\
\mathbf{E}^{d} &= \sum_{lm} C_{lm}\mathbf{K}_{lm}(j,k_2r) + C'_{lm}\mathbf{K}'_{lm}(j,k_2r) + D_{lm}\mathbf{L}_{lm}(j,k_2r) + D'_{lm}\mathbf{L}'_{lm}(j,k_2r) \\
\mathbf{H}^{d} &= -im_2\sum_{lm} C_{lm}\mathbf{L}_{lm}(j,k_2r) + C'_{lm}\mathbf{L}'_{lm}(j,k_2r) + D_{lm}\mathbf{K}_{lm}(j,k_2r) + D'_{lm}\mathbf{K}'_{lm}(j,k_2r)\,,
\end{aligned} \tag{81}$$

where $m_1 = \sqrt{\frac{\epsilon_1}{\mu_1}}$ and $m_2 = \sqrt{\frac{\epsilon_2}{\mu_2}}$.

For the incoming radiation we use eqns.(58) and adapt it to the renewed notation:

$$\begin{aligned}
\mathbf{E}^{in} &= \mathcal{E}\sum_{l=1}^{\infty} E_l(\mathbf{K}_{l1}(j,k_1r) + i\mathbf{L}'_{l1}(j,k_1r)) \\
\mathbf{H}^{in} &= m_1\mathcal{E}\sum_{l=1}^{\infty} E_l(\mathbf{K}'_{l1}(j,k_1r) - i\mathbf{L}_{l1}(j,k_1r))\,.
\end{aligned} \tag{82}$$

We remind the reader that $E_l = \frac{i^{l+1}(2l+1)}{l(l+1)}$.

To take care of the third and fourth of the interface conditions (71), we take the vector product with $\mathbf{e}_r$ and then use the orthogonality relations (80). First, let us consider $m \neq 1$. Taking inner products with appropriate choices of $\mathbf{K}_{lm} \times \mathbf{e}_r$ and $\mathbf{L}_{lm} \times \mathbf{e}_r$ we obtain among others the following equations:

$$A_{lm}X_l^h(k_1a) = C_{lm}X_l^j \qquad m_1A_{lm}h_l(k_1a) = m_2m_2C_{lm}j_l(k_2a)\,. \tag{83}$$

But unless

$$m_2j_l(k_2a)X_l^h(k_1a) - m_1h_l(k_1a)X_l^j(k_2a) = 0\,. \tag{84}$$

the only solution is $A_{lm} = C_{lm} = 0$. Equation (84) can never be satisfied for general values of a, only a special value of a can make this expression to vanish, but there are infinitely many values of l. Also, for $m = 1$ we will see that the expression on the right hand side has to be nonzero in order for a solution to exist. From a more physical perspective, if no radiation is incoming, we do not expect some radiation to be scattered, and hence the equations that are not altered by the presence of the incoming radiation – here the presence of terms proportional to E_l – are expected to have the same solutions as without incoming radiation. We thus arrive at the conclusion that all coefficients vanish unless $m = 1$; indeed, for the other coefficients one finds similar equations for $m \neq 1$.

We rewrite the coefficients slightly; as the coefficients A_{lm} and others vanish for $m \neq 1$, we write $A_{l1} = A_l$ and similarly for the others – we thus omit the index 1. The equations to solve now are only those for $m = 1$ and taking appropriate inner products with $\mathbf{K}_{l1} \times \mathbf{e}_r$ and others we arrive at the following set of equations:

$$
\begin{aligned}
\mathcal{E} E_l X_l^j(k_1 a) + A_l X_l^h(k_1 a) &= C_l X_l^j(k_2 a) \\
m_1 B_l X_l^h(k_1 a) &= m_2 D_l X_l^j(k_2 a) \\
A_l' X_l^h(k_1 a) &= C_l' X_l^j(k_2 a) \\
i m_1 \mathcal{E} E_l X_l^j(k_1 a) + m_1 B_l' X_l^h(k_1 a) &= m_2 D_l' X_l^j(k_2 a) \\
B_l h_l(k_1 a) &= D_l j_l(k_2 a) \\
m_1 \mathcal{E} E_l j_l(k_1 a) + m_1 A_l h_l(k_1 a) &= m_2 C_l j_l(k_2 a) \\
i \mathcal{E} E_l j_l(k_1 a) + B_l' h_l(k_1 a) &= D_l' j_l(k_2 a) \\
m_1 A_l' h_l(k_1 a) &= m_2 C_l' j_l(k_2 a)
\end{aligned}
\tag{85}
$$

From those equations without E_l, i.e., from the second, third and fifth, we see that, using a reasoning similar to above, that $B_l = D_l = 0$ and $A_l' = C_l' = 0$. The remaining equations are linear equations and hence are easily solved, although the algebra can be a little messy:

$$
\begin{aligned}
A_l &= \frac{m_1 j_l(x_1) X_l^j(x_2) - m_2 j_l(x_2) X_l^j(x_1)}{m_2 j_l(x_2) X_l^h(x_1) - m_1 h_l(x_1) X_l^j(x_2)} \mathcal{E} E_l \\
C_l &= \frac{m_1 j_l(x_1) X_l^h(x_1) - m_1 h_l(x_1) X_l^j(x_1)}{m_2 j_l(x_2) X_l^h(x_1) - m_1 h_l(x_1) X_l^j(x_2)} \mathcal{E} E_l \\
B_l' &= i \frac{m_2 j_l(x_1) X_l^j(x_2) - m_1 j_l(x_2) X_l^j(x_1)}{m_1 j_l(x_2) X_l^h(x_1) - m_2 h_l(x_1) X_l^j(x_2)} \mathcal{E} E_l \\
D_l' &= i \frac{m_1 j_l(x_1) X_l^h(x_1) - m_1 h_l(x_1) X_l^j(x_1)}{m_1 j_l(x_2) X_l^h(x_1) - m_2 h_l(x_1) X_l^j(x_2)} \mathcal{E} E_l \, .
\end{aligned}
\tag{86}
$$

We can now apply some cosmethics to obtain some nicer form. First we define the refraction index

$$
n = \sqrt{\frac{\epsilon_2 \mu_2}{\epsilon_1 \mu_1}} \, . \tag{87}
$$

We remark that $x_2 = k_2 a = k_1 a n = x_1 n$. Multiplying denominator and nominator in the above expressions with $x_1 x_2$ and using the definitions

$$
Q_l(x) = x h_l(x) \, , \quad S_l(x) = x j_l(x) \tag{88}
$$

and

$$
A_l = \alpha_l \mathcal{E} E_l \, , \quad C_l = n \gamma_l \mathcal{E} E_l \, , \quad B_l' = i \beta_l \mathcal{E} E_l \, , \quad D_l' = i n \delta_l \mathcal{E} E_l \tag{89}
$$

we then find

$$
\begin{aligned}
\alpha_l &= \frac{m_1 S_l(x_1) S_l'(x_2) - m_2 S_l(x_2) S_l'(x_1)}{m_2 S_l(x_2) Q_l'(x_1) - m_1 Q_l(x_1) S_l'(x_2)} \\
\gamma_l &= \frac{m_1 S_l(x_1) Q_l'(x_1) - m_1 Q_l(x_1) S_l'(x_1)}{m_2 S_l(x_2) Q_l'(x_1) - m_1 Q_l(x_1) S_l'(x_2)} \\
\beta_l &= \frac{m_2 S_l(x_1) S_l'(x_2) - m_1 S_l(x_2) S_l'(x_1)}{m_1 S_l(x_2) Q_l'(x_1) - m_2 Q_l(x_1) S_l'(x_2)} \\
\delta_l &= \frac{m_1 S_l(x_1) Q_l'(x_1) - m_1 Q_l(x_1) S_l'(x_1)}{m_1 S_l(x_2) Q_l'(x_1) - m_2 Q_l(x_1) S_l'(x_2)}
\end{aligned}
\tag{90}
$$

and the expansions of the scattered fields and the fields inside the droplets now finally become:

$$
\begin{aligned}
\mathbf{E}^{out} &= \mathcal{E} \sum_l E_l \Big(\alpha_l \mathbf{K}_{l1}(h, k_1 r) + i \beta_l \mathbf{L}_{l1}'(h, k_1 r) \Big) \\
\mathbf{H}^{out} &= m_1 \mathcal{E} \sum_l E_l \Big(\beta_l \mathbf{K}_{l1}'(h, k_1 r) - i \alpha_l \mathbf{L}_{l1}(h, k_1 r) \Big) \\
\mathbf{E}^{d} &= n \mathcal{E} \sum_l E_l \Big(\gamma_l \mathbf{K}_{l1}(j, k_2 r) + i \delta_l \mathbf{L}_{l1}'(j, k_2 r) \Big) \\
\mathbf{H}^{d} &= m_2 n \mathcal{E} \sum_l E_l \Big(\delta_l \mathbf{K}_{l1}'(j, k_2 r) - i \gamma_l \mathbf{L}_{l1}(j, k_2 r) \Big) \,.
\end{aligned}
\tag{91}
$$

The expressions (90) and (91) are our final expressions, those that we wanted to deduce in a systematic and explicit way. We will now perform some checks, to verify the correctness of the result. The first check is to take $\epsilon_1 = \epsilon_2$ and $\mu_1 = \mu_2$, so effectively to elimenate the droplet. We then have $m_1 = m_2$, $n = 1$ and $x_1 = x_2$. We then see that $\alpha_l = \beta_l = 0$ and $\gamma_l = \delta_l = E_l$. As expected, then the outgoing fields vanish and the fields inside the droplet are then just the continuation of the incoming wave to the inside.

A second check consists in the first two of the interface conditions (71). Using the constitutive relations $\epsilon_1 \mathbf{E}^{out} = \mathbf{D}^{out}$, $\epsilon_2 \mathbf{E}^{d} = \mathbf{D}^{d}$, $\mu_1 \mathbf{H}^{out} = \mathbf{B}^{out}$ and $\mu_2 \mathbf{H}^{d} = \mathbf{B}^{d}$ these radial interface conditions become

$$
\begin{aligned}
&\epsilon_1 (\mathbf{E}^{in} + \mathbf{E}^{out}) \cdot \mathbf{e}_r = \epsilon_2 \mathbf{E}^d \cdot \mathbf{e}_r \quad \text{at} \quad r = a \\
&\mu_1 (\mathbf{H}^{in} + \mathbf{H}^{out}) \cdot \mathbf{e}_r = \mu_2 \mathbf{H}^d \cdot \mathbf{e}_r \quad \text{at} \quad r = a \,.
\end{aligned}
\tag{92}
$$

Since $\mathbf{L}_{lm} \cdot \mathbf{e}_r = \mathbf{L}_{lm}' \cdot \mathbf{e}_r = 0$ and

$$
\begin{aligned}
\mathbf{K}_{lm}(w, x) \cdot \mathbf{e}_r &= \frac{w_l(x)}{x} l(l+1) P_{lm}(\xi) \cos(m\varphi) \\
\mathbf{K}_{lm}'(w, x) \cdot \mathbf{e}_r &= \frac{w_l(x)}{x} l(l+1) P_{lm}(\xi) \sin(m\varphi)
\end{aligned}
\tag{93}
$$

and by the orthogonality of the associated Legendre functions, we find the radial interface conditions reduce to

$$\epsilon_1\alpha_l\frac{h_l(x_1)}{x_1}+\epsilon_1\frac{j_l(x_1)}{x_1}=\epsilon_2 n\gamma_l\frac{j_l(x_2)}{x_2} \tag{94}$$

and

$$\mu_1 m_1\beta_l\frac{h_l(x_1)}{x_1}+\mu_1 m_1\frac{j_l(x_1)}{x_1}=\mu_2 m_2 n\delta_2\frac{j_l(x_2)}{x_2}\,. \tag{95}$$

Using identities like $\epsilon_1 n m_2 = m_1\epsilon_2$, $\mu_1 m_1 n = \mu_2 m_2$ and $x_2 = n x_1$ and inserting the explicit expressions of α_l, β_l, γ_l and δ_l one finds that the interface conditions are satisfied as well.

8 Aftermath

To calculate the ingoing and outgoing power, we cannot simply apply $\mathbf{S} = \mathbf{E}\times\mathbf{H}$, since our fields are complex. Considering the real parts of fields $\mathbf{E}e^{-i\omega t}$ and $\mathbf{H}e^{-i\omega t}$ one finds that the vector

$$\mathbf{S}=\frac{1}{2}\mathrm{Re}\Big(\mathbf{E}\times\mathbf{H}^*\Big) \tag{96}$$

describes the Poynting vector in the complex case.

The ingoing power is then given by $P^{in} = \mathrm{Re}\Big(\mathbf{e}_z\cdot\mathbf{E}^{in}\times\mathbf{H}^{in*}\Big)\pi a^2$. From $\mathbf{E}^{in}$, given in eqn.(39), one finds

$$\mathbf{H}^{in*} = m_1\mathbf{e}_y\mathcal{E}e^{-ikz}\,. \tag{97}$$

Therefore the ingoing power is $P^{in} = \frac{1}{2}m_1\pi a^2\mathcal{E}^2$.

The outgoing energy flux is $I^{out} = \int_{S^2}\mathbf{e}_r\cdot\mathrm{Re}\Big(\mathbf{E}^{out}\times\mathbf{H}^{out*}\mathrm{d}\Omega\Big)$. We will consider a large sphere of radius R around the droplet. We need then the large x behavior of $X_l^*h_l$ and its complex conjugate. One finds that $X_l(x)^*h_l(x)\approx\frac{-i}{x^2}$ plus terms that go faster to zero than x^{-2}. Working out the products $\mathbf{e}_r\cdot\mathbf{K}_{l1}\times\mathbf{K}_{l'1}$, $\mathbf{e}_r\cdot\mathbf{K}_{l1}\times\mathbf{L}_{l'1}$ and all others one finds that the only relevant nonzero integrals are

$$\int_{S^2}\mathrm{d}\Omega\,\mathbf{e}_r\cdot\mathbf{K}_{lm}\times\mathbf{L}^*_{lm}\,,\qquad\int_{S^2}\mathrm{d}\Omega\,\mathbf{e}_r\cdot\mathbf{K}'_{lm}\times\mathbf{L}^{*'}_{lm}\,, \tag{98}$$

and their complex conjugates.

We find

$$\int_{S^2}\mathrm{d}\Omega\mathbf{e}_r\cdot\mathbf{E}^{out}\times\mathbf{H}^{out*}=im_1\mathcal{E}^2\int_{S^2}\mathrm{d}\Omega\sum_{l=1}^{\infty}|E_l|^2\mathbf{e}_r\cdot\Big(|\alpha_l|^2\mathbf{K}_{l1}\times\mathbf{L}^*_{l1}-|\beta_l|^2\mathbf{K}^*_{l1}\times\mathbf{L}'_{l1}\Big)\,, \tag{99}$$

from which, after some algebra, follows that the outgoing power is given by

$$P^{out} = \frac{4\pi^2 m_1 \mathcal{E}^2}{k_1^2} \sum_{l=1}^{\infty} (2l+1)\Big(|\alpha_l|^2 + |\beta_l|^2\Big)\,. \tag{100}$$

From this, one easily finds the ratio P^{out}/P^{in}, which is the total cross section.

The sun does not radiate through planar waves. To correctly describe the physics of rainbows at least two aspects need to be taken into account: (1) The radiation of a distant pointlike object is more of the form $\frac{e^{ikR}}{R}$ than a plane wave, where R the distance to the pointlike object, (2) the sun is not a pointlike object. Suppose one has taken the first point into account. Then one could try and write the incident radiation as a sum the radiation of several pointlike distant objects. We therefore give some thoughts on the first point.

We could proceed as follows: In one step we find adequate spherical solutions to the Maxwell equations, and then relate two spherical coordinate systems to each other. Then we expand the transformed solutions in a series of the parameter $\frac{1}{R}$ keeping only the lowest orders. This then gives a deformed expression for the incident radiation, which can – at least in principle – be expanded in terms of the spherical vector harmonics.

Spherical electromagnetic waves can be constructed as follows:

$$\mathbf{E}(x,y,z) = \nabla \times \Big(h_0(kr)\mathbf{e}_x\Big) = -k\mathbf{L}_{11}\,, \tag{101}$$

where the last equality follows from $\nabla \times (\sin\vartheta \cos\varphi \mathbf{r}) = \mathbf{e}_x \times \mathbf{e}_r$. Calculating $\mathbf{H}$ from eqn.(101) one finds an expression proportional to $\mathbf{K}_{11}$. Then one finds that

$$\mathbf{S}\mathbf{e}_r = \frac{1}{2}\mathrm{Re}\Big(\mathbf{E} \times \mathbf{H}^*\big) \sim \frac{1}{r^2}\Big(\sin^2\varphi + \cos^2\varphi \cos^2\vartheta\Big) + O(r^{-3})\,, \tag{102}$$

which shows that the radiation intensity indeed has the right r-dependence.

One could choose the two coordinate systems as follows: One has two coordinate systems at the pointlike sun, C' is a cartesian coordinate system at the sun and S' is a spherical coordinate system at the sun, where the usual relations eqns.(1) hold, but then with primes coordinates r', ϑ', φ', x', y' and z'. On the earth we choose one water droplet, and take a cartesian coordinate C and a spherical coordinate system S at the center of the droplet. We choose the x-axis to be parallel to the x'-axis, the y-axis parallel to the y'-axis and the z-axis parallel to the z'-axis and furthermore, the droplet is located at $z' = R$, $x' = y' = 0$.

That is, the z-axis and the z'-axis are the same line. One then finds

$$
\begin{aligned}
\cos\vartheta' &= \frac{\epsilon\cos\vartheta + 1}{\sqrt{1+\epsilon^2+2\epsilon\cos\theta}} \\
\sin\vartheta' &= \frac{\epsilon\sin\vartheta}{\sqrt{1+\epsilon^2+2\epsilon\cos\theta}} \\
\varphi' &= \varphi \\
r' &= R\sqrt{1+\epsilon^2+2\epsilon\cos\theta}\,,
\end{aligned}
\tag{103}
$$

where $\epsilon = \frac{r}{R}$. Using the relations between C' and C and the expressions eqns.(1) for both the sun and the droplet, one arrives sfter some algebra at the following relations between the basis vectors

$$
\begin{aligned}
\mathbf{e}^{'}_{r'} &= \frac{(\cos\vartheta+\epsilon)\mathbf{e}_r - \sin\vartheta\mathbf{e}_\vartheta}{\sqrt{1+\epsilon^2+2\epsilon\cos\theta}} \\
\mathbf{e}^{'}_{\vartheta'} &= \frac{\sin\vartheta\mathbf{e}_r + (\cos\vartheta+\epsilon)\mathbf{e}_\vartheta}{\sqrt{1+\epsilon^2+2\epsilon\cos\theta}} \\
\mathbf{e}^{'}_{\varphi'} &= \mathbf{e}_\varphi\,.
\end{aligned}
\tag{104}
$$

The square root is related to the generating function of the Legendre polynomials

$$
\frac{1}{\sqrt{1+x^2-2x\xi}} = \sum_{n=0}^{\infty} P_n(\xi)x^n\,. \tag{105}
$$

It is therefore tedious, but straightforward to obtain expansions in ϵ to any order of all quantities involved. Hence the incoming electromagnetic wave can be described by a power series expansion in ϵ, where the coefficients are vectors, which are to be expanded in the vector spherical harmonics.

The functions $S_l(x)$ and $Q_l(x)$ defined in eqns. (88) are called Bessel–Riccati functions. The functions $\psi_l(x) = xw_l(x)$ satisfy the differential equation

$$
x^2\psi_l''(x) + (x^2 - l(l+1))\psi_l(x) = 0\,. \tag{106}
$$

In several problems the Bessel–Riccati equation shows up. For example, if one tries to solve the Schrödinger equation for a particle in a three-dimensional, infinite, spherical potential well of radius a by the same method of separation of variables discussed in section 3, one finds that the radial part of the wave function $\psi = R(r)\Theta(\vartheta)\Phi(\varphi)$ can be described by a solution u of the Bessel–Riccati equations that satisfies the normalization constraint $\int_0^a |u(r)|^2 \mathrm{d}r = 1$, where the relation between R and u is given by $R(r) = ru(r)$.

Finally, we remark on rainbow-like phenomena. Various material properties, in this note incorporated in ϵ_2 and μ_2 may describe on the frequency. Therefore the coefficients α_l, β_l,

γ_l and δ_l will vary with ω. This then means that certain intensity extremal directions may vary with ω. Hence, if any intensity variations with respect to angles ϑ may occur, they will vary with color, giving rise to effects like the rainbow.

Many open questions remain. For sure we haven't taken the effort to plot typical scattering patterns, which would have completed this note. Our aim was to write down, how the basic equations of Mie theory can be derived. Now it is up to the reader to find out more, and fill in the gaps.

9 Notational glossary

Except for the spherical vector harmonics $\mathbf{L}'_{lm}$ and $\mathbf{K}'_{lm}$ a prime will be used for a function of a single variable to denote its derivative with respect to the variable.

$\triangle$:	Laplacian, p.4
a:	radius of water droplet
$\Gamma^{\pm}_{lm}$:	special functions on the sphere, p. 15
c_{lm}:	$c_{lm} = \pi \frac{2l(l+1)}{2l+1} \frac{(l+m)!}{(l-m)!}$, 16
$\delta_{l,l'}$, $\delta_{\alpha,\beta}$:	Kronecker Delta, p. 6
$\Delta^{\pm}_{lm}$:	special functions on the sphere, p. 15
$\mathbf{D}$:	displacement field, p.3
ϵ, ϵ_1, ϵ_2:	permittivity, p. 3
$\mathcal{E}$:	strength of ingoing electromagnetic fields, p. 10
E_l:	coefficient in expansion if ingoing radiation $E_l = \frac{i^{l+1}(2l+1)}{l(l+1)}$, p. 13
$\mathbf{E}^d$:	electric field in the droplet, section 7
$\mathbf{E}^{in}$:	incoming electric field, section 6
$\mathbf{E}^{out}$:	outgoing electric field, section 6
$\mathbf{e}_x$. $\mathbf{e}_y$, $\mathbf{e}_z$:	cartesian unit basis vectors, p. 2
$\mathbf{e}_r$. $\mathbf{e}_\vartheta$, $\mathbf{e}_\varphi$:	spherical unit basis vectors, p. 2
$\mathbf{E}$:	electric field, p.3
$\mathbf{H}^d$:	magnetic field in the droplet, section 7
$\mathbf{H}^{in}$:	incoming magnetic field, section 6
$\mathbf{H}^{out}$:	outgoing magnetic field, section 7

h_l: spherical Bessel function $j_l + iy_l$, p. 9

j_l: regular spherical Bessel function, p. 9

$k,\ k_1,\ k_2$: wave number, p. 3

$\mathbf{K}_{lm}$, $\mathbf{K}'_{lm}$: vector spherical harmonic, p. 7

$\mathbf{L}_{lm}$, $\mathbf{L}'_{lm}$: vector spherical harmonic, p. 7

$\mu,\ \mu_1,\ \mu_2$: permeability, p. 3

m_1, m_2: $\sqrt{\frac{\epsilon_i}{\mu_i}}$, p. 18

n: refraction index $n = \sqrt{\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}}$, p. 19

$\Pi_l^{\pm}$: orthogonal functions, p. 11

P_l: Legendre polynomial, p. 7

P_{lm}: associated Legendre functions, p. 8

$Q_l(x)$: second Bessel–Riccati function $xh_l(x)$, p. 19

r, ϑ, φ: spherical coordinates, p. 2

$S_l(x)$: first Bessel–Riccati function $xj_l(x)$, p. 19

t: time

w_l: (generic) spherical Bessel function, p. 5, 9

ξ: $\cos\vartheta$, argument of associated Legendre functions, 5

X_l: special function $X_l(x) = \frac{1}{x}(xw_l(x))'$, p. 11

X_l^j, X_l^h: special function $X_l^j(x) = \frac{1}{x}(xj_l(x))'$, $X_l^h(x) = \frac{1}{x}(xh_l(x))'$, p. 17

y_l: irregular spherical Bessel function, p. 9

ω: angular frequency of radiation, p. 3