On the number of alternating permutations

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1 Introduction

Combinatorics is a field of mathematics that deserves little attention in high-school education. One reason is probably that creativity of students is necessary to get to some result. Therefore a success is not directly guaranteed, or only accessible, when sufficient time is present; in high-school education the educational goals should be at reach within a certain, but limited, amount of time. Although creativity is needed in real-life situations, educational aims are dominated by algorithmic and programmatic themes, which often allow a testable and reachable scheme of skills. As a consequence, combinatorics is only tought in some minimal form. We think that mathematics education should also try to offer contact hours, where nontestable skills that require some creativity are developped. The goal of education should not be a list of test results, but more a preparation for a future life.

This little note will not change the world, neither is this note about some fundamental results in combinatorics. We present some thoughts on a question/exercise posed by Martin Erickson in his book Beautiful Mathematics (Mathematical Associaiton of America, Spectrum; german version: Mathematische Appetithäppchen, Springer Verlag). In a little chapter on the expansions for the tangens and secans functions he relates the expansions to the number of alternating permutations. An alternating permutation of the numbers $\{1, 2, \ldots, n\}$ is a permutation $a_1 a_2 \cdots a_n$ of the integers 1 to n, such that a_2 is larger than a_1, a_3 is smaller than a_2, a_4 is larger than a_3 and so on. In other words, these are the permutations where one first goes up, then goes down, goes up, goes down, and so on. In this little note we present a proof, why the expansions of secans and tangens give the numbers of the alternating permutations.

We hope that some mathematicians or mathematics teachers reading this note feel encouraged to once and a while take the opportunity to show some of the elegance of combinatorics to non-mathematicians. Also, some pupils or students might find this note worthwile reading and feel encouraged to spend once and a while some time on combinatorial problems. The material in this note can be understood by high-school students in their final years; some preparation by the teacher is needed. Maybe the teacher should expand some on the idea of giving a power series expansion of and on the compact notation. The idea of reducing a problem to earlier solved problems might be worth discussing with pupils even in a nonmathematical context.

2 Series expansions and recursive formulae

To study the behavior of a function f around some point one can try to give an approximating expression, which contains enough information, but is easier to handle than the function f itself. The idea of Taylor expansion of a function f is to give a polynomial expression up to arbitrary degree that approximates f very well. As the degree is arbitrary, one takes the limit and arrives at a power series expression. The point around which one wants to study f we now choose as the origin. A Taylor expansion is thus of the form

$$f(x) = \sum_{k=0}^{\infty} \frac{f_k}{k!} x^k$$

where we included $k!$ in the denominator for convenience. Matching all derivatives at $x = 0$ requires $f_k = f^{(k)}(0)$. If f satisfies $f(x) = f(-x)$ then $f_k = 0$ for all odd k, and if f satisfies $f(x) = -f(-x)$ then $f_k = 0$ for all even k.

We now consider the functions $\tan(x)$ and $\sec(x) = \frac{1}{\cos(x)}$. Differentiating one obtains $\tan'(x) = \sec^2(x)$ and $\sec'(x) = \tan(x)\sec(x)$. Putting

$$\tan(x) = \sum_{k=0}^{\infty} \frac{t_k}{k!} x^k\,, \qquad \sec(x) = \sum_{k=0}^{\infty} \frac{s_k}{k!} x^k\,,$$

and comparing coefficients we arrive at

$$t_k = \sum_{p=0}^{k-1} \binom{k-1}{p} s_p s_{k-1-p}\,, \qquad s_k = \sum_{p=0}^{k-1} \binom{k-1}{p} t_p s_{k-1-p}\,.$$

Since $t_0 = s_1 = 0$ and $t_1 = s_0 = 1$ these formulae can be used to recursively find all other s_k and t_k. Since $t_k = 0$ for all even k and $s_k = 0$ for all odd k the sums in the formulae contain many zeros; in fact the running index p can be taken to take two steps at a time. Using the recursion formulae we find:

$$s_2 = \binom{1}{0} t_0 s_1 + \binom{1}{1} t_1 s_0 = 0 + 1 = 1$$

$$t_3 = \binom{2}{0} s_2 s_0 + \binom{2}{1} s_1 s_1 + \binom{2}{2} s_0 s_2 = 1 + 0 + 1 = 2$$

$$s_4 = \binom{3}{9} t_0 s_3 + \binom{3}{1} t_1 s_2 + \binom{3}{2} t_2 s_1 + \binom{3}{3} t_3 s_0 = 0 + 3 + 0 + 2 = 5$$

and proceeding $t_5 = 16$, $s_6 = 61$, $t_7 = 272$. Note again that $0 = t_0 = t_2 = t_4 = \ldots$ and $0 = s_1 = s_3 = s_5 = \ldots$, a fact that is respected by the recursive formulae.

3 Main result

We define a function $a : \mathbb{N} \to \mathbb{N}$ as follows: We put $a_0 = a_1 = 1$ and define $a(n) = s_n$ if n is even and $a(n) = t_n$ if n is odd. The main result is the interesting fact, which Martin Erickson noted in his book, that $a(n)$ is the number of alternating permutations of the set $\{1, 2, \ldots, n\}$ starting with going up. Let us first give some examples to calculate this number of alternating permutations and give some remarks. To make the language easier to read we denote $b(n)$ the number of alternating permutations of $\{1, 2, \ldots, n\}$ starting with going up.

If $n = 2$ the set has only two elements and thus the only allowed permutation is (12). If $n = 3$ we find two alternating permutations starting with going up: (132) and (231). If $n = 4$ we find 5: (1324), (1423), (2314), (2413) and (3412). Thus at least for $n = 1, 2, 3, 4, 5$ the numbers $a(n)$ agree with the number of alternating permutations starting with going up.

If $S = \{a_1, a_2, \ldots, a_n\}$ is a set of integers with $a_1 < a_2 < \cdots < a_n$ then the number of alternating permutations of S starting with going up equals $b(n)$. This is true since only the ordering matters. In particular, the number of alternating permutations of $\{2, 3, \ldots, n\}$ starting with going up equals $b(n-1)$.

We could have equally well have chosen to let the alternating permutations start with going down. Their number equals $b(n)$ as is easily seen by applying the transformation $k \mapsto n - k$.

We will shortly show that $b(n) = a(n)$ by shoing that $b(n)$ satisfies the recursive relation satisfied by $a(n)$. To get the idea we will first treat two examples. We will calculate $b(5)$ in two ways and then calculate $b(6)$.

Any permutation of $\{1, 2, 3, 4, 5\}$ can be obtained by a permutation of $\{1, 2, 3, 4\}$ and then adding the 5. To find $b(5)$ we note that the 5 can only be at the second or the fourth place in an alternating permutation since elsewise it cannot start with going up. Schematically this can be seen as follows: We only look for permutations of the form

$$X \nearrow X \searrow X \nearrow X \searrow X$$

and clearly the 5 cannot follow a going down $\searrow$. This leaves only two places. We thus look for permutations of the form $X5YYY$ and $XXX5Y$. By symmetry their number is equal. To construct such an alternating permutation we first have to choose the X and the Y, this can be done in $\binom{4}{1} = 4$ ways. On X in $X5YYY$ we have no restriction, on the YYY in $X5YYY$ we know that the allowed number of their permutations is $b(3)$, a.k.a. t_3. We thus find $b(5) = 2 \cdot 4 \cdot 2 = 16$.

Instead of matching the 5 into a permutation of $\{1, 2, 3, 4\}$ we could equally well fit a 1 into a permutation of $\{2, 3, 4, 5\}$. Then the 1 can only be at the first, the third and the fifth place. We thus look for alternating permutations starting with going up of the form $1XXXX$, of which there are $s_4 = 5$, or of the form $XX5XX$, of which there are $\binom{4}{2} s_2 s_2 = 6$ or of the form $XXXX1$ of which there are again 5. Adding these numbers we arrive again at $5 + 6 + 5 = 16$, but we note that the way we obtained this number resembles the recursive formula for t_5.

Now we look for $b(6)$. We start as in the previous paragraph, we note that the 1 can only be in the first, the third and the fifth place, as we look for permutations of the form

$$X \nearrow X \searrow X \nearrow X \searrow X \nearrow X \,.$$

We then split our job into counting the allowed permutations of the form $1XXXXX$, $XX1XXXX$, $XXXX1X$. The first form is easy, there are $b(5) = 16$ of these. To obtain an allowed permutation of the second form, we first have to choose 2 out of 5, permute the two in the right way and also find a good permutation for the remaining three. Thus there are $\binom{5}{2} \cdot b(2) \cdot b(3) = \binom{6}{2} \cdot s_2 \cdot t_3 = 20$. For the last form we first need to select 4 out of 5 and then permute the four in the right way. Thus we find $\binom{5}{4} \cdot s_4 = 25$ of them. In total we thus find

$$b(6) = \binom{5}{0} t_5 s_0 + \binom{5}{2} s_2 t_3 + \binom{5}{4} t_1 s_4 = 61 \,.$$

A similar counting for $b(7)$ reveals

$$b(7) = \binom{6}{0} s_6 s_0 + \binom{6}{2} s_4 s_2 + \binom{6}{4} s_4 s_2 + \binom{6}{6} s_6 s_0$$

and then the idea is clear; the $b(n)$ satisfy the same recursion relation as the $a(n)$, hence, being equal for $n = 1, 2, 3, 4, 5$ they are equal for all n.

Let us try to make a formal proof using an induction argument. Assume $a(k) = b(k)$ holds for $1 \leq n \leq n$. We consider first the case, where $n + 1$ is even, so that n is odd. We consider allowed permutations of the form

$$1(X \cdots X)_n \,, \quad XX1(X \cdots X)_{n-2} \,, \quad \ldots (X \cdots X)_{2k} 1 (X \cdots X)_{n-2k} \,, \ldots, (X \cdots X)_{n-1} 1X$$

where $(X \cdots X)_p$ is a short-hand for p concatenated X-symbols. The number of allowed permutations is thus

$$s_0 t_n + \binom{n}{2} s_2 t_{n-2} + \ldots + \binom{n}{n-1} s_{n-1} t_1 = \sum_{p=0}^{n} \binom{n}{p} s_p t_{n-p} \,.$$

In the case where $n+1$ is odd and thus n is even, we have the allowed permutations

$$1(X\cdots X)_n\,,\quad XX1(X\cdots X)_{n-2}\,,\quad \ldots(X\cdots X)_{2k}1(X\cdots X)_{n-2k}\,,\ldots,(X\cdots X)_n1$$

and their number is

$$s_0s_n+\binom{n}{2}s_2s_{n-2}+\ldots+\binom{n}{n}s_ns_0=\sum_{p=0}^{n}\binom{n}{p}s_ps_{n-p}\,.$$

In both expressions we used that s_k vanishes if k is odd, and t_k vanishes if k is even. Having shown that $b(n+1)$ satisfies the same recursive relations as $a(n+1)$ and knowing that $a(n)=b(n)$ for $n=1$ (and if fact for $n=2,3,4,5,6$ as shown by examples) we thus arrive at the conclusion $a(n)=b(n)$ for all n. We thus succeeded in the exercise posed by Martin Erickson.

4 Aftermath and ackowledgements

The arguments given are all rather basic and could in principle be adapted for a (master-)class mathematics in high-school. Leaving the formal notation aside and working by examples and without induction, pupils might appreciate the different way of thinking used in combinatorics.

The worked out example shows that some power series have coefficients that have a certain meaning. In fact, in number theory more expansions, even those with zero radius of convergence are used to find formulae. We hope that we have encouraged the reader to think of some other power series and relations with combinatorics.

We would like to thank Martin Erickson for his nice book on mathematics that ignited us to work out this example. Unfortunately, he passed away too early in 2013. We would have liked to have had the opportunity to enjoy the way how Martin Erickson presented mathematics; he was a gifted mathematician and a gifted teacher.

We would like to thank the colleagues of mathematics at the high school Wenzgasse at Vienna for a nice atmosphere, in which discussions about mathematical problems, easy and hard ones, which works motivating to write up the solved problems, even the easy ones.