Alternating harmonic series through integrals of power series

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We consider the following identities:

$$\begin{aligned} O &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \frac{\pi}{4} \\ H &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \ln(2) \end{aligned} \tag{1}$$

The second series is know as the alternating harmonic series. We prove both identities and rely heavily on the geometric series:

$$\sum_{n=0}^{\infty} s^n = \frac{1}{1-s}\,, \qquad \sum_{n=0}^{\infty} (-1)^n s^n = \frac{1}{1+s}\,,$$

which are valid for $|s| < 1$.

Remark: We will not proof that we can interchange summation and integration. The interested reader is urged to apply the Dominated Convergence Criterium.

1 Integrals

Consider the integral I defined by

$$I = \int_0^{\infty} \frac{\mathrm{d}x}{1+e^x}$$

which clearly exists. We first remark that the integrand is the derivative of the funtion $f(x) = -\ln(1+e^{-x})$. Thus we have $I = f(\infty) - f(0) = -\ln(1) + \ln(2) = \ln(2)$. Now consider the expansion

$$\frac{1}{1+e^x} = \frac{e^{-x}}{1+e^{-x}} = \sum_{n=0}^{\infty} (-1)^n e^{-(n+1)x}$$

so that we also find the following expression for I:

$$I = \int_0^\infty \sum_{n=0}^{\infty} (-1)^n e^{-(n+1)x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\,.$$

We have thus proven the second identity in (1).

Now we consider the well-known integral $2J$ given by

$$2J = \int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{2}\,.$$

We first remark that if we put $t = 1/x$ we have

$$\frac{\mathrm{d}x}{1+x^2} = \frac{\mathrm{d}t}{1+t^2}$$

from which it follows that

$$J = \int_1^\infty \frac{\mathrm{d}x}{1+x^2} = \int_0^1 \frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{4}\,.$$

Also, one directly finds $\int_0^1 \frac{\mathrm{d}x}{1+x^2} = \arctan(1) - \arctan(0) = \pi/4$. A direct conclusion is that the Cauchy distribution is invariant under inversions $x \mapsto 1/x$. We now expand

$$\frac{1}{1+x^2} = \frac{1}{x^2}\frac{1}{1+\frac{1}{x^2}} = \sum_{n=0}^{\infty} \frac{(-1)}{x^{2n+2}}$$

and integrate term by term to obtain

$$J = \sum_{n=0}^{\infty} \int_1^\infty \mathrm{d}x \frac{(-1)}{x^{2n+2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\,.$$

This proves the first identity of (1).

2 Expansions of known functions

Consider the series

$$a(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

which converges for all $|x| < 1$. Differentiating $a(x)$ one obtains

$$a'(x) = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}\,.$$

We thus have

$$a(x) = a(0) + \int_0^x \frac{\mathrm{d}t}{1-t} = -\ln(1-x)\,,$$

which exists for all $|t| < 1$ – for $x = 1$ we see that the harmonic series diverges, a well-known fact. In fact, the limit $t \to -1$ exists and we thus find

$$a(-1) = -\ln(2) = -1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \dots\,,$$

which alternatively proves the second identity of (1). We can in fact do more. The function $a(x)$ is analytic inside the unit disk and in many points on the boundary the limit exists. We rewrite the above and obtain

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots = -\frac{\ln(1-x)}{x}\,.$$

If we insert $x = i$ we obtain

$$1 + \frac{i}{2} - \frac{1}{2} - \frac{i}{4} + \frac{1}{5} + \frac{i}{6} - \frac{1}{7} + \dots = -\frac{\ln(1-i)}{i} = i\ln(1-i)\,. \tag{2}$$

By inspection we see that the left-hand side of (2) reduces to

$$\frac{i}{2}H + O\,.$$

On the other hand $1 - i = \sqrt{2}e^{-\frac{i\pi}{4}}$ so that the right-hand side of (2) reduces

$$-\frac{\ln(1-i)}{i} = \frac{\pi}{4} + i\ln(\sqrt{2}) = \frac{\pi}{4} + \frac{i}{2}\ln(2)\,.$$

We thus immediately read off that $O = \frac{\pi}{4}$ and $H = \ln(2)$.

We have thus shown in several ways that the identities in (1) hold.