# Calabi–Yau manifolds for dummies

Dennis*

December 1, 2008

**Abstract**

In the text below we try to introduce the concept of a Calabi–Yau manifold. We start by defining vector bundles and complex manifolds, then introduce Kähler manifolds, review the most rudimentary concepts of Chern classes and then define Calabi–Yau manifolds. We also comment on the importance of Calabi–Yau manifolds in physics. As an example to guide us on the way, we have chosen the two-sphere, which is Kähler but not Calabi–Yau, to test all concepts developed. The text is aimed at students of mathematics and physics with some background in differential and algebraic geometry, although we assume they might have forgotten most of what they once knew.

## Contents



*The greatest dummy of all.

# 1 Vector bundles

We do assume the reader has heard of what a manifold is made of, but in a few words and by means of the sphere we give the definition and set the notation for both the remaining general text and the maps and charts for the two-sphere. NB: sometimes we are a bit sloppy in the notation; if $\psi : A \to B$ is some map and $C \subset A$ we denote the restriction of $\psi$ to $C$ just by $\psi : C \to B$, but of course strictly speaking the $\psi' s$ are not the same and we should write $\psi|_C$. Sometimes the notation gets too clumsy and we appeal to the reader's common sense to note the inprecision and to understand what is meant and to appreciate our choice in trying to make the notation understandable and managable and clear.

The following phrase is a bit long, but it is what it is: A real smooth manifold of dimension $n$ is a paracompact Hausdorff topological space $M$, such that for each point $p \in M$, there is an open neighborhood $U$ of $p$ and a homeomorphism $f_U : U \to U' \subset \mathbb{R}^n$, where $U'$ is an open subset of $\mathbb{R}^n$ and such that for two such map $f_U : U \to U'$ and $f_V : V \to V'$ with $U \cap V \neq \emptyset$ the maps $f_U \circ f_V^{-1} : f_V^{-1}(V \cap U) \to f_U^{-1}(U \cap V)$ and $f_V \circ f_U^{-1} : f_U^{-1}(V \cap U) \to f_V^{-1}(U \cap V)$ are diffeomorphisms (in the smooth setting, whereas in the algebraic setting one requires biregular morphisms). So, in short: a manifold of dimension $n$ locally looks like $\mathbb{R}^n$ and the open sets that are homeomorphic to some open subset of $\mathbb{R}^n$ are patched together in a nice way, that is, using diffeomorphisms. The maps $f_U$ are called chart maps, the open sets $U$ are called charts and the maps $f_U \circ f_V^{-1}$ (or actually their restrictions) are called transition maps, or coordinate changes. The whole set of charts is given the name atlas.

Consider the sphere $S^2$, defined by

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\right\} .$$

There are two obvious open sets

$$U = \left\{(x, y, z) \in S^2 \,|\, z \neq 1\right\} , \tag{1}$$

$$V = \left\{(x, y, z) \in S^2 \,|\, z \neq -1\right\} , \tag{2}$$

and we have two maps

$$f_U : U \to \mathbb{R}^2 , \quad f_U : (x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) , \tag{3}$$

$$f_V : V \to \mathbb{R}^2 , \quad f_U : (x, y, z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right) . \tag{4}$$

The reader is urged to check that the image of both maps is indeed the whole $\mathbb{R}^2$. We call the image of $f_U$ the $U$-plane and the image of $f_V$ the $V$-plane. We use coordinates $(u_1, u_2)$ on the $U$-plane and we use coordinates $(v_1, v_2)$ on the $V$-plane.

The inverse morphisms are given by

$$f_U^{-1} : (u_1, u_2) \mapsto \Big( \frac{2u_1}{1+||u||^2}, \frac{2u_2}{1+||u||^2}, \frac{-1+||u||^2}{1+||u||^2} \Big) , \tag{5}$$

$$f_V^{-1} : (v_1, v_2) \mapsto \Big( \frac{2v_1}{1+||v||^2}, \frac{2v_2}{1+||v||^2}, \frac{1+||v||^2}{1+||v||^2} \Big) , \tag{6}$$

where $||u||^2 = u_1^2 + u_2^2$ and similarly for $||v||^2$. Indeed, on the domain where they these functions are defined, they are $C^\infty$-functions.

**Definition 1.1.** *Let $E$ and $M$ be manifolds and $F$ a vector space. We call a smooth map $\pi : E \to M$ a vector bundle with fibre $F$ if $\pi$ is surjective and all preimages $\pi^{-1}(x)$ are isomorphic to the vector space $F$ such that*

*(i) There exists a set $\mathcal{A}$ of open sets that cover $M$ such that there exist isomorphisms $\varphi_U : \pi^{-1}(U) \to U \times F$ for all $U \in \mathcal{A}$. The isomorphisms $\varphi_U$, where $U$ ranges over the elements of $\mathcal{A}$, are called local trivializations. The elements of $\mathcal{A}$ are called trivializing charts.*

*(ii) The isomorphisms $\varphi_U$ for $U \in \mathcal{A}$ satisfy* $\mathrm{proj}_1 \circ \varphi_U = \pi|_{\pi^{-1}(U)}$ *where* $\mathrm{proj}_1 : U \times F \to U$ *is the projection onto the first factor. In other words, the diagram*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_U} & U \times F \\ & {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle \mathrm{proj}_1} \\ & & U \end{array} \tag{7}$$

*commutes.*

*(iii) If $U, V \in \mathcal{A}$ and $U \cap V \neq \emptyset$ we have an induced map $\varphi_{UV} : F \to F$ given by*

$$\varphi_V \circ \varphi_U^{-1}(p, v) = (p, \varphi_{UV}(p)(v)) , \tag{8}$$

*where $p \in U$ and $v \in V$. The functions $\varphi_{UV}$ are sometimes called transitions functions of the bundle $E \to M$. We require that the transition functions $\varphi_{UV}$ define smooth functions $U \cap V \to GL_{\mathbb{R}}(F)$ given by $p \mapsto \varphi_{UV}(p)$. In particular, if $F$ is isomorphic to $\mathbb{R}^r$ we get smooth maps $U \cap V \to GL_{\mathbb{R}}(r)$.*

**Exercise 1.2.** *Why in eqn.(8) does the point $p$ not change to another point in $U \cap V$ under $\varphi_V \circ \varphi_U^{-1}$?*

Let $E \to M$ and $E' \to M$ be two vector bundles over $M$, a map $f : E \to E'$ is a morphism of vector bundles over $M$ if the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow \quad \swarrow & \\ & M & \end{array} \tag{9}$$

commutes. We call two vector bundles $E \to M$ and $E' \to M$ isomorphic if there are morphisms of vector bundles over $M$, $f : E \to E'$ and $g : E' \to E$ such that $f \circ g = \mathrm{id}_{E'}$ and $g \circ f = \mathrm{id}_E$.

An example of a vector bundle over $M$ is given by taking $E = F \times M$ and $\pi$ to be the projection to $M$. Any bundle of this kind or isomorphic to a bundle of this kind is called trivial. If the fiber $F$ is a vector space of rank $r$ we call $E \to M$ a vector bundle of rank $r$. Often one write $E_p$ for the fibre $\pi^{-1}(p)$ over $p$. Another example of a vector bundle is given by $C^\infty(M)$, the space of all smooth functions on $M$. We write $C^\infty(U)$ for the vector space of smooth functions on some open set $U \subset M$. By fibrewise multiplication we see that any vector bundle is a $C^\infty(M)$-module. The bundle $C^\infty(M)$ has one-dimensional fibers and is therefore called a line bundle.

Given a vector bundle $\pi : E \to M$ over $M$ with fiber $F$, then we call a section over an open set $U \subset M$ a smooth map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$. In words, a section assigns in a smooth way to each $p \in U$ an element of the fiber $\pi^{-1}(p) \cong F$. We write $\Gamma(E, U)$ for the $C^\infty(U)$-module of sections of $E$ over $U$; we write $C^\infty(E)$ for the set of all smooth sections of $E$ (hence forgetting the domain; $\sigma$ is in $C^\infty(E)$ if there is an open subset $U \subset M$ such that $\sigma \in \Gamma(E, U)$). A global section is a section over $M$, i.e., an element of $\Gamma(E, M)$. In general it is not guaranteed that global sections exist. If there exist enough nowhere vanishing global sections, then the bundle is trivial:

**Lemma 1.3.** *Let $\pi : E \to M$ be a vector bundle of rank $r$. Then $E \to M$ is trivial if and only if there exist $r$ linearly independent nowhere vanishing global sections.*

*Proof.* Let $E \to M$ be trivial, and suppose $f : E \to M \times F$ is the isomorphism. Choose any basis $\{e_i\}_i$ of $F$, then the $\sigma_i : p \mapsto f^{-1}(p, e_i)$ define $r$ global nowhere vanishing section that are linearly independent. Conversely, let $\{\sigma_i\}_i$ be a set of $r$ linearly independent nowhere vanishing global sections, then any section $\tau \in \Gamma(E, U)$ can be expended as $\tau = \sum_i \alpha_i \sigma_i$ and $\alpha_i \in C^\infty(U)$. In particular, $q \in \pi^{-1}(p)$ can be written as $q = \sum \beta_i \sigma_i(p)$ where $\beta_i \in \mathbb{R}$. This gives the isomorphism. □

**Exercise 1.4.** *Give a description of the bundle $C^\infty(M)$ in terms of the definition 1.1. What are the trivializing charts and what are the transition functions? Show that in differential geometry there are always many global sections of this bundle (hint: partition of unity or bump functions).*

We now come to an important result in the theory of vector bundles, which we will not prove. The prove is not difficult and is left as an exercise, for which the reader might consult the literature. The theorem is about the transition functions. It turns out that these completely determine a vector bundle up to isomorphism.

**Theorem 1.5.** *Let $\pi : E \to M$ be a vector bundle over $M$ with fibres $F$. Suppose that $\{U_\alpha\}$ are trivializing charts that cover $M$ and suppose $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ are local trivializations from which the transition functions $f_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_{\mathbb{R}}(F)$ are deduced. Then the transition functions satisfy*

$$f_{\alpha\alpha}(p) = \mathrm{id}\,, \qquad \forall p \in U_\alpha\,, \tag{10}$$

$$f_{\alpha\beta}(p) f_{\beta\alpha}(p) = \mathrm{id}\,, \qquad \forall p \in U_\alpha \cap U_\beta\,, \tag{11}$$

$$f_{\alpha\beta}(p) f_{\beta\gamma}(p) f_{\gamma\alpha} = \mathrm{id}\,, \qquad \forall p \in U_\alpha \cap U_\beta \cap U_\gamma\,, \tag{12}$$

*where concatenations of the transition functions at $p$ means the matrix product and* id *is the identity matrix. Conversely, given an open cover $\{U_\alpha\}$ then any set of transition functions eqns.(10,11,12) determines up to isomorphism one unique vector bundle over $M$ with fibres $F$ and trivializing charts $U_\alpha$ and transition functions $f_{\alpha\beta}$.*

Maybe the best way to remember the contents of theorem 1.5 is that a vector bundle is determined up to isomorphism by the open cover, the fibre and the transition functions.

## 1.1 Structure groups

Now that we have seen what a bundle is, we can immediately understand some interesting phenomena. If $E \to M$ is a vector bundle with fibre $F$, then the transition functions take values in $GL_{\mathbb{R}}(F)$. The smallest subgroup $G \subset GL_{\mathbb{R}}(F)$ for which the transition functions all take values in $H$ is called the structure group of $E$. For fixed fibre $F$ and $M$ we can try and structure group $H \subset GL_{\mathbb{R}}(F)$ we can classify the vector bundles over $M$ with structure group $H$. For a fixed vector bundle we can again try to fit all transition functions into a small as possible subgroup of $GL_{\mathbb{R}}(F)$. If this is possible and one does indeed so, then one says one has reduced the structure group. We will see an explicit reduction of the structure group in section 3.

The remainder of this section can be skipped at a first reading.

For the tangent bundle (if you don't know what this is, read this section later) on a Riemannian manifold (same remark applies) of dimension $n$, the structure group can be chosen inside $O(n)$, and if the manifold is orientable, even in $SO(n)$. One thus says that an orientable Riemannian $n$-dimensional manifold admits a reduction of the structure group to $SO(n)$. A $G_2$-manifold is a seven dimensional manifold such that the structure group of the tangent bundle can be chosen to lie in the $G_2$-subgroup of $SO(7)$. These manifolds are of importance in string theory and supergravity theory for compactifications of M-theory, which lives in 11 spacetime dimensions.

Let $E \to M$ be a Hermitian vector bundle over a complex manifold $M$ of rank $n$, which means that the vector bundle has fibres that are complex vector spaces and there is a global nondegenerate section of the bundle $\bar{E}^* \otimes E^* \to M$, which for each fibre is a Hermitian form on the fibre. Here $\bar{E}^*$ means the complex conjugate bundle of $E^*$, which can be obtained by taking as transition functions the complex conjugates of those of $E^*$. In short, we have a shortly varying Hermitian metric on the fibres. Then the structure group can be chosen to lie in $U(n)$.

**Exercise 1.6.** *Try to prove the claims made in the last two paragraphs of this subsection.*

## 1.2 Manufacturing bundles

Given a vector bundle $E \to M$ and $E' \to M$, is there a way to combine these two? Since any bundle is up to isomorphism determined by an open cover and a set of transition functions, we can try to get new bundle from old ones by combining and manipulating transition functions.

**Direct sum.** Let $E \to M$ and $E' \to M$ be two vector bundles over $M$ with fibres $F$ and $F'$ respectively. By refining the open covers of $E$ and $E'$, we may assume that we have a cover of $M$ such that all open sets of the cover are trivializing for both $E$ and $E'$. Denote the open sets of the cover by $U_\alpha$, where $\alpha$ ranges over some index set. Furthermore, denote the local trivializations of $E$ and $E'$ by$\varphi_\alpha$ and $\varphi'_\alpha$ respectively and denote the transition functions of $E$, which are deduced from $\varphi_\beta \circ \varphi_\alpha^{-1}$ by $f_{\alpha\beta}$ and those of $E'$ by $f'_{\alpha\beta}$. Then consider the vector bundle $E \oplus E' \to M$ obtained by taking as fibres $F \oplus F'$ and as transition functions $f_{\alpha\beta} \oplus f'_{\alpha\beta}$, which sends $(v, v') \in F \oplus F'$ to $(f_{\alpha\beta} v, f'_{\alpha\beta} v')$. The vector bundle obtained in this way is the direct sum of $E$ and $E'$. Another name for the direct sum bundle is the Whitney sum.

**Tensor Product.** With the same notation as in the preceding paragraph, consider the vector bundle obtained by taking as fibres $F \otimes F'$ and as transition functions $f_{\alpha\beta} \otimes f'_{\alpha\beta}$, which sends $v \otimes v' \in F \otimes F'$ to $f_{alpha\beta} v \otimes f'_{\alpha\beta} v'$. The tensor product is here meant as tensor product of $C^\infty(M)$-modules; if $\sigma$ and $\sigma$ are sections of $E$ and $E'$ respectively over some open subset $V \subset M$ and $h \in C^\infty(V)$ then we identify $h\sigma \otimes \sigma'$ with $\sigma \otimes h\sigma'$. The tensor product obtained in this way is the tensor product of $E$ and $E'$.

**Dual bundle.** With the same notation as above, although we only consider $E$. We want the dual bundle to consist of the smooth sections that eat up sections of $E$ producing a smooth section of $C^M$. We are forced to take as fibres the dual of $F$, for which we write $F^*$. Since $C^M$ is a trivial bundle over $M$[1], we need that the

[1]Oops! Is this an answer to an exercise we gave?

transition functions of the dual bundle are precisely the inverses of those of $E$. Thus, we can construct the dual bundle $E^* \to M$ by gluing the fibres $F^*$ together with the transition functions $(f_{\alpha\beta})^{-1}$, where the inverse is pointwise matrix inversion (since as a function $f_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_{\mathbb{R}}(F)$ is in general not invertible and if so, it does not give what we want). With this construction, we have obtained the (isomorphism class) of the dual bundle. Later, when considering tangent and cotangent bundles, we get back to the dual bundle and its construction, but then in a more specified context.

**Determinant bundle.** Since we know how to make tensor products of bundles, we also know how to make antisymmetric tensor products: given a vector bundle $E \to M$ of rank $r$ we can consider for any $l$ with $1 \le l \le r$ the vector bundle $\bigwedge^l E \to M$ where the transition functions are the same as for $E^{\otimes l}$, but where the fibres are the subspaces of $E^{\otimes l}$ that consist of the antisymmetric tensors. The rank of $\bigwedge^l E$ is $\binom{r}{l}$ and thus for $l = r$ we find a line bundle. It is easy to find the transition functions of this line bundle; they are the determinants of the matrices that represent the transition functions of the vector bundle. Therefore the bundle $\bigwedge^r E \to M$ is called the determinant bundle. For the determinant bundle of the (holomorphic) cotangent bundle one reserves the name canonical bundle. As the canonical bundle is a line bundle, for algebraic varieties there exists a corresponding Cartier divisor. Such a divisor is called a canonical divisor and the equivalence class (thus inside the Picard group) is called the canonical class. For Calabi–Yau manifolds the canonical class is an important part of the definition.

## 1.3 The pull-back bundle

Suppose we have a vector bundle $\pi : E \to M$ with fibre $F$ and a map $g : N \to M$ of smooth manifolds. We can then construct a vector bundle over $N$ with fibres $F$ as follows: Within the category of smooth manifolds we consider the pull-back of the following diagram:

$$\begin{array}{ccc} & & E \\ & & \big\downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\;g\;} & M \end{array} \tag{13}$$

We assume the reader is familiar with the universal property of pull-back diagrams, but for completeness we give a short description: We look for an object $P$, which is thus a smooth manifold, together with two maps $a : P \to E$ and $b : P \to N$ such that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\;a\;} & E \\ {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\;g\;} & M \end{array} \tag{14}$$

such that if $P'$ is another smooth manifold with maps $a' : P' \to E$ and $b' : P' \to N$ making the diagram commutative, then there exists a unique map $c : P' \to P$ such that $a' = c \circ a$ and $b' = c \circ b$. We can construct the pull-back as follows: The direct product $E \times N$ is a smooth manifold with two natural maps $\mathrm{pr}_E : E \times N$ and $\mathrm{pr}_N : E \times N$ making the diagram (14) commutative. To get the actual pull-back we consider the submanifold of $E \times N$ consisting of the points $(e, n) \in E \times N$ such that

$\pi(e) = g(n)$. In general one writes the pull-back as $E \times_M N$;

$$E \times_M N = \{(e, n) \in E \times N | \pi(e) = g(n)\} , \tag{15}$$

and the two maps $a : E \times_M N \to E$ and $b : E \times_M N \to N$ are the restrictions of $\mathrm{pr}_E$ and $\mathrm{pr}_N$ respectively to $E \times_M N$. We want to make $b : E \times_M N \to N$ into a vector bundle over $N$, which we will call the pull-back bundle induced by $g : N \to M$. One writes $g^*E$ for the pull-back bundle $E \times_M N \to N$.

To verify that the pull-back bundle is indeed a vector bundle, we leave as an exercise for the reader:

**Exercise 1.7.** *Let $\pi : E \to M$ be a vector bundle with local trivializations $f_\alpha : \pi^{-1}(U_\alpha) \to F \times U_\alpha$, where $M = \cup_\alpha U_\alpha$ is an open cover of $M$. Let $g : N \to M$ be a smooth map of manifolds. Denote $\mathrm{pr}_1 : N \times_M E \to N$ and $\mathrm{pr}_2 : N \times_M E \to N$ the canonical projections. Let $q_\alpha : F \times U_\alpha \to F$ be the projection to $F$. In this exercise you are asked to show that the pull-back bundle is indeed a vector bundle, with fibre $F$.*

*(i) Show that $\mathrm{pr}_1^{-1}(g^{-1}(U_\alpha)) \cong g^{-1}(U_\alpha) \times_{U_\alpha} \pi^{-1}(U_\alpha)$.*

*(ii) Introduce the functions $\phi_\alpha : g^{-1}(U_\alpha) \times_{U_\alpha} \pi^{-1}(U_\alpha) \to g^{-1}(U_\alpha) \times F$ defined by $\phi_\alpha : (n, e) \mapsto (n, q_\alpha(f_\alpha(e)))$. Show that the functions $\phi_\alpha$ provide a local trivialization and that above a point $n \in N$ the transition functions on the fibre are the same as at $g(n)$.*

## 1.4 Tangent and cotangent bundles

When dealing with differential forms, one needs to symmetrize and antisymmetrize on occasion. We denote $S_n$ the group of permutations of $n$ objects. An elementary permutation is an element of $S_n$ that interchanges two elements and leaves the rest unchanged. There exists a well-defined group homomorphism $\mathrm{sgn} : S_n \to \mathbb{Z}_2$, sending $\sigma \in S_n$ to the sign of the permutations, which is $(-1)^{n(\sigma)}$, where $n(\sigma)$ is the number of elementary permutations needed to write $\sigma$ as a product of elementary permutations. For example for $n = 3$, the permutation $\sigma$ that sends $(1, 2, 3)$ to $(3, 2, 1)$ has sign $\mathrm{sgn}(\sigma) = -1$.

It turns out to be convenient to know a little on $GL_n(\mathbb{R})$ representations as they appear a lot in both physics and in mathematics. We will keep it on a low level and very basic. However, we assume the reader know what the group $GL_{\mathbb{R}}(n)$ is and what a representation is. The fundamental representation of $GL_{\mathbb{R}}(n)$ is the $n$-dimensional representation defining $GL_{\mathbb{R}}(n)$ and thus every element of $GL_{\mathbb{R}}(n)$ is represented by an invertible $n \times n$-matrix. We denote this representation by $\mathbf{n}$ and it is a trivial but conceptually good exercise to verify that the representation $\mathbf{n}$ is irreducible. Sometimes one calls $\mathbf{n}$ the vector representation. The Lie algebra of $GL_{\mathbb{R}}(n)$ is denoted $\mathfrak{gl}_{\mathbb{R}}(n)$ and is the Lie algebra of $n \times n$-matrices. For the $n$-dimensional representation of $\mathfrak{gl}_{\mathbb{R}}(n)$ we also use the notation $\mathbf{n}$. Since $\dim(\mathfrak{gl}_{\mathbb{R}}(n)) = n^2$, there exists no faithfull representation of $GL_{\mathbb{R}}(n)$ of smaller dimension. Even more, given any representation of $GL_{\mathbb{R}}(n)$ we can look at the kernel, which is a normal subgroup. But $GL_{\mathbb{R}}(n)$ has no normal subgroups except for the abelian subgroup formed by multiples of the identity matrix. Since $(n-1)^2 < n^2 - 1$ there are no irreducible representations of $GL_{\mathbb{R}}(n)$ of smaller dimension than $n$ except the trivial representation, which we denote $\mathbf{1}$. In physics a trivial representation is called a singlet. It turns out that all representations of $GL_{\mathbb{R}}(n)$ and of $\mathfrak{gl}_{\mathbb{R}}(n)$ can be obtained from taking tensor products of $\mathbf{n}$ and then decomposing into irreducible parts. In more mathematical terms: every irreducible representation of $GL_{\mathbb{R}}(n)$ is a submodule of some tensor product $\mathbf{n} \otimes \cdots \otimes \mathbf{n}$. To find the irreducible components of an $r$-fold tensor product $\mathbf{n}^{\otimes r}$, one uses symmetrization and antisymmetrization and the irreducible components can be classified by means of Young tableau's. We will not go into details here; the interested reader is refered to the vast literature on this subject. For us the most interesting irreducible components of an $r$-fold tensor product $\mathbf{n}^{\otimes r}$ are the totally symmetric and the totally antisymmetric part. We can define two projection operators that act in the representation $\mathbf{n}^{\otimes r}$ as follows

$$P_A(v_1 \otimes \cdots \otimes v_r) = \frac{1}{r!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} , \tag{16}$$

$$P_S(v_1 \otimes \cdots \otimes v_r) = \frac{1}{r!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} . \tag{17}$$

The projection $P_A$ makes from a tensor a totally antisymmetric tensor and $P_S$ makes from a tensor a totally symmetric tensor:

$$\sigma \circ P_A = \mathrm{sgn}(\sigma) P_A , \quad \sigma \circ P_S = P_S .$$

We denote the image of $P_A$ in $\mathbf{n}^{\otimes r}$ by $\bigwedge^r \mathbf{n}$.

An $r$-tensor is in the setting of representations a finite linear sum of terms of the form $v_1 \otimes \cdots \otimes v_r$. When we choose a basis $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$ we can write any $r$-tensor $T$ as

$$T = \sum T_{i_1 \cdots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} ,$$

where the sum is over all nonzero components. In physics one write $T_{i_1 \cdots i_r}$ for a tensor; the stress is on the components. Antisymmetrizing and symmetrizing is thus done on the level of the components: one writes $T_{[a_1 \cdots a_r]}$ for the antisymmetrized tensor $T$ and $T_{(a_1 \cdots a_r)}$ for the symmetrized tensor. To get some feeling for how antisymmetrization works, we urge the reader checks the following identities:

$$2T_{[ab]} = T_{ab} - T_{ba} , \tag{18}$$

$$2T_{(ab)} = T_{ab} + T_{ba} , \tag{19}$$

$$6T_{[abc]} = T_{abc} + T_{bca} + T_{cab} - T_{bac} - T_{acb} - T_{cba} . \tag{20}$$

An interesting phenomena when different tensors are multiplied and then antisymmetrized: if $T$ is an antisymmetric two-tensor and $S$ is just a vector one easily verifies:

$$3S_{[a}T_{bc]} = S_a T_{bc} + S_b T_{ca} + S_c T_{ab} .$$

This ends the discussion on $GL_{\mathbb{R}}(n)$ representations.

**Tangent vectors.** There are several ways to introduce tangent vectors, one of which uses the notion of differentiating along a curve. The more intrinsic and suited approach is via derivations of $C^\infty(M)$. But first a few words on the first approach. Consider a smooth curve $\gamma : [-1, 1] \to M$ with $\gamma(0) = p \in M$. We call the derivative $\gamma'(0)$ a tangent vector at $p$ tangent to $\gamma$. On the set of tangent vectors one can define an equivalence relation, for example by saying that two tangent vectors at $p$ along $\gamma$ and $\widetilde{\gamma}$ are equivalent of they induce the same tangent vector in a chart inside $\mathbb{R}^n$. After this one can use existence of differential equations to introduce a vector space structure on the equivalence classes to obtain a vector space of tangent vectors. The drawback of this approach is that one has to show the independence of the chosen curve all the time.

The other approach is as follows. For any point $p \in M$ we call $C^\infty(M)_p$ the set of germs of $C^\infty(M)$, that is, $C^\infty(M)_p$ is the set of equivalence classes of sections of $C^\infty(M)$ that are defined in an open set containing $p$ and two such functions are called equivalent if they agree on some open subset containing $p$. A tangent vector at $p$ is a linear map $X_p : C^\infty(M)_p \to \mathbb{R}$ satisfying the Leibniz rule. We denote $T_pM$ the set of all tangent vectors at $p$. It is easily checked that $T_pM$ is in a natural way a vector space over $\mathbb{R}$. Similarly for each open set $U \subset M$ we can look at the set $\mathrm{Der}(U)$ of all $\mathbb{R}$-linear maps $X : \Gamma(C^\infty(M), U) \to \Gamma(C^\infty(M), U)$ that satisfy the Leibniz rule. It is easy to see that $\text{Der}(U)$ is a $\Gamma(C^\infty(M), U)$-module. Note that we can add sections of $\text{Der}(U)$ for all $U$. Taking the inductive limit $\lim_{U\ni p}\text{Der}(U)$ we obtain $T_pM$. Hence each fibre is a vector space over $\mathbb{R}$. One can prove that if $U$ is contained in some chart each section of $\text{Der}(U)$ has a chart expression

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \,,$$

where the $X^i$ are smooth function on the image of the chart of $U$. Hence for $p \in U \subset M$ we find that a tangent vector $X_p$ at $p$ we can locally express (by which we mean by means of charts getting an expression for some open subset of $\mathbb{R}^n$) as

$$X_p = \sum_{=1}^{n} X(p)^i \frac{\partial}{\partial x^i}\Big|_p \,.$$

Where now $X(p)^i$ are just components of a vector in $\mathbb{R}^n$, that is, they are numbers. In our notation we have already anticipated the following: if $X$ is a smooth vector field over $U \subset M$ and $p \in U$ then we can evaluate $X$ in $p$ by looking at local coordinates $x^\mu$ around $p$ such that $p$ corresponds to the point where all the $x^\mu$ are zero. Then we let $X$ act on all the coordinate functions: $X(p)^\mu = X(x^\mu)$. We leave the verification of this procedure to the reader.

Now we are in a position to discuss the vector bundle structure of the tangent bundle of the manifold $M$. We denote the tangent bundle by $TM$ and we want a bundle $\pi : TM \to M$ such that all its sections are precisely the smooth vector fields on $M$. It is clear how to do this: cover $M$ with charts $M = \cup_i U_i$, such that each $U_i$ is homeomorphic to some open set in $\mathbb{R}^n$. For each $i$ we define $\pi^{-1}(U_i) = U_i \times \mathbb{R}^n$. For each point $p \in U_i$ the fibre is now $\mathbb{R}^n$, and each tangent vector at $p$ gives a vector in the fibre and conversely (although not obvious) given any point in the fibre - that is thus a vector in $\mathbb{R}^n$ - we can find a small neighborhood $V_p$ of $p$ and a vector field on $V_p$ such that the value of this vector field in $p$ precisely equals the value of the chosen vector in the fibre above $p$. We now need to give the transition functions. Suppose $p \in U \cap V$ where $U$ and $V$ are two charts. Denote $U'$ and $V'$ the images of $U$ and $V$ respectively in $\mathbb{R}^n$. We use local coordinates $x^1, \ldots, x^n$ in $U'$ and local coordinates $y^1, \ldots, y^n$ in $V'$. There is a diffeomorphism $\varphi : U' \to V'$ going from the $U$-chart to the $V$-chart; we can write $\varphi : (x^1, \ldots, x^n) \mapsto (\varphi^1(x), \ldots, \varphi^n(x))$. Any vector field $X$ around $p$ can be taken to be a vector field in some open subset of $U \cap V$ by restricting the domain of $X$. Let us express $X$ in the local coordinates of $U'$ as the vector field

$$X_U = \sum_i X_U^i(x) \frac{\partial}{\partial x^i}$$

and as

$$X_V = \sum_a X_V^a(y) \frac{\partial}{\partial y^a} \,.$$

When we take a function $f$ defined around $p$, we can use its coordinate expresions $f_U(x)$ and $f_V(y)$ and $f_V(y) = f_U(\varphi(x))$. To obtain the transition functions we let $X$ act on $f$ and demand that we get the same result. Using the chain rule we get

$$X_V^a(y) = \sum_i \frac{\partial \varphi^a(x)}{\partial x^i}\Big|_{x=\varphi^{-1}(y)} X_U^i(\varphi^{-1}(x)) \,. \tag{21}$$

**Exercise 1.8.** *Verify equation (21).*

In practice one writes the diffeomorphism $\varphi$ in the form of $y^a = y^a(x)$ and simply writes

$$X_V = \frac{\partial y}{\partial x} X_U \,, \tag{22}$$

where $\frac{\partial y}{\partial x}$ is the matrix with entries

$$\frac{\partial y^a}{\partial x^i}.$$

The matrices $\frac{\partial y}{\partial x}$ are the transition functions. For each point $p \in U \cap V$ they are elements of $GL_{\mathbb{R}}(n)$.

**Lie bracket.** An important construction in differential geometry (but can also be constructed for algebraic geometry), is the Lie bracket. The Lie bracket combines two vector fields $X, Y$ into a new vector field, written $[X, Y]$; in other words, the Lie bracket is a map $[,] : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$. We first note that the consecutive action of two vector fields does not define a new vector field: for any $X, Y \in C^\infty(TM)$ and $f, g \in C^\infty(M)$ we have

$$X(Y(fg)) = X(Y(f)g + fY(g)) = X(Y(f))g + Y(f)X(g) + X(f)Y(g) + fX(Y(g)),$$

which does not equal $X(Y(f))g + fX(Y(g))$ in general. However, the antisymmetric part $X \circ Y - Y \circ X$ does define a vector field:

**Exercise 1.9.** *Verify that* $(X \circ Y - Y \circ X)(fg) = (X \circ Y - Y \circ X)(f)g + f(X \circ Y - Y \circ X)(g)$.

We denote the vector field defined by $X \circ Y - Y \circ X$ with $[X, Y]$, that is, $[X, Y](f) = X(Y(f)) - Y(X(f))$ and call it the Lie bracket of $X$ and $Y$.

**Exercise 1.10.** *For two vector fields $X$ and $Y$ find the local expression: take a chart $U$ parametrized by coordinates $x^\mu$ and write $X = \sum_\mu X^\mu \partial_\mu$ and $Y = \sum_\mu Y^\mu \partial_\mu$, where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Then prove that*

$$[X, Y] = \sum_{\mu,\nu} \Big( X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu \Big) \partial_\nu .$$

Now we consider the cotangent bundle $T^*M$. For each open set $U$ in $M$ we define the set $\Gamma(T^*M, U)$ to be the set of all $C^\infty(M)$-linear maps $\mathrm{Der}(U) \to C^\infty(M)$. We call the elements of $\Gamma(T^*M, U)$ differentials or a differential form. It is clear that $\Gamma(T^*M, U)$ is an $C^\infty(M)$-module. When $U$ is a chart and $U'$ is the image of $U$ in $\mathbb{R}^n$ with coordinates $x^\mu$, then any vector field can be expanded in the basis elements $\partial/\partial x^\mu$. We define a dual basis to the set of the $\partial/\partial x^\mu$ by introducing the objects $\mathrm{d}x^\nu$ satisfying

$$\mathrm{d}x^\mu \Big( \frac{\partial}{\partial x^\mu} \Big) = \delta_\mu^\nu , \tag{23}$$

where $\delta_\mu^\nu$ is the Kronecker delta, which has the numerical value 1 if $\mu = \nu$ and zero otherwise. A differential form $\omega \in \Gamma(T^*M, U)$ can locally be expended as

$$\omega = \sum_\mu \omega_\mu \mathrm{d}x^\mu .$$

The coefficients $\omega_\mu$ are smooth functions on $U$. The action of $\omega$ on a vector field $X$ with a common domain is given by

$$\omega(X) = \sum_{\mu\nu} \omega_\mu \mathrm{d}x^\mu \Big( X^\nu \frac{\partial}{\partial x^\nu} \Big) = \sum_\mu \omega_\mu X^\mu .$$

Dually, we define an action of $\mathrm{Der}(U)$ on $\Gamma(T^*M, U)$ by $X \cdot \omega = \omega(X)$ for a vector field $X$ and a differential form $\omega$. We want to build a bundle $T^*M$ with sections over $U$ precisely $\Gamma(T^*M, U)$. By all the preceding it is clear what to do: the trivializing charts for $T^*M$ are precisely the coordinate charts and the transition functions are given by the inverses - as matrices, or as $GL_{\mathbb{R}}(n)$-elements - of the tangent bundle $TM$. More precise, using the notation preceding equation (21), the transition functions

$\mathbb{R}^n \to \mathbb{R}^n$ from the $U$-chart local trivialization to the $V$-chart local trivialization are given by

$$\omega_{V,\nu} = \sum_\mu \Big( \frac{\partial \varphi^\nu(x)}{\partial x^\mu}\Big|_{x=\varphi^{-1}(y)}\Big)^{-1} \omega_{U,\mu}(\varphi^{-1}(x))$$

where $\omega_U$ and $\omega_V$ are the local expressions of a differential form $\omega$ defined on an open subset of $U \cap V$. Again we will write this more shortly as

$$\omega_V = \Big(\frac{\partial y}{\partial x}\Big)^{-1} \omega_U\,.$$

Although the expression for the transition functions looks more difficult than for the tangent bundle, but in fact it is more easy:

**Exercise 1.11.** *With the notation as above and $\omega$ a differential form, show that the image under the transition function from the $U$-chart local trivialization to the $V$-chart local trivialization can be written as*

$$\sum_\mu \omega_{V,\mu}(y)\mathrm{d}y^\mu = \sum_\mu \omega_{U,\mu}(y(x))\mathrm{d}y^\mu(x)\,,$$

*where it is inderstood that*

$$\mathrm{d}y^\mu(x) = \sum_\nu \frac{\partial y^\mu}{\partial x^\nu}(x)\mathrm{d}x^\nu\,.$$

Having discussed the tangent and the cotangent bundle, we are ready for tensors in differential geometry. A $(r,s)$-tensor is a section of the bundle $TM^{\otimes r} \otimes T^*M^{\otimes s}$. Alternatively, one can describe an $(r,s)$-tensor as a $C^\infty(M)$-linear map from $T^*M^{\otimes r} \otimes TM^{\otimes s}$ to $C^\infty(M)$.

**Exercise 1.12.** *Describe the transition function for the bundle $TM^{\otimes r} \otimes T^*M^{\otimes s}$. What does this have to do with representation theory of $GL_{\mathbb{R}}(n)$.*

A prominent role is played by the differential $p$-forms. A differential $p$-form, or shortly $p$-form is a section of the bundle $\bigwedge^p T^*M$, which is by definition the antisymmetrized $p$-fold tensor product. We know that under $GL_{\mathbb{R}}(n)$ these form an irreducible module. We have a bundle map $P_A : T^*M^{\otimes p} \to \bigwedge^p T^*M$ given by the fiberwise action of the projection operator $P_A$ defined in eqn.(16).

When we use local charts on some open set $U$, we can expand a differential form $\omega \in \bigwedge^p T^*M$ as follows

$$\omega = \sum_{\mu_1,\dots,\mu_p} \omega_{\mu_1\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p}\,, \tag{24}$$

where by definition

$$\mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p} = \sum_{\sigma \in S_p} (-1)^{\mathrm{sgn}(\sigma)} \mathrm{d}x^{\mu_{\sigma(1)}} \otimes \dots \otimes \mathrm{d}x^{\mu_{\sigma(2)}}\,. \tag{25}$$

We can now define a linear operator $\mathrm{d}_p : \bigwedge^p T^*M \to \bigwedge^{p+1} T^*M$ for each $p$ by defining

$$\mathrm{d}_p \sum_{\mu_1,\dots,\mu_p} \omega_{\mu_1\cdots\mu_p} \mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_p} = \sum_{\mu_1,\dots,\mu_p,\nu} \partial_\nu\omega_{\mu_1\cdots\mu_p}\mathrm{d}x^\nu\wedge\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_p}\,. \tag{26}$$

We will in general omit the suffix $p$ on $\mathrm{d}_p$ and just write $d$, since this cause little confusion and is useful in calculations. We can thus view d as an operator $d : \bigwedge^\bullet T^*M \to \bigwedge^\bullet T^*M$.

**Lemma 1.13.** *The operator $\mathrm{d} : \bigwedge^\bullet T^*M \to \bigwedge^\bullet T^*M$ has the following properties:*

*(i)* $\mathrm{d}\circ\mathrm{d} = 0\,.$

*(ii) Given $X \in C^\infty(TM)$ and $f \in C^\infty(M)$ then we have $\mathrm{d}f(X) = X(f)$.*

Note that the lemma justifies the notation $\mathrm{d}x^\mu$ since we can really identify $\mathrm{d}x^\mu$ with the result of d acting on the smooth section of $C^\infty(M)$ $x^\mu$.

*Proof.* Applying the definition on any $p$-form $\omega$, we get

$$\mathrm{dd}\omega = \sum_{\nu_i,\mu_j} \partial_{\nu_1}\partial_{\nu_2}\omega_{\mu_1\cdots\mu_p}\mathrm{d}x^{\nu_1} \wedge \mathrm{d}x^{\nu_2} \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}\,, \tag{27}$$

but $\mathrm{d}x^{\nu_1} \wedge \mathrm{d}x^{\nu_2} = -\mathrm{d}x^{\nu_2} \wedge \mathrm{d}x^{\nu_1}$ and $\partial_{\nu_1}\partial_{\nu_2} = \partial_{\nu_2}\partial_{\nu_1}$ and thus $\mathrm{dd}\omega = -\mathrm{dd}\omega = 0$.

For the second part, we just write out in some local coordinates $X(f) = X^\mu\partial_\mu(f)$ and $\mathrm{d}f = \partial_\nu f \mathrm{d}x^\nu$ so that $\mathrm{d}f(X) = (\partial_\nu f)X^\mu \mathrm{d}x^\nu(\partial_\mu) = X^\mu\partial_\mu f = X(f)$. □

**Lemma 1.14.** *Let $\chi \in C^\infty(T^*M)$ and $\omega \in C^\infty(\bigwedge^2 T^*M)$ be given, then for all vector fields $X, Y$ and $Z$ we have*

$$(i)\ (\mathrm{d}\chi)(X,Y) = X(\chi(Y)) - Y(\chi(X)) - \chi([X,Y])\,, \tag{28}$$

$$(ii)\ (\mathrm{d}\omega)(X,Y,Z) = X(\omega(Y,Z)) + Y(\omega(Z,X)) + Z(\omega(X,Y)) - \omega([X,Y],Z) - \omega([Z,X],Y) - \omega([Y,Z],X) \tag{29}$$

*Proof.* The proof is a classical case of writing and working. Hence we just sketch the lines, although we prove $(i)$ almost in full detail, to give the flavor.

We write out $\mathrm{d}\chi(X,Y)$ in local coordinates:

$$\begin{aligned}\mathrm{d}\chi(X,Y) &= \sum_{\mu,\nu}(\partial_\mu\chi_\nu - \partial_\nu\chi_\mu)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu(X,Y) \\ &= \sum_{\mu\nu} X^\mu Y^\nu(\partial_\mu\chi_\nu - \partial_\nu\chi_\mu)\,.\end{aligned} \tag{30}$$

On the other hand, we also have

$$X(\chi(Y)) = \sum_{\mu,\nu} X^\mu\partial_\mu(\chi_\nu Y^\nu) = \sum_{\mu,\nu} X^\mu Y^\nu\partial_\mu\omega_\nu + X^\mu\omega_\nu\partial_\mu Y^\nu$$

and similarly for $Y(\chi(X))$. For $\chi([X,Y])$ we find

$$\chi([X,Y]) = \sum_{\mu,\nu} \chi_\nu X^\mu\partial_\mu Y^\nu - \chi_\nu Y^\mu\partial_\mu Y^\nu\,.$$

Adding up to $X(\chi(Y)) - Y(\chi(X)) - \chi([X,Y])$ we find that it equals the expression (30).

For the second identity, we note that

$$3\partial_{[\mu}\omega_{\nu\lambda]} = \partial_\mu\omega_{\nu\lambda} + \partial_\nu\omega_{\lambda\mu} + \partial_\lambda\omega_{\mu\nu}\,,$$

as $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Then $\mathrm{d}\omega(X,Y,Z)$ is nothing more than

$$\sum_{\mu,\nu,\lambda}\Big(\partial_\mu\omega_{\nu\lambda} + \partial_\nu\omega_{\lambda\mu} + \partial_\lambda\omega_{\mu\nu}\Big)X^\mu Y^\nu Z^\lambda\,. \tag{31}$$

The tedious part lies in writing out

$$X(\omega(Y,Z)) + Y(\omega(Z,X)) + Z(\omega(X,Y)) - \omega([X,Y],Z) - \omega([Z,X],Y) - \omega([Y,Z],X)$$

and noting that many terms cancel and that the result equals the expression 31. □

## 1.5 Induced maps for (co-) tangent bundle

If $M$ and $N$ are smooth manifolds and $f : M \to N$ is a smooth map, by which we mean that in each chart $U$ of $N$ the expressions for $f$ in terms of coordinates for any chart in $f^{-1}(U) \subset M$ are smooth. We get induced maps between the tangent bundles and cotangent bundles, which we will shortly describe.

Clearly we get a pull-back map for functions: $f^* : C^\infty(N) \to C^\infty(M)$ given by $g \mapsto g \circ f$. The reader is urged to check that if $g$ is smooth than also $g \circ f$ is smooth.

If $X$ is a tangent vector at $p$, then $X$ corresponds to the tangent derivative of some path $\gamma : [-1, 1] \to M$ with $\gamma(0) = p$: $X = \frac{d}{dt}\gamma(t)\|_0$. By composing $\gamma$ with $f$ we get a path $\tilde{\gamma} : [-1, 1] \to N$ with $\tilde{\gamma}(0) = f(p)$. We define the tangent map of $f$ at $p$ to be the map $T_p f : T_p M \to T_{f(p)}N$ that assigns to $X$ the tangent vector at $f(p)$ given by

$$T_p f(X) = \frac{d}{dt} f \circ \gamma(t)\Big|_0 .$$

There is another desciption of the map $Tf : TM \to TN$: if $X$ is a tangent vector field, then $X$ corresponds to a derivation on $C^\infty(M)$. We define the map $Tf : TM \to TN$ to be the map that assigns to a vector field $X$ on $M$ the vector field $Tf(X)$ on $N$ that sends a smooth function $g \in C^\infty(N)$ to

$$Tf(X)(g) = X(f^*(g)) = X(g \circ f) .$$

**Exercise 1.15.** *Check that the given descriptions of the tangent map $Tf$ coincide.*

For differential forms we get a map $f^* : T^*N \to T^*M$ (since functions are 0-forms, one uses the same notation for the induced map), which is described as follows:

$$(f^*\omega)(X) = \omega(Tf(X)) , \quad \omega \in C^\infty(T^*N) , X \in C^\infty(TM) . \tag{32}$$

We extend $f^*$ to $f^* : T^*N^{\otimes r} \to T^*M^{\otimes r}$ by

$$(f^*\omega)(X_1, \dots, X_r) = \omega(Tf(X_1), \dots, Tf(X_r)) , \quad \omega \in C^\infty(T^*N) , X_i \in C^\infty(TM) .$$

Hence we obtain a map $f^* : \bigwedge^r T^*N \to \bigwedge^r T^*M$ by restricting to the antisymmetric elements of $T^*N^{\otimes r}$. The following lemma contains useful information, but since we won't need it that much, the proof is left as an exercise.

**Lemma 1.16.** *Let $f : M \to N$ be a smooth map and denote $f^*$ the induced map on differential forms*

*(i) Proof that $f^*$ is an algebra map: $f^*(\alpha \wedge \beta)$ for any differential forms on $N$.*

*(ii) Proof that $f^*$ commutes with $\mathrm{d}$: $f^*(\mathrm{d}\omega) = \mathrm{d}f^*(\omega)$.*

*Proof.* Is an exercise. Hint: For the first part, proof that if $\alpha$ is a $p$-form and $\beta$ a $q$-form, then

$$(\alpha\wedge\beta)(X_1, \dots, X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sgn}(\sigma)\alpha(X_{\sigma(1)}, \dots, X_{\sigma(p)})\beta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}) .$$

For the second part, use local coordinates. The reader might also consult [2] or [3]. □

## 1.6 Cohomology: the minimum one should to know

In the above section we constructed for an $n$-dimensional manifold a series of bundles $\bigwedge^p T^*M$ for $p = 0, 1, 2, \dots$. The series terminates after $p = n$, an aspect that will be irrelevant for the following discussion. The bundles $\bigwedge^p T^*M$ are connected via the differential $\mathrm{d}_p : \bigwedge^p T^*M \to \bigwedge^{p+1} T^*M$ and the differential has the property that $\mathrm{d}_{p+1} \circ \mathrm{d}_p = 0$ for all $p$. In this section we develop some formal technical machinery to deal with such situations.

**Definition 1.17.** *A chain complex* $\mathcal{A} = (A_k, \mathrm{d}_k)_k$ *is a series of abelian groups* $\{A_k\}_{k\in\mathbb{Z}}$ *together with morphisms of abelian groups* $\mathrm{d}_k : A_k \to A_{k+1}$ *such that* $\mathrm{d}_{k+1} \circ \mathrm{d}_k = 0$ *for all* $k$.

**Exercise 1.18.** *Verify that the bundles* $\bigwedge^p T^*M$ *together with the differential* $\mathrm{d}$ *constitue a chain complex.*

For a general chain complex $\mathcal{A} = (A_k, \mathrm{d}_k)_k$ we write $Z_k$ for the kernel of $\mathrm{d}_k$ and $B_k$ for the image of $\mathrm{d}_{k-1}$. We thus have $B_k \subset Z_k$. For the case of differential forms one uses the following language:

**Definition 1.19.** *Let* $M$ *be manifold and* $\omega$ *a differential* $p$*-form on the open subset* $U \subset M$. *We say that* $\omega$ *is closed on* $U$ *if* $\mathrm{d}\omega = 0$. *We say that* $\omega$ *is exact on* $U$ *if there exists a* $(p-1)$*-form* $\chi$ *such that* $\omega = \mathrm{d}\chi$. *Thus all exact forms on* $U$ *are closed on* $U$. *If no reference to an open subset* $U$ *is made, one means* $U = M$.

We call a chain complex an exact sequence if $B_k = Z_k$. Exact sequences have very nice properties and are studied in great detail both in algebraic geometry and differential geometry. We give an example:

**Lemma 1.20.** *Let* $U$ *be the open unit ball in* $\mathbb{R}^n$, *that is,*

$$U = \{x \in \mathbb{R}^n | ||x|| < 1\} ,$$

*then if* $\omega$ *is a closed* $p$ , *there exists a* $(p-1)$*-form* $\chi$ *such that* $\omega = \mathrm{d}\chi$. *Hence all closed forms on* $U$ *are exact.*

We won't proof the lemma, which goes under the name of Poincaré's lemma, as it is not the main theme of the exposition and as it can be found in almost all books on differential geometry. The case $n = 2$ we present as an exercise:

**Exercise 1.21.** *Consider in* $\mathbb{R}^2$ *the open disk* $D$ *containing the points* $(x, y)$ *with* $x^2 + y^2 < 1$. *Let* $\omega = a(x, y)\mathrm{d}x + b(x, y)\mathrm{d}y$ *where* $a$ *and* $b$ *are smooth function on* $D$. (1) *Express* $\mathrm{d}\omega = 0$ *in terms of* $a$ *and* $b$. (2) *We define a function* $\chi$ *as follows:*

$$\chi(x, t) = \int_0^1 dt \left( a(tx, ty)x + b(tx, ty)y \right) .$$

*Verify that* $\mathrm{d}\chi = \omega$. *Hint: Partial integration.* (3) *If* $f$ *is any smooth function on* $D$ *with* $\mathrm{d}f = 0$, *then show that* $f$ *is constant.* (4) *If* $\omega$ *is a two-form, then of course* $\mathrm{d}\omega = 0$. *Write* $\omega = a(x, y)\mathrm{d}x \wedge \mathrm{d}y$ *and verify*

$$\mathrm{d}\Big( \int_0^1 a(tx, ty)tx \, dt\mathrm{d}y - \int_0^1 a(tx, ty)ty \, dt\mathrm{d}x \Big) = \omega .$$

(5) *If we modify the disk by taking out the origin then we can get* 1*-forms that are not exact. Show that on* $D - \{0\}$ *the* 1*-form* $\omega$ *defined by*

$$\omega = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2} ,$$

*is closed but not exact. Hint: show that* $\int_\gamma \omega \neq 0$ *if* $\gamma$ *is a noncontractible loop around* 0.

To measure how far a chain complex $\mathcal{A} = (A_k, \mathrm{d}_k)_k$ away is from being a exact, one introduces the cohomology groups $H^i(\mathcal{A})$, which are defined as

$$H^i(\mathcal{A}) = Z_k/B_k = \mathrm{Ker}\mathrm{d}_k/\mathrm{Im}\mathrm{d}_{k-1} .$$

A chain complex $\mathcal{A}$ is an exact sequence if and if all cohomology groups vanish identically. When $U$ is an open set of a manifold $M$ one writes $H^i(\bigwedge T^*M, U)$ for the cohomology groups associated to the chain complex with abelian groups $A_k =$

$C^\infty(\bigwedge^k T^*M, U)$ and with connecting maps $\mathrm{d}_k = \mathrm{d}$. Often one just writes $H^i(U)$ for $H^i(\bigwedge T^*M, U)$. The name for the cohomology groups $H^i(\bigwedge T^*M, U)$ that appears in the literature is de Rahm cohomology groups and the chain complex is called the de Rahm complex. The most interesting cohomology groups for our purposes are the $H^i(M) = H^i(\bigwedge T^*M, M)$; that is the globally defined closed $i$-forms modulo the exact ones.

Clearly we have a map $Z_k \mapsto H^i(\mathcal{A})$ and for a closed differential $i$-form $\omega$ one writes $[\omega]$ for the image in $H^i(U)$. A non-trivial element of $H^i(U)$ can thus be represented by a closed $i$-form that is not exact. As we have seen in the exercise above, the content of the de Rahm cohomology groups depends on the topology of the space.

## 1.7 Connection and curvature

Let $E \to M$ be a vector bundle over $M$. In any trivializing chart one can differentiate a section by differentiating the component functions. However, the transition functions depend in general on the points and thus differentiation is not well-defined in this way; it depends on the chart chosen. A physicist would say that bluntly differentiating the components of a section is not covariant. The way out of this problem is the invention of the connection:

**Definition 1.22.** *Let $E \to M$ be a vector bundle over $M$. We call a map $\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M)$ a connection if*

*(i) $\nabla$ is additive; for $\sigma, \tau \in \Gamma(E, U)$ for some open subset $U \in M$ we have*

$$\nabla(\sigma + \tau) = \nabla(\sigma) + \nabla(\tau)\,.$$

*(ii) $\nabla$ satisfies a Leibniz rule; for $\sigma \in \Gamma(E, U)$ and $f \in C^\infty(U)$ for some open subset $U \in M$ we have*

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma)\,.$$

Since for any connection $\nabla$ the result of acting on a section $\sigma$ is an element of $C^\infty(E \otimes T^*M)$, we can act with $\nabla\sigma$ on a vector field $X$ to get a section of $E$: $\nabla\sigma(X) \in C^\infty(E)$. Indeed, we can write

$$\nabla\sigma = \sum_k \tau_k \otimes \omega_k\,, \quad \tau_k \in C^\infty(E)\,,\ \omega_k \in C^\infty(T^*M)\,,$$

so that we have

$$\nabla\sigma(X) = \sum_k \tau_k \omega_k(X)\,.$$

We in general write $\nabla\sigma(X) = \nabla_X\sigma$ and in physics we call this the covariant derivative of $\sigma$ along $X$. More general, in physics the notion of a connection is called a covariant derivative. Later we will see that on a local coordinate chart we can write $\nabla \sim \mathrm{d} + A_\mu \mathrm{d}x^\mu$ and then physicists call $A_\mu \mathrm{d}x^\mu$ a gauge field, which might look different in every coordinate chart. The transition of going from one expression for the gauge field in the one chart to an expression for the gauge field in the other coordinate chart are called gauge transformations. We can reformulate the definition of the connection using the above:

**Definition 1.23.** *Let $E \to M$ be a vector bundle over $M$. A connection is an $\mathbb{R}$-linear map $\nabla : C^\infty(E) \times C^\infty(TM) \to C^\infty(E)$ satisfying for any $X, Y \in C^\infty(TM)$, $\sigma, \tau \in C^\infty(E)$ and $f \in C^\infty(M)$:*

*(i) $\nabla_{fX+Y}\sigma = f\nabla_X\sigma + \nabla_Y\sigma$,*

*(ii) $\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma$,*

*(iii) $\nabla_X(\sigma + \tau) = \nabla_X\sigma + \nabla_X\tau$,*

*where $\nabla_X \sigma$ means $\nabla(\sigma, X)$.*

**Exercise 1.24.** *Prove that definition 1.22 and definition 1.23 are equivalent.*

It is to be noted that $\nabla_X$ is not a tensor as it is not a $C^\infty(M)$-linear mapping. Experience tells us that it is convenient to look for tensors that we can deduce from $\nabla$.

**Lemma 1.25.** *Let $E \to M$ be a vector bundle over $M$ with a connection $\nabla$. For any $X, Y \in C^\infty(TM)$ the following $\mathbb{R}$-linear mapping $C^\infty(E) \to C^\infty(E)$ is a tensor:*

$$R(X,Y) : \sigma \mapsto R(X,Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma \,. \tag{33}$$

*Proof.* We leave the detials as an exercise: we only have to check that $R(X,Y)(f\sigma) = fR(X,Y)\sigma$ for $f \in C^\infty(M)$, which is a trivial writing exercise. □

**Definition 1.26.** *We define curvature associated as the tensor $R$ that maps $X, Y \in C^\infty(TM)$ and $\sigma \in C^\infty(E)$ to $R(X,Y)\sigma \in C^\infty(E)$. Sometimes to stress the relation with the connection we write $R^\nabla$.*

**Exercise 1.27.** *Show that the curvature $R$ is really a tensor, i.e., that we have $R(fX, gY)\sigma = fgR(X,Y)\sigma$ for $f, g \in C^\infty(M)$.*

**Remark 1.28.** *There are numerous examples of calculations for curvatures. Time is too short to treat them. In general they involve some time to really make the formulae work nicely. Therefore we only treat the Levi–Civitá connection on the related Riemann curvature (both to be defined later) on $S^2$ in later sections. If time permits I add a short example later. For now we urge the reader to do some research in the vast literature on for example gauge fields in physics where connections are abundant; a nice exercise is to explicitly find the relation between the concepts from Maxwell theory (electric field, magnetic field, potential, vector potential, gauge transformations) and the introduced notions of connection and curvature and transtition functions.*

We call a connection flat if the associated curvature vanishes identically: $R \equiv 0$. In general flat connections might not exist.

**Lemma 1.29.** *If $E \to M$ is a trivial vector bundle, then $E$ admits a flat connection.*

*Proof.* Let $f : E \to M \times F$ be an isomorphism. It suffices to show that $M \times F$ admits a flat connection, since if $R$ is the curvature on $F \times M$ then we get a connection on $E$ as follows $\tilde{R}(X,Y)\sigma = f^{-1}(R(X,Y)(f(\sigma)))$. On $M \times F$ we can define a connection $\nabla$ by looking in any chart $U$ on $M$ with coordinates $x^\mu$ and then for any section $\sigma$ defining $\nabla\sigma = \partial_\mu \sigma \mathrm{d}x^\mu$, where $\partial_\mu \sigma$ can be evaluated by choosing a basis for $F$ and then differentiating componentwise. Since there are no transition functions, the connection $\nabla$ is well-defined. It is trivial algebra that for any smooth vector fields $X, Y$ we have $R(X,Y)\sigma = 0$ (since partial derivatives commute). □

To find a suitable connection is a nontrivial task. It is not guaranteed that a connection exists. If connections exists there might be many and to choose any one of them becomes an issue of arbitrariness. In some cases one can require a connection to be compatible with some additional structure. For example if the vector bundle is equipped with a metric, one gets a handle on the connections. We note that a metric on $E \to M$ is a global nowhere degenerate section $h$ of the bundle $E^* \otimes E^* \to M$; at each point $h$ is a metric on the fibre $F$ and varies smoothly from point to point. Metric compatibility then means that a connection $\nabla$ satisfies for any two sections $\sigma, \tau \in C^\infty(E)$ and any smooth vector field $X$

$$X(h(\sigma,\tau)) = h(\nabla_X \sigma, \tau) + h(\sigma, \nabla_X \tau)\,.$$

In general metric compatibility does not fix a metric. For the tangent bundle the situation becomes a lot nicer (or less boring, depending on your point of view).

Therefore from here, the focus in this section is on the tangent bundle, although many concepts can be developped for general vector bundles.

Assume the tangent bundle $TM \to M$ is equipped with a connection $\nabla$. Let us introduce some notation that is also used in physics literature. Let $U$ be a chart on a manifold $M$ and suppose $U' \subset \mathbb{R}^n$ is the image of $U$. Then we can introduce coordinates $x^1, \ldots, x^n$ on $U'$ and parametrize $U$ by means of the $x^\mu$. Every vector field on $U$ can be expanded in the basis elements $e_\mu = \partial/\partial x^\mu$. If we know how a connection $\nabla$ acts on the $e_\mu$ we have completely fixed the action of $\nabla$ on sections over $U$. We therefore introduce coefficients, called Christoffel symbols, defined by

$$\nabla e_\nu = \sum_{\mu,\lambda} \Gamma^\lambda_{\mu\nu} \mathrm{d}x^\mu \otimes e_\lambda \,, \tag{34}$$

that is

$$\nabla_{e_\mu} e_\nu = \sum_\lambda \Gamma^\lambda_{\mu\nu} e_\lambda \,. \tag{35}$$

On occasion we use the abbreviation $\nabla_\mu = \nabla_{e_\mu}$. Now let $X$ be a smooth vector field on $U$, then we can expand $X = \sum_\mu X^\mu e_\mu$ and when we use the notation $\nabla_\mu X^\nu = (\nabla_\mu X)^\nu$, which again means $\nabla_\mu X = (\nabla_\mu X)^\nu e_\nu$, we obtain

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda \,. \tag{36}$$

**Exercise 1.30.** *Verify eqn.(36).*

Having a connection $\nabla$ on the tangent bundle $TM$, we get an induced connection on $T^*M$ and on all tensor product bundles $TM^{\otimes r} \otimes T^*M^{\otimes s}$ as follows (warning: we use the same symbol $\nabla$ for this connection). Since the bundle $C^\infty(M)$ is trivial, we want that the connection on $C^\infty(M)$ remains d. Hence if $\omega$ is a smooth differential one-form and $X, Z$ smooth vector fields, then we want

$$Z(\omega(X)) = (\nabla_Z \omega)(X) + \omega(\nabla_Z X) \,. \tag{37}$$

We let the reader verify that one indeed gets a connection:

**Exercise 1.31.** *(a) Verify that $(\nabla_Z(f\omega))(X) = Z(f)\omega(X) + f(\nabla_Z \omega(X))$.*

*(b) Introduce local coordinates and expand $\sum_\mu \omega_\mu \mathrm{d}x^\mu$ and (using the same notation as above for vector fields) derive the local expression:*

$$\nabla_\mu \omega_\nu := (\nabla_\mu \omega)_\nu = \partial_\mu \omega_\nu - \sum_\lambda \Gamma^\lambda_{\mu\nu} \omega_\lambda \,.$$

*(c) Conclude that the connection $\nabla$ on $T^*M$ is uniquely determined by eqn.(37).*

## TODO check self too

Now it is not too hard too extend the obtained connection on $TM$ and $T^*M$ to arbitrary tensors. Let $S \in C^\infty(TM^{\otimes r} \otimes T^*M^{\otimes s})$, then we can view $S$ as a linear map $S : \prod_{i=1}^s C^\infty(TM) \otimes \prod_{j=1}^r C^\infty(T^*M) \to C^\infty(M)$. We then, as above, require for $Z$ and $X_i$ smooth vector fields and $\omega_i$ smooth differential one-forms that

$$Z(S(X_1, \ldots, X_s, \omega_1, \ldots, \omega_r) = S(\nabla_Z X_1, \ldots, \omega_r) + \ldots + S(X_1, \ldots, \nabla_Z \omega_r) \,. \tag{38}$$

**Exercise 1.32.** *Work out eqn.(38) in local coordinates, verify that it is a connection, find the analogue of eqn.(36) for $(r,s)$-tensors. Conclude that eqn.(38) uniquely fixes the connection on $TM^{\otimes r} \otimes T^*M^{\otimes s}$.*

We now know how to extend a given connection on $TM$ to a connection on the tensor bundles. So, we have more or less reduced a whole possible family of connections on different natural bundles to the family of connections on $TM$. The two properties that are fixing a connection uniquely is metric compatibility and zero torsion. The concept of torsion is a difficult one, and why one might want to set it to zero is not less mysterious.

**Definition 1.33.** *Consider the tangent bundle $TM \to M$. Then for each connection $\nabla$ on $TM$ we define the torsion associated to $\nabla$ as the tensor $\mathcal{T} \in C^\infty(TM \otimes \bigwedge^2 T^*M)$ by*

$$\mathcal{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\,. \tag{39}$$

**Exercise 1.34.** *Verify that the torsion associated to some connection is indeed an element of $C^\infty(TM \otimes \bigwedge^2 T^*M)$: explicitly, show that $\mathcal{T}(X,Y) = -\mathcal{T}(Y,X)$ and $\mathcal{T}(fX + X', Y) = f\mathcal{T}(X,Y) + \mathcal{T}(X',Y)$.*

Since clearly $[e_\mu, e_\nu] = 0$ we find a local expression for the torsion on $U$ given by

$$T = \sum_{\mu,\nu,\lambda} \Gamma^\lambda_{[\mu\nu]} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes e_\lambda\,. \tag{40}$$

**Exercise 1.35.** *Verify eqn.(40). Suppose that if $V$ is another charts having nonempty intersection with $U$, use coordinates $y^1, \ldots, y^n$ to parametrize $V$, and find the relation between the Christoffel symbols as expressed in terms $U$ coordinates and $V$ coordinates. Verify that the Christoffel symbols do not define a tensor.*

From the exercise it is clear that the nontensorial property of the Christoffel symbols lies in the symmetric part $\Gamma^\lambda_{(\mu\nu)}$. If a connection is such that the torsion tensor vanishes, we call the connection a torsion-free connection. In this case the Christoffel symbols are symmetric. Torsion-free connections play a big role in differential geometry and physics for reasons that we will not explain (partially due to lack of knowledge).

**Remark 1.36.** *Let $\nabla$ be a torsion-free connection on $M$, then for any two vector fields $X, Y \in C^\infty(TM)$ we have*

$$\nabla_X Y - \nabla_Y X = [X,Y]\,, \tag{41}$$

*as follows from the definition of torsion.*

**Definition 1.37.** *Let $M$ be a manifold. We say that $M$ is a Riemannian if the tangent bundle is equipped with a metric, that is, a global smooth section $g$ of the symmetric tensor product $T^*M \otimes_{sym} T^*M$ such that $g$ is at each point positive-definite and nondegenerate. We say that $M$ is pseudo-Riemannian if the tangent bundle is equipped with a global smooth section $g$ of the symmetric tensor product $T^*M \otimes_{sym} T^*M$ such that $g$ is at each point nondegenerate.*

**Remark 1.38.** *The definition of a metric in physics is what mathematicians might call a pseudo-metric, which is a symmetric bilinear nondegenerate form. The difference with a metric is the positive definiteness: a pseudo-metric can have negative eigenvalues. Then the definition of a pseudo-Riemannian metric can be rephrased by using the word pseudo-metric. We leave the paraphrasing as an exercise.*

A symmetric bilinear form on a vector space can be expressed as a symmetric matrix, hence on a manifold we have such an expression for each point. Let $U$ be a chart on $M$ and choose coordinates $x^1, \ldots, x^n$ to parametrize $U$. Then the vectors $e_\mu = \partial/\partial x^\mu$ make up a local trivialization of the tangent bundle on the chart $U$. We define $g_{\mu\nu} = g(e_\mu, e_\nu)$; we can represent $g$ on $U$ by the symmetric matrix $(g_{\mu\nu})$, which has entries that are $C^\infty$-functions on $U$. It is important to note that a metric is not a matrix: under a coordinate transformation $x \mapsto Bx$ a matrix and a bilinear form transform differently; a matrix $C$ transforms as $C \mapsto BCB^{-1}$ and a bilinear form $D$ as $D \to BDB^T$. In the physics literature the notation $g_{\mu\nu}$ is used for the metric; the advantage for keeping the indices is to remember what kind of a tensor the metric is and what $GL_{\mathbb{R}}(n)$ representation the metric corresponds to. The metric is a symmetric $(0,2)$-tensor, but the symmetric tensor product representation of $GL_{\mathbb{R}}(n)$ is not irreducible. Therefore, when physicists need to do a counting of degrees of freedom, the metric has to be reduced to an irreducible representation, which amounts to subtracting the trace[2] with the result that a metric on an $n$-dimensional manifold corresponds to $\frac{1}{2}n(n+1)-1$ degrees of freedom.

As a metric is nondegenerate, at each point as a matrix, it can be inverted. Taking again a chart $U$ parametrized by coordinates $x^1, \dots, x^n$, it makes sense to talk of $g^{-1}$; it is the $(2,0)$ tensor that can be represented locally on $U$ by the symmetric matrix $(g_{\mu\nu})^{-1}$. The expression for the inverse matrix involves taking determinants and subdeterminants of the original symmetric matrix $g_{\mu\nu}$; hence the inverse metric is a smooth function of the coordinates on $U$. Furthermore, one can verify that the inverse metric defined in this way is compatible with the transition functions of the bundle $TM \otimes_S TM$, where the $\otimes_S$ denotes the symmetrized tensor product, and thus the inverse metric is a well-defined global section of $TM \otimes_S TM$.

**Exercise 1.39.** *Verify that the inverse metric defined locally in one chart in the preceding paragraph is compatible with the transition functions of the bundle $TM \otimes_S TM$. That is, you have to check that if $U_1$ and $U_1$ are two coordinate charts and $U_1 \cap U_2$ is nonempty that when $h^{(i)}_{\mu\nu}$ is the expression for the components of the inverse metric on $U_i$, $i = 1, 2$, then $h^{(1)}_{\mu\nu} = \sum_{\lambda,\rho} \Phi_{\mu\lambda}\Phi_{\nu\rho}h^{(2)}_{\lambda\rho}$, where $\Phi_{\mu\nu}$ are components of the transition functions of the tangent bundle. Hint: use the explicit expression for the inverse of a matrix.*

In the physics literature, the inverse of the metric is denoted with the same $g$, but with the indices 'upstairs': $g^{\mu\nu}$ denotes the inverse metric. Therefore an often appearing formula is

$$\sum_\nu g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda \,,$$

where (1) a summation is understood for a repeated index that appears once upstairs and once downstairs (this is called Einstein convenstion) and where (2) $\delta_\mu^\lambda$ is the Kronecker delta, which has the numerical value if $\mu = \lambda$ and zero otherwise. We will not go into the details of the notation and its usefulness at this point.

**Proposition 1.40** (Levi–Civitá Connection)**.** *Let $M$ be a (pseudo-) Riemannian manifold with metric $g$. Then there exists a unique connection $\nabla$ on $TM$ with the properties:*

*(i) $\nabla$ is metric-compatible: $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for all smooth vector fields $X, Y$ and $Z$.*

*(ii) $\nabla$ has no torsion.*

Before we plunge into the proof of the proposition, we offer the following easy but (we think) useful exercise:

**Exercise 1.41.** *Prove that metric compatibility of the connection means that the metric is covariantly constant. Use local coordinates to find that metric compatibility is equivalent to requiring locally that*

$$\partial_\mu g_{\nu\lambda} - \sum_\rho \Gamma^\rho_{\mu\nu} g_{\rho\lambda} - \sum_\rho \Gamma^\rho_{\mu\lambda} g_{\nu\rho} = 0 \,.$$

[2] If you don't understand this, read on; you won't need it in the sequel. The remark is only intended for those that once encountered a counting of degrees of freedom in the physics literature.

*Proof.* We present two methods of proof. The first one is the physicist's way: one just uses the assumptions to calculate the Christoffel symbols, which fixes the connection. We use local coordinates $x^\mu$ on some chart $U$ and the short-hand notation $e_\mu = \partial/\partial x^\mu$, $g_{\mu\nu} = g(e_\mu, e_\nu)$ and $\nabla_\mu = \nabla_{e_\mu}$. Metric compatibility means that the metric is covariantly constant $\nabla_\mu g_{\nu\lambda} = 0$. The trick is to write this equation down three times, but with indices permuted cyclically and using the symmetry of the Christoffel symbols and the metric;

$$\nabla_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma^\rho_{\mu\nu} g_{\rho\lambda} - \Gamma^\rho_{\mu\lambda} g_{\nu\rho} = 0 \tag{42}$$

$$\nabla_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^\rho_{\nu\lambda} g_{\rho\mu} - \Gamma^\rho_{\nu\mu} g_{\lambda\rho} = 0 \tag{43}$$

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\mu\rho} = 0\,. \tag{44}$$

Now one calculates (42)+(43)-(44). One then obtains:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} \sum_\lambda g^{\lambda\rho} \Big( \partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \Big)\,. \tag{45}$$

The mathematical method is contained in the identity (46) of the following lemma:

**Lemma 1.42** (Koszul's formula)**.** *Let $\nabla$ be a torsion-free connection on a $M$. Then we have the following formula*

$$\begin{aligned} 2g\Big(\nabla_X Y, Z\Big) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) \\ &\quad - g([X,Z],Y) - g([Y,Z],X)\nabla_X Y - \nabla_Y X\,. \end{aligned} \tag{46}$$

*Proof.* The proof can be done by writing out using the definitions and assuming the absence of torsion, whence $[X,Y] = \nabla_X Y - \nabla_Y X$ and the metric compatibility $X(g(Y,Z)) = g(\nabla_X Y, Z) + g)Y, \nabla_X Z)$. Using the metric compatibility one expands the right-hand side of the Koszul formula and one sees that all covariant derivatives appear in terms combining into Lie derivatives, except the $\nabla_X Y$-terms; they add up. The algebraic and trivial calculation is left as an exercise. [End of proof of lemma] □

One than sees that if a connection exists it is unique. Indeed for two connections $\nabla$ and $\widetilde{\nabla}$ we have that for each pair of vectors $X, Y$ the vector $v = \nabla_X Y - \widetilde{\nabla}_X Y$ satisfies $g(v, Z) = 0$ for all $Z$. Hence $v = 0$. One then can calculate for each pair of vector fields $X, Y$ the object $\nabla_X Y$ as follows: introduce a local basis $e_\mu$ of vector fields on an open chart $U$, then find a dual basis $f_\mu$ of vector fields, that is, a set of vector fields with the property $g(e_\mu, f_\nu) = 1$ if $\mu = \nu$ and zero otherwise. Then $\nabla_X Y = \sum g(\nabla_X Y, f_\mu) e_\mu$. □

**Exercise 1.43.** *Using local coordinates, express the Riemann tensor in terms of the Christoffel symbols: Define coefficients $R^\rho_{\lambda\mu\nu}$ by $R(e_\mu, e_\nu)e_\lambda = \sum_\rho R^\rho_{\lambda\mu\nu} e_\rho$ and calculate the $R^\rho_{\lambda\mu\nu}$ in terms of the Christoffel symbols.*

**Definition 1.44.** *Let $U$ be a coordinate chart with coordinate $x^\mu$ and put $e_\mu = \partial/\partial x^\mu$. Then we define the Ricci tensor Ric as the following contraction of the Riemann tensor*

$$Ric := Ric_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu\,, \quad Ric_{\mu\nu} = \sum_\lambda R^\lambda \mu\lambda\nu\,,$$

*where as in the exercise we used $R(e_\mu, e_\nu)e_\lambda = \sum_\rho R^\rho_{\lambda\mu\nu} e_\rho$.*

**Exercise 1.45.** *Find a local expression for the Ricci tensor.*

If the metric $g$ is such that the Ricci tensor of the Levi–Civitá connection vanishes, we say that $g$ is Ricci flat. The property of being Ricci flat will play an important role in defining Calabi–Yau manifolds.

## 1.8 Tangent bundle of $S^2$

In order to describe the tangent bundle we need to have the chart transition functions. Using eqns.(3,4,5,6) we find that $f_V \circ f_U^{-1} : \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$ is given by

$$f_V \circ f_U^{-1} : (u_1, u_2) \mapsto \frac{1}{||u||^2}(u_1, u_2)\,, \tag{47}$$

and $f_U \circ f_V^{-1} : \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$ is given by

$$f_U \circ f_V^{-1} : (v_1, v_2) \mapsto \frac{1}{||v||^2}(v_1, v_2)\,. \tag{48}$$

We find that for the tangent bundle the transition matrices are given

$$\Phi_{UV} = \frac{\partial(v_1, v_2)}{\partial(u_1, u_2)} = \frac{1}{||u||^4}\begin{pmatrix} u_2^2 - u_1^2 & -2u_1u_2 \\ -2u_1u_2 & u_1^2 - u_2^2 \end{pmatrix}\,, \tag{49}$$

and of course the inverse is $\Phi_{VU}$ for which we find

$$\Phi_{VU} = \frac{\partial(u_1, u_2)}{\partial(v_1, v_2)} = \frac{1}{||v||^4}\begin{pmatrix} v_2^2 - v_1^2 & -2v_1v_2 \\ -2v_1v_2 & v_1^2 - v_2^2 \end{pmatrix}\,. \tag{50}$$

We can express the transition functions in terms of $x$, $y$ and $z$ as follows

$$\Phi_{UV} = \frac{1}{(1+z)^2}\begin{pmatrix} y^2 - x^2 & -2xy \\ -2xy & x^2 - y^2 \end{pmatrix}\,, \quad \Phi_{VU} = \frac{1}{(1-z)^2}\begin{pmatrix} y^2 - x^2 & -2xy \\ -2xy & x^2 - y^2 \end{pmatrix}\,.$$

Geometrically, a tangent vector to the sphere at the point $(x, y, z)$ is a vector perpendicular to $(x, y, z)$. We can use this geometric picture to get a metric on the sphere: for the inner product of two tangent vectors seen in one of the charts, we find the related three-dimensional vectors that are tangent to the sphere and calculate the inner-product in $\mathbb{R}^3$. Also for later reference, it is convenient to be able to switch from three-dimensional vectors tangent to the sphere to their images of vector in one of the charts.

First we consider the $U$-chart. Look at the vector

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,,$$

at the point $(u_1, u_2)$ in the $U$-plane. Then $e_1$ corresponds to the derivative

$$\frac{d}{dt}\Big|_0 (u_1 + t, u_2)\,.$$

We can map the path $t \mapsto (u_1 + t, u_2)$ and get the corresponding path on the sphere by applying $f_U^{-1}$. Taking the derivative we obtain the vector $Tf_U^{-1}(e_1)$;

$$Tf_U^{-1}(e_1) = \frac{2}{(1 + ||u||^2)^2}\Big(1 + u_2^2 - u_1^2, -2u_1u_2, 2u_1\Big)\,.$$

Similarly we obtain for the vector $e_2$ defined by

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,,$$

the vector $Tf_U^{-1}(e_2)$ on the sphere given by

$$Tf_U^{-1}(e_2) = \frac{2}{(1 + ||u||^2)^2}\Big(-2u_1u_2, 1 + u_1^2 - u_2^2, 2u_2\Big)\,.$$

Similarly, we find in the $V$-chart:

$$Tf_V^{-1} : \begin{pmatrix}1\\0\end{pmatrix} \mapsto \frac{2}{(1+||v||^2)^2}\Big(1 - v_1^2 + v_2^2, -2v_1v_2, -2v_1\Big)\,, \tag{51}$$

$$Tf_V^{-1} : \begin{pmatrix}0\\1\end{pmatrix} \mapsto \frac{2}{(1+||v||^2)^2}\Big(-2v_1v_2, 1 + v_1^2 - v_2^2, -2v_2\Big) \tag{52}$$

**Exercise 1.46.** *Find the tangent maps $Tf_U$ and $Tf_V$, given by (3) and (4). What is their kernel at each point of the sphere? Are they surjective?*

Having found the tangent vectors at the sphere, as embedded in $\mathbb{R}^3$, that correspond to the unit vectors in the $U$- and $V$-plane, we get the metric on the sphere. For any tangent vectors $X, Y$ in the $U$-plane at the point $p = (u_1, u_2)$ we take the inner product to be the corresponding inner product on the sphere:

$$g_p(X, Y) = \langle Tf_U^{-1}(X), Tf_U^{-1}(Y)\rangle\,,$$

where $\langle,\rangle$ denotes the standard inner product in $\mathbb{R}^3$. For tangent vectors in the $V$-plane we do the same thing. We find for the $U$-chart that the metric at $p = (u_1, u_2)$ is represented by the matrix

$$g_{U,p} = \frac{4}{(1+||u||^2)^2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\,, \tag{53}$$

and similar for the $V$-chart.

**Exercise 1.47.** *Calculate the expression for the metric in the $V$-chart.*

We can now calculate the Levi–Civitá connection and the curvature. We will do so by calculating the Christoffel symbols. Since the metric is of a very simple form, we find it convenient to first let the reader do some calculation:

**Exercise 1.48.** *Suppose that in some chart with coordinates $x^\mu$ the metric is given by $g_{\mu\nu} = \frac{1}{f(x)}\delta_{\mu\nu}$, where $\delta_{\mu\nu}$ is 1 if $\mu = \nu$ and zero otherwise. (That is, the metric is a multiple of the standard metric, which is represented by the identity matrix.) Show that*

$$\Gamma^\rho_{\mu\nu} = \frac{-1}{2f(x)}\Big(\delta_{\rho\nu}\partial_\mu f + \delta_{\mu\rho}\partial_\nu f - \delta_{\mu\nu}\partial_\rho f\Big)\,.$$

From the exercise (or just calculating directly) we find that for the sphere we have the following Christoffel symbols in the $U$-chart

$$\Gamma^1_{11} = \frac{-2u_1}{1+||u||^2}\,,\quad \Gamma^1_{12} = \frac{-2u_2}{1+||u||^2}\,,$$
$$\Gamma^1_{22} = \frac{2u_1}{1+||u||^2}\,,\quad \Gamma^2_{11} = \frac{-u_2}{1+||u||^2}\,,$$
$$\Gamma^2_{12} = \frac{-2u_1}{1+||u||^2}\,,\quad \Gamma^2_{22} = \frac{-2u_2}{1+||u||^2}\,.$$

We thus find for the Levi–Civitá connection $\nabla$:

$$\nabla\frac{\partial}{\partial u_1} = \frac{2}{1+||u||^2}\Big(-u_1\frac{\partial}{\partial u_1}\otimes \mathrm{d}u_1 + u_2\frac{\partial}{\partial u_2}\otimes \mathrm{d}u_1 - u_2\frac{\partial}{\partial u_1}\otimes \mathrm{d}u_2 - u_1\frac{\partial}{\partial u_2}\otimes \mathrm{d}u_1\Big)$$

$$\nabla\frac{\partial}{\partial u_2} = \frac{2}{1+||u||^2}\Big(-u_2\frac{\partial}{\partial u_1}\otimes \mathrm{d}u_1 - u_1\frac{\partial}{\partial u_2}\otimes \mathrm{d}u_1 + u_1\frac{\partial}{\partial u_1}\otimes \mathrm{d}u_2 - u_2\frac{\partial}{\partial u_2}\otimes \mathrm{d}u_1\Big)$$

When we introduce the notation $\partial_i = \partial/\partial u_i$ and $\nabla_i = \nabla_{\partial_i}$ for $i = 1, 2$ we obtain:

$$\nabla_1\partial_1 = -\nabla_2\partial_2 = \frac{-2u_1\partial_1 + 2u_2\partial_2}{1+||u||^2}$$
$$\nabla_1\partial_2 = \nabla_2\partial_1 = \frac{-2u_1\partial_2 - 2u_2\partial_1}{1+||u||^2}\,.$$

**Exercise 1.49.** *Use the connection given above to find the curvature in the $U$-chart. That is, you are asked to calculate $R(\partial_1, \partial_2)\partial_1$ and $R(\partial_1, \partial_2)\partial_2$. Explain why these two quantities determine the Riemann tensor and explain why the formula for the curvature reduces to $R(\partial_1, \partial_2)\partial_k = \nabla_1\nabla_2\partial_k - \nabla_2\nabla_1\partial_k$.*

An interesting question is whether there exist nowhere vanishing global sections of the tangent bundle. In the algebraic setting the question is not too hard to answer; there exists no such section. Using the transition functions this is not too hard too show, and we leave this as an exercise (hint: vector fields are just given by polynomial components). In the $C^\infty$-case one has to do a bit more work. Using the theorem of Hopf one can show that the number of zero's of a vector field is related to the Euler characteristic. Since the Euler characteristic of the sphere is $-2$ any vector field that is defined globally needs to have zero's. One concludes that the tangent bundle of the sphere is not trivial.

## 1.9 Holonomy

Let $\gamma : [0, 1] \to M$ be a smooth map $\gamma(0) = p \in M$ and $\gamma(1) = q \in M$. Thus $\gamma$ is a smooth path connecting $p$ and $q$. We note that the derivative of $\gamma$, which we denote by $\gamma'(t)$ consists of a smoothly varying tangent vector over $\gamma(t)$. (One can see this as a section of the pull-back bundle $\gamma^*TM$ over a $[0, 1]$; this provides a nice unified picture. See for a nice explanation of holonomy from this perspective in [7].) Since $M$ is a smooth manifold being path connected and connected is the same thing. Therefore, if we assume $M$ is connected (which we will do when we need $M$ to be connected), then any two points can be connected through a smooth path. If $E \to M$ is a vector bundle over $M$ we are interested into how we can transport a vector above $p$ to a vector above $q$. It turns out that the most interesting way in doing so lies in the following definition:

**Definition 1.50.** *Let $\gamma : [0, 1] \to M$ be a smooth curve connecting $p$ to $q$ on $M$ and denote $\gamma'(t)$ the tangent vector field to the curve. Let furthermore $X$ be an element of the fibre at $p$ of the vector bundle $E \to M$ over $M$. The parallel transport of $X$ from $p$ to $q$ is the solution of the differential equation*

$$\nabla_{\gamma'(t)}\widetilde{X} = 0\,, \quad \widetilde{X}(0) = X\,. \tag{54}$$

**Remark 1.51.** *One has to be a bit carefull in reading the differential equation; the equation only depends on the value of $\gamma'(t)$ on the path $\gamma(t)$; using a local small chart on each point around the path one can find an extension of $\gamma(t)$. The differential equation does not depend on how one extends $\gamma'(t)$. The nicer approach is to pull-back all to the interval $[0, 1]$.*

**Exercise 1.52.** *Let $E \to M$ be a vector bundle over $M$ and take an open set $U$ around a point $p \in M$ such that $U$ is homeomorphic to some open subset in $\mathbb{R}^n$ - and hence we can use coordinates on $U$ - and such that there are linearly independent sections $e_i \in \Gamma(E, U)$ that span $\Gamma(E, U)$. Find a local differential equation for parallel transport.*

For us the most interesting case is when $E \to M$ is the tangent bundle and when the two endpoints of the curve $\gamma$ coincide;$\gamma(0) = \gamma(1) = p$. In this case, each path $\gamma$ induces a linear map $P^\gamma : T_pM \to T_pM$. It is not too hard to see that if we turn the part around $\tilde{\gamma}(t) = \gamma(1-t)$ we have $P^{\tilde{\gamma}} \circ P^\gamma X = X$ for all $X \in T_pM$, and thus the linear transformations $P^\gamma$ make up a group, which we call the holonomy group $\mathrm{Hol}_p(\nabla)$ of $\nabla$ at $p$. We assume $M$ is a Riemannian manifold with metric $g$, and we assume $TM$ is equipped with the Levi–Civitá connection. We write $\mathrm{Hol}_p(g)$ for the holonomy group at $p$ associated to the Levi–Civitá connection of the metric $g$. Let

$X(t)$ be the parallel transport of a vector around a loop starting and ending at $p$, then we have

$$\frac{d}{dt}g(X(t),X(t)) = \gamma'(t)(g(X(t),X(t))) = 2g(\nabla_{\gamma'(t)}X(t),X(t)) = 0\,.$$

We conclude that parallel transport on a Riemannian manifold with respect to the Levi–Civitá connection preserves the length of a vector.

**Proposition 1.53.** (*i*) *The holonomy group* $\mathrm{Hol}_p(\nabla)$ *is a Lie subgroup of* $GL_r(\mathbb{R})$ *for a vector bundle of rank* $r$. (*ii*) *Let* $(M,g)$ *be a Riemannian manifold of dimension* $n$*, then holonomy group* $\mathrm{Hol}_p(g)$ *is a Lie subgroup of the orthogonal group* $O(n)$*. If in addition* $M$ *is orientable, then the holonomy group is a Lie subgroup of* $SO(n)$*.*

*Proof.* The first statement we won't proof, but refer to ... . The second proof is immediate from the previous paragraph. The third part follows from the definition of orientable: a manifold is orientable if the determinant bundle of $T^*M$ is trivial. ☐

**Definition 1.54.** *Let* $E \to M$ *be any vector bundle with connection* $\nabla$*. The identity component* $\mathrm{Hol}_p(\nabla)_0$ *of the holonomy group at* $p \in M$ *is called the restricted holonomy group.*

If $M$ is connected, then any two points $p$ and $q$ can be connected by a path and thus $\mathrm{Hol}_p(\nabla) \cong \mathrm{Hol}_q(\nabla)$. Then one can omit the suffix $p$ and one speaks of the holonomy group of $M$. Therefore, from now one, we assume that $M$ is connected as for the nonconnected case, one can calculate the holonomy group of each connected component. If $M$ is furthermore simply connected, then the holonomy group is connected. This follows as any path in homotopic to the constant map $\gamma_0 : t \mapsto p$ and hence for any path $\gamma$ we can deform $P^\gamma$ to the identity. A less trivial fact is the following:

**Theorem 1.55. Ambrose–Singer Theorem***: Let* $M$ *be a connected manifold and* $E \to M$ *a vector bundle with a connection* $\nabla$ *and associated curvature* $R$*. Then the Lie algebra of the holonomy group at* $r \in M$ *is generated by the matrices* $R(v,w) \in \mathrm{End}(E)$*, where* $v, w$ *run over the elements of the tangent space at* $p$*.*

*Proof.* We won't prove theorem 1.55. The original statement and proof can be found in [8]. ☐

## 1.10 Calculating the holonomy group on $S^2$

In exercise 1.49 you were aksed to calculate the components of the Riemann tensor. When all went right, you should have found

$$R(\partial_1,\partial_2)\partial_1 = \frac{-4}{(1+||u||^2)^2}\partial_2\,,\quad R(\partial_1,\partial_2)\partial_2 = \frac{4}{(1+||u||^2)^2}\partial_1\,.$$

Thus we can write in matrix form:

$$R(\partial_1,\partial_2) = \frac{4}{(1+||u||^2)^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,.$$

We see that $R(X,Y) = (X_1Y_2 - X_2Y_1)R(\partial_1,\partial_2)$, where $X$ and $Y$ are two tangent vectors at $p = (u_1,u_2)$. The Lie algebra generated by such matrices is precisely the Lie algebra of $SO(2)$. Since the sphere is simply connected and $SO(2)$ is connected, we know that the holonomy group lies within $SO(2)$. But each element $m$ of $SO(2)$ can be written as

$$m = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}\,.$$

Therefore, the holonomy group is precisely $SO(2)$.

Since $SO(2)$ is a rather simple group and $S^2$ a rather simple manifold, we did not need the Ambrose–Singer theorem. Take any point $p$ on a manifold $M$ and pick two nonparallel tangent vectors $X$ and $Y$ at $p$. Then $X$ and $Y$ can be used to trace out a infinitesimal parallellogram starting at $p$ with sides parallel to $X$ and $Y$. Running around the parallellogram and parallelly transporting a tangent vector $Z$ around, one sees that the action of the Lie algebra of the holonomy group on $Z$ is precisely $R(X,Y)Z$, see e.g. [13] for an elementary explanation guided by an explicit calculation. Since the sphere is Riemannian and the Levi–Civitá connection preserves length, the holonomy group is contained in $SO(2)$. But the exponential map $\mathrm{Exp} : \mathfrak{so}(2) \to SO(2)$ is surjective. Hence there are only two possible options: (1) Either $R(X,Y) = 0$ for all tangent vectors $X, Y$ for all points. Then the holonomy group is trivial. 2 Or $R(X,Y)$ does not vanish identically for all $X$ and $Y$ and then defines at some point a nonzero Lie algebra element, whence the whole Lie algebra (which is one-dimensional). Exponentiating we thus get the whole of $SO(2)$. Since $S^2$ is simply connected, then for each point the holonomy group is $SO(2)$.

## 1.11 Spin bundles

In order to understand how physicists found out that Calabi–Yau were interesting, one has to know a bit about spinors. In this subsection we introduce a minimum in order that the reader can read the original paper [1] without being too puzzled about the spinors.

Elementary particles come in two species: fermions and bosons. Bosons correspond to vectorial representations of the Lie algebra $\mathfrak{so}(n)$ and fermions correspond to spinorial representations. The fundamental representation of the Lie algebra $\mathfrak{so}(n)$ is the $n$-dimensional vector representation. All representations that can be obtained by taking direct sums, tensor products and subrepresentations of the vector representation are called vectorial. It turns out that these are not all representations of $\mathfrak{so}(n)$; there exists a spinorial representation that cannot be obtained from the vector representation as a subrepresentation of some tensor product. Taking tensor products, direct sums and subrepresentations of the vector representation and the spinor representation one obtains all representations of $\mathfrak{so}(n)$[9]. For the important consequences on the physics side, we refer the reader to the vast amount of literature on quantum field theory or statistical physics. Physicists use the names spinors and fermions for the (quantum) fields that describe fermions. Mathematically token, a spinor is a section from a spin bundle. Therefore, we introduce the notion of a spin bundle.

Let $M$ be an $n$-dimensional orientable Riemannian manifold with metric $g$. Then the structure group of the tangent bundle is $SO(n)$. The universal cover group of $SO(n)$ is called the spin group $Spin(n)$ and one may wonder whether there exists a lifting of the transition functions to the spin group in a nice way such that we can find a bundle where the transition functions are in fact taking values in the spin group. It turns out that if the second Stiefel–Whitney class (just as the Chern class some algebraically calculable differential form) vanishes, then it is possible to find a lift of the structure functions to the spin group. The bundle on then obtains is called a spin bundle; as fibres one takes the spinorial representation of $Spin(n)$ (which is just obtained by exponentiating the spinorial representation of $\mathfrak{so}(n)$ as $Spin(n)$ is simply connected). For fermions to exist in nature, one thus needs the vanishing of the second Stiefel–Whitney class.

Let $M$ be an orientable pseudo-Riemannian manifold of dimension $p + q = n$ with metric $g$ of signature $(p, q)$ - which means that the number of positive eigenvalues of $g$ is $p$ and the number of negative is $q$. Locally for each point we can find an open neighborhood $U$ parametrized with coordinates $x^\mu$ and vector fields $E_a = E_a^\mu$ such that

$$g(E_a, E_b) = \eta_{ab}\,, \quad 1 \leq a, b \leq n\,,$$

where $\eta_{ab}$ is the $ab$th entry of a diagonal matrix with on the diagonal $\pm 1$. To prove that this is possible, one uses that any symmetric matrix $B$ can be brought to diagonal form $D = ABA^T$, where $D$ has the same signature as $B$. On $U$ we can introduce the dual 1-forms $e^a$ defined by $e^a(E_b)$ is 1 is $a = b$ and zero otherwise. But then we have

$$g|_U = \sum \eta_{ab} e^a \otimes e^b\,.$$

We assume $M$ is equipped with the Levi–Civitá connection. Then for the covariant derivative of $e^a$ we can write $\nabla e^a = \sum_b {\omega^a}_b e^b$ but as the metric is covariantly constant we have

$$\nabla e^a - \sum_b {\omega^a}_b \wedge e^b = 0\,.$$

Since the Levi–Civitá connection preserves the metric and the signature is $(p,q)$, the so-called spin connection coefficients ${\omega^a}_b$ are elements of $C^\infty(T^*U) \otimes \mathfrak{so}(p,q)$, where $\mathfrak{so}(p,q)$ is the Lie algebra of matrices that preserve a metric of signature $(p,q)$. For constructing a connection on the spin bundle one uses the spin connection coefficients; we have a bundle of $\mathfrak{so}(p,q)$-elements and thus we can lift them to spin representation. For those not very familiar with Clifford algebras we advise to skip the upcoming paragraph and to except that with the spin connection coefficients one can lift the Levi–Civitá connection to a connection on the spin bundle (if $M$ admits a spin bundle).

For a more pedestrian introduction to spinors and Clifford algebras, see [11] and [12], Appendix C. Consider the free algebra over $\mathbb{R}$ over the objects $\Gamma^a$ for $1 \leq a \leq n$. We now impose the relation

$$\Gamma^a \Gamma^b + \Gamma^b \Gamma^a = 2\eta^{ab}\,.$$

The algebra we obtain in this way is called the Clifford algebra of signature $(p,q)$. It turns out that there are not many irreducible nontrivial representations of the Clifford algebra; most of them are 1-dimensional. It turns out that the nontrivial representations precisely give the spinorial representations of $\mathfrak{so}(p,q)$. The relation is as follows: from the defining relation one finds that the objects defined by

$$2\Sigma^{ab} = \Gamma^a \Gamma^b - \Gamma^b \Gamma^a$$

satisfy the commutation relations of the standard basis elements of the Lie algebra $\mathfrak{so}(p,q)$. Hence the nontrivial representation of the Clifford algebra gives a nontrivial representation of $\mathfrak{so}(p,q)$. One can check that the representation is irreducible and has the dimension of the spinorial representation; hence it is the spinorial representation. We now define the one-forms $\omega_{ab} = \sum_c {\omega^a}_c \eta^{cb}$, so that we have $\omega_{ab} = -\omega_{ba}$. The connection on the spin bundle is then given by

$$\nabla \psi = \sum_\mu \partial_\mu \psi \otimes \mathrm{d}x^\mu + \sum_{ab} \kappa \Sigma^{ab} \psi \otimes \omega_{ab}\,,$$

where $\kappa$ is some constant that has to be adjusted to fit with the connection on the tangent bundle and where $\psi$ is a spinor.

# 2 Complex manifolds

A classical text on complex manifolds is [5]. For our presentation we have also relied heavily on [10].

We give a short and cryptic definition of what a complex manifold is and let the reader go through the details of polishing the definition.

**Definition 2.1.** *An $n$-dimensional complex manifold is a (paracompact and Hausdorff) topological space with charts that are homeomorphic to open sets (in the usual 'analytic' topology) of $\mathbb{C}^n$ and such that the transition functions are holomorphic functions of the coordinates of $\mathbb{C}^n$.*

To distinguish between the other notion of manifolds, which we modelled by open sets of $\mathbb{R}^n$, we denote the latter by real manifolds. By viewing $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ we see that every complex manifold is also a real manifold. The converse does in general not hold; for a real manifold to be complex, we at least need that the dimension is even. Suppose $M$ is an $n$-dimensional complex manifolds with some charts $U_i$, where $i$ runs over some index set, and with transition functions $f_{ij} : O_i \to O_j$, where the $O_i \subset \mathbb{C}^n$ are the images of the charts. Then we can view the $O_i$ as open sets in $R^{2n}$ and use coordinates $x^\mu$ and $y^\mu$ defined by $z^\mu = x^\mu + iy^\mu$. We write $f_{ij} = u_{ij} + iv_{ij}$; the functions $u_{ij}$ and $v_{ij}$ are real-analytic and thus $\mathbb{C}^\infty$. Furthermore, they satisfy the Cauchy–Riemann equations:

$$\frac{\partial u_{ij}}{\partial x^\mu} = \frac{\partial v_{ij}}{\partial y^\mu}\,, \quad \frac{\partial u_{ij}}{\partial y^\mu} = -\frac{\partial v_{ij}}{\partial x^\mu}\,.$$

Therefore we can rephrase the definition of a complex manifold as a real manifold locally homeomorphic to $\mathbb{R}^{2n}$, described by coordinates $(x^\mu, y^\mu)$ with real-analytic transition functions satisfying the Cauchy–Riemann equations.

**Definition 2.2.** *Let $M$ be a real manifold. We call a tensor $J \in C^\infty(TM \otimes T^*M) \cong C^\infty(\mathrm{End}(TM))$ an almost complex structure if at each point $p$ we have $J_p^2 = -\mathrm{id}_{T_pM}$. When $M$ has an almost complex structure we say that $M$ is an almost complex manifold.*

Let $M$ be a real manifold of dimension $m$ with an almost complex structure $J$. In any coordinate system we can write $J$ locally as

$$J = J^\mu_\nu \frac{\partial}{\partial x^\mu} \otimes \mathrm{d}x^\nu\,.$$

At each point the almost complex structure $J$ can be represented by a real square matrix that squares to the identity, hence is nondegenerate. Thus $(\det J)^2 > 0$ and since $\det J^2 = (-1)^m$ it follows that $m$ must be even, $m = 2k$. At each point $p$ the matrix $J$ can be brought to the form

$$J(p) = \begin{pmatrix} 0 & \mathbb{1}_k \\ -\mathbb{1}_k & 0 \end{pmatrix}\,. \tag{55}$$

One should note that although we can choose this matrixform at each point, it is not guaranteed that in a chart around $p$ we can perform a coordinate change such that $J$ has for each point in the chart the matrixform (55). Each point $p$ on an even dimensional manifold admits a neighborhood $U_p$ such that one can define a tensor that takes the matrixform (55) on $U_p$. The problem of finding an almost complex structure lies in the patching together of the locally defined tensors.

**Lemma 2.3.** *Each complex manifold is almost complex and the almost complex structure $J$ can be chosen to take the matrixform of (55) globally.*

*Proof.* Let $U$ be a coordinate chart with coordinates $z^\mu$. Then we define

$$J_U = i\sum_\mu \frac{\partial}{\partial z^\mu} \otimes \mathrm{d}z^\mu - i\sum_\mu \frac{\partial}{\partial \bar{z}^\mu} \otimes \mathrm{d}\bar{z}^\mu\,.$$

Then on the patch $U$, $J_U$ is well-defined. Suppose $U$ and $V$ are two overlapping coordinate patches where we have defined a $J_U$ and a $J_V$ in this way and suppose $V$

is parametrized with coordinates $w^\mu$. Since the transition functions are holomorphic we have

$$\frac{\partial \bar{w}}{\partial z} = 0\,, \quad \frac{\partial w}{\partial \bar{z}} = 0\,,$$

and with this we find that $J_U$ and $J_V$ define the same tensor on the overlap $\cap V$. Hence we get a globally defined tensor. It is trivial to check that it squares to $-1$. □

It is convenient to complexify the tangent bundle of an almost complex manifold $M$. We thus consider vector fields of the form $X + iY$ where $X$ and $Y$ are smooth real vector fields. In order to do this, we also have to complexify the bundle of smooth functions and the cotangent bundle. We extend the actions of tensors by $\mathbb{C}$-linearity and also the Lie bracket is extended by $\mathbb{C}$-linearity. We write $TM_{\mathbb{C}}$ for the complexified tangent bundle. We introduce the global sections $P^\pm \in C^\infty(TM_{\mathbb{C}} \otimes T^*M_{\mathbb{C}}) \cong C^\infty(\mathrm{End}TM_{\mathbb{C}})$, defined by

$$P^\pm = \frac{1}{2}(1 \mp iJ)\,,$$

and which have the properties $P^+ \circ P^- = 0$, $(P^\pm) = P^\pm$ and $P^+ + P^- = 1$, where 1 denotes the identity element of $C^\infty(\mathrm{End}TM_{\mathbb{C}})$. Hence the sections $P^\pm$ define projection operators with eigenspace decomposition $T_pM_{\mathbb{C}} = T_pM_C^+ \bigoplus T_pM_{\mathbb{C}}^-$ where for each point $p$ and where $T_pM_C^\pm$ consists of the complex tangent vectors $v$ at $p$ satisfying $J(p)v = \pm iv$. Since $P^\pm$ are smooth tensors one can write for any complex vector field $X$ the decomposition $X = X^+ + X^-$, with $X^\pm = P^\pm$ and thus bundle $TM_{\mathbb{C}}$ splits as $TM_{\mathbb{C}} = TM_{\mathbb{C}}^+ \oplus TM_{\mathbb{C}}^-$. One calls a section of $TM_{\mathbb{C}}^+$ a holomorphic vector field and a section of $TM_{\mathbb{C}}^-$ an anti-holomorphic vector field.

We can extend the action of the projection operators $P^\pm$ to one-forms by putting $(P^\pm\omega)(X) = \omega(P^\pm X)$. For differential $r$-forms we define the projection operators $P^{(p,q)}$ with $r = p + q$ as follows: for two-forms we put

$$(P^{(2,0)}\omega)(X,Y) = \omega(P^+X, P^+Y)\,, \quad (P^{(0,2)}\omega)(X,Y) = \omega(P^-X, P^-Y)\,,$$
$$(P^{(1,1)}\omega)(X,Y) = \omega(P^+X, P^-Y) - \omega(P^+Y, P^-X) = \omega(P^+X, P^-Y) + \omega(P^+X, P^-Y)\,.$$

Thus for general $r$-forms we put

$$P^{(p,q)}\omega(X_1, \ldots, X_r) = \sum_{\sigma_i = \pm, \# + = p} \omega(P^{\sigma_1}X_1, \ldots, P^{\sigma_r}X_r)\,,$$

where the summation runs over all possible distinct choices of the signs $\sigma_i$ and where the number of plus signs is $p$ (and thus the number of minus signs is $q$).

**Exercise 2.4.** *Verify that for any $r$-form $\omega$ with $p + q = r$, the $r$-form $P^{(p,q)}\omega$ is an section of $\bigwedge^r T^*M$, that is, show that*

$$P^{(p,q)}\omega(X_1, \ldots, X_r)$$

*is antisymmetric in the $X_i$. Furthermore show that $\sum_{p+q=r} P^{(p,q)}\omega = \omega$.*

We call an $r$-form $\omega$ an $(p,q)$-form with $p + q = r$ if $P^{(p,q)}\omega = \omega$. We write $T^*M^{(p,q)}$ for the bundle of $(p,q)$-forms. We thus have

$$\bigwedge^r T^*M_{\mathbb{C}} = \sum_{p+q=r} T^*M^{(p,q)}\,.$$

It is an easy exercise in linear algebra to verify that if $\alpha$ is an $(p_1, q_1)$-form and $\beta$ an $(p_2, q_2)$-form, then $\alpha \wedge \beta$ is an $(p_1 + p_2, q_1 + q_2)$-form. It follows that each $(p,q)$-form can at least locally be written as a sum of products of $p$ $(1,0)$-forms and $q$ $(0,1)$-forms. If $\{\theta_i\}_{i=1}^k$ is a local basis for $T^*M^{(1,0)}$ over $\mathbb{C}$, then with some linear algebra (at the end of this section there is huge exercise on linear algebra with almost complex structures) one can see that the complex conjugates $\bar{\theta}_i$ form a basis for $T^*M^{(0,1)}$. Since we have $T^*M_{\mathbb{C}} = T^*M^{(1,0)} \oplus T^*M^{(0,1)}$ it follows that locally we have $\bigwedge^2 T^*M = T^*M^{(1,0)} \wedge T^*M^{(1,0)} \oplus T^*M^{(1,0)} \wedge T^*M^{(0,1)} \oplus T^*M^{(0,1)} \wedge T^*M^{(0,1)}$ and thus we can write

$$\mathrm{d}\theta_i = \sum_{j,k} a_{i,jk}\theta_j \wedge \theta_k + b_{i,jk}\theta_j \wedge \bar{\theta}_k + c_{i,jk}\bar{\theta}_j \wedge \bar{\theta}_k \,.$$

**Exercise 2.5.** *Verify that if $\omega$ is an $(p,q)$-form, then*

$$\mathrm{d}\omega \in T^*M^{(p-1,q+2)} \oplus T^*M^{(p,q+1)} \oplus T^*M^{(p+1,q)} \oplus T^*M^{(p+2,q-1)} \,.$$

**Definition 2.6.** *Let $M$ be an almost complex manifold. If the Lie bracket of two holomorphic vector fields is again holomorphic, we say that the complex structure is integrable.*

We remark that if $J$ is integrable, then the Lie bracket of two antiholomorphic vector fields is antiholomorphic. This follows since, if $X$ is antiholomorphic, then the complex conjugate is holomorphic (remember that $J\bar{X} = \overline{JX}$) and since the Lie bracket is compatible with complex conjugation; $\overline{[X,Y]} = [\bar{X},\bar{Y}]$ for any two smooth vector fields $X$ and $Y$.

**Exercise 2.7.** *Verify that a complex structure is integrable if and only if for any two vector fields $X, Y$ we have $P^-[P^+X, P^+Y] = 0$.*

Later we will find it convenient to use the following tensor for testing whether an almost complex manifold might be given the structure of a complex manifold.

**Definition 2.8.** *Let $M$ be an almost complex manifold with almost complex structure $J$, then we define the Nijenhuis tensor $N_J \in C^\infty(\bigwedge^2 T^*M \otimes TM)$ by*

$$N_J(X,Y) = [X,Y] + J[X,JY] + J[JX,Y] - [JX,JY] \,,$$

*for all smooth vector fields $X, Y$.*

**Exercise 2.9.** *Verify that $N_J$ is a tensor (thus $N_J(fX,gY) = fgN_J(X,Y)$ and additivity) and that $N_J(X,Y) = -N_J(Y,X)$. Furthermore, choose some local coordinates and find a local expression for $N_J$.*

**Theorem 2.10.** *An almost complex structure is integrable if and only if $N_J(X,Y) = 0$ for any two smooth vector fields $X$ and $Y$.*

*Proof.* Suppose the Nijenhuis tensor vanishes. Let $X$ and $X$ be two holomorphic vector fields; $JX = iX$ and $JY = iY$. Then $N_J(X,Y) = 0$ implies

$$0 = N_J(X,Y) = 2[X,Y] + 2iJ[X,Y] \,,$$

and thus $J[X,Y] = i[X,Y]$. Now suppose that $J$ is integrable. Take $X, Y$ any two vector fields and write $X = Z + \bar{Z}$ and $Y = W + \bar{W}$ where $Z, W$ are holomorphic and $\bar{Z}, \bar{W}$ are antiholomorphic. Then one checks easily that $N_J(Z,\bar{W}) = N_J(\bar{Z},W) = 0$. Then we find

$$N_J(X,Y) = 2[Z,W] + 2iJ[Z,W] + 2[\bar{Z},\bar{W}] - 2i[\bar{Z},\bar{W}] \,,$$

which vanishes by holomorphicity of $Z, W$ and antiholomorphicity of $\bar{Z}, \bar{W}$. □

The following theorem is very important and easy to remember but we won't proof it:

**Theorem 2.11** (Nirnberg–Newlander - 1957)**.** *Let $M$ be an almost complex manifold with an almost complex structure $J$. If $J$ is integrable, then $M$ is a complex manifold.*

**Exercise 2.12.** *Verify that if $M$ is a complex manifold then $N_J$ is identically zero.*

Let $M$ be an almost complex manifold with almost complex structure $J$. If $J$ is integrable then it follows from lemma ... that for any $(1,0)$-form $\chi$ we have

$$\mathrm{d}\chi \in T^*M^{(2,0)} \oplus T^*M^{(1,1)} .$$

**Exercise 2.13.** *Verify that on a complex manifold (that is, an almost complex manifold with integrable almost complex structure) we have for any $(p,q)$-form $\omega$*

$$\mathrm{d}\omega \in T^*M^{(p,q+1)} \oplus T^*M^{(p+1,q)} .$$

We can now define the Dolbeault operators $\partial$ and $\bar{\partial}$ as follows: for any $(p,q)$-form $\omega$ we write $\mathrm{d}\omega = \partial\omega + \bar{\partial}\omega$ with $\partial\omega = P^{(p+1,q)}\mathrm{d}\omega$ and $\bar{\partial}\omega P^{(p,q+1)}\mathrm{d}\omega$.

**Exercise 2.14.** *Verify that $\partial\partial = 0$ and $\bar{\partial}\bar{\partial} = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$.*

Using the Dolbeault operators we can define a cohomology for complex manifolds: one defines the abelian groups $H^{p,q}(M)$ as follows

$$H^{p,q}(M) = \mathrm{Ker}(\bar{\partial} : H^{p,q} \to H^{p,q+1})/\mathrm{Im}(\bar{\partial} : H^{p,q-1} \to H^{p,q}) .$$

One can show that $H^{p,q} = \bar{H^{q,p}}$ and $H^{p,q} \cong H^{k-p,k-q}$ if $M$ is $2k$-dimensional and compact. We define the Hodge numbers $h^{p,q}(M)$ as the dimension of $H^{p,q}(M)$. Very often the Hodge numbers are represented in a diamond-like figure called the Hodge diamond. We have

$$h^r(M) = \mathrm{dim}H^r(M) = \sum_{r=p+q} h^{p,q}(M) ,$$

where the numbers $h^r(M)$ are called the Betti-numbers.

We end this discussion on complex manifolds with a nice result, which is easy to remember and which puts complex algebraic curves on a nice pedestal:

**Lemma 2.15.** *Every orientable two-dimensional Riemannian manifold is a complex manifold.*

*Proof.* Let $g$ be the metric on the manifold. We know $g$ is positive definit at each point. Hence we can choose the charts small enough such that the metric on each chart is given by a positive function times the identity matrix. Let $U$ and $V$ be two overlapping charts parametrized by coordinates $(u_1, u_2)$ and $(v_1, v_2)$ respectively. In the sequel, a suffix $U$ $(V)$ refers to the $U$-chart ($V$-chart). In the $U$- and $V$-chart we get the following expressions $g_U$ and $g_V$ for the metric:

$$g_U = \lambda_U^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad g_V = \lambda_V^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for some smooth functions $\lambda_U$ and $\lambda_V$. The given form of the metrics gives us two equations for the transition functions for the coordinates:

$$\frac{\partial v_1}{\partial u_1}\frac{\partial v_1}{\partial u_2} + \frac{\partial v_2}{\partial u_1}\frac{\partial v_2}{\partial u_2} = 0 \tag{56}$$

$$\left(\frac{\partial v_1}{\partial u_1}\right)^2 + \left(\frac{\partial v_2}{\partial u_1}\right)^2 = \left(\frac{\partial v_1}{\partial u_2}\right)^2 - \left(\frac{\partial v_2}{\partial u_2}\right)^2 \tag{57}$$

where (56) follows from the vanishing of the diagonal components of $g_V$ and where (57) follows from the fact that the diagonal components change in the same way. We introduce complex coordinates $z = u_1 + iu_2$ and $w = v_1 + iv_2$. We calculate

$$\frac{\partial w}{\partial z} = \frac{\partial(v_1 + iv_2)}{\partial u_1} - i\frac{\partial(v_1 + iv_2)}{\partial u_2} \tag{58}$$

and also $\frac{\partial \bar{w}}{\partial z}$ and multiply them and using (56) and (57) we find

$$\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial z} = 0\,.$$

If $\frac{\partial \bar{w}}{\partial z} = 0$ we are done, since then the transition function is holomorphic. In the other case we use (58) and obtain that

$$\det\frac{\partial(v_1, v_2)}{\partial(u_1, u_2)} = \det\begin{pmatrix} \frac{\partial v_1}{\partial u_1} & \frac{\partial v_1}{\partial u_2} \\ \frac{\partial v_1}{\partial u_2} & -\frac{\partial v_1}{\partial u_1} \end{pmatrix} < 0\,. \tag{59}$$

But since the manifold is orientable we have a two-form $\omega$ that vanishes nowhere and in the $U$-chart can be expressed as

$$\omega_U = f_U^2(u_1, u_2)\mathrm{d}u_1 \wedge \mathrm{d}u_2\,,$$

for some smooth function $f$. We can find a similar expression in the $V$-chart and in order that these expressions are compatible we must have that the determinant (59) is positive. Therefore, $w$ must be a holomorphic function of $z$. □

## 2.1 Linear algebra for almost complex structures

This subsection is devoted to some linear algebra of vector spaces with an almost complex structure. We let $V$ be a real vector space of dimension $2m$ and let $J$ be an endomorphism of $V$ with $J^2 = -1$. We write $V_{\mathbb{C}}$ for the complexified vector space $V \otimes_{\mathbb{R}} \mathbb{C}$. We write $V^*$ for the dual of $V$ and $V_{\mathbb{C}}^*$ for the dual of $V_{\mathbb{C}}$. We use the same notation as above for alternating forms and projection operators.

**Exercise 2.16.** *(i) Show that $\omega$ is an $(1,0)$-form on $V_{\mathbb{C}}$ if and only if $\omega(Jv) = i\omega(v)$ for all $v$.*

*(ii) Let $f_1, \ldots, f_{2m}$ be a basis for $V_{\mathbb{C}}^*$. Show that the $P^+ f_i$ span $V_{\mathbb{C}}^{*(1,0)}$. Then we may assume that the first $m$ of the $P^+ f_i$ are linearly independent over $\mathbb{C}$ and are a basis for $V_{\mathbb{C}}^{*(1,0)}$.*

*(iii) Write $P^+ f_i = \lambda_i + i\mu_i$ for $1 \le i \le m$ and where $\lambda_i, \mu_i$ are elements of $V^*$. Show that*

$$\lambda_i(Jx) = -\mu_i(x)\,, \quad \mu_i(Jx) = \lambda_i(x)\,, \quad \forall x\,.$$

*(iv) Show that the $\lambda_i$ and the $\mu_i$ together form a basis for $V^*$. Hint: you have to show that the are linearly independent over $\mathbb{R}$; to do so define $\overline{P^+ f_i} = \lambda_i - i\mu_i$ and show that they are in $V_{\mathbb{C}}^{*(0,1)}$. Express a linear dependence in $P^+ f_i$ and $\overline{P^+ f_i}$, apply to $v \in V$ and to $Jv$ and use that the $P^+ f_i$ are linearly independent over $\mathbb{C}$.*

*(v) Conclude that the elements $\overline{P^+ f_i} = \lambda_i - i\mu_i$ form a basis for $V_{\mathbb{C}}^{*(0,1)}$.*

*(vi) Prove that $\Omega = \bigwedge_i P^+ f_i \wedge \bigwedge_i \overline{P^+ f_i} \neq 0$.*

*(vii) Perform a basis change and observe that $\Omega$ changes by a multiplication with a positive real number. Conclude that an almost complex manifold is orientable.*

*(viii) Suppose we forget about $J$ and are given a decomposition of $V_{\mathbb{C}}^* = W \oplus \overline{W}$, where $\overline{W}$ is the complex conjugate of $W$. Let $f_1, \ldots, f_m$ be a basis for $W$ and write $f_i = a_i + ib_i$ with $a_i, b_i \in V^*$. Show that the $a_i$ and $b_i$ are linearly independent over $\mathbb{R}$.*

*(ix) We define an endomorphism $J^t$ on $V_{\mathbb{C}}^*$ by putting $J^t\varphi = i\varphi$ if $\varphi \in W$ and $J^t\varphi = -i\varphi$ if $\varphi \in \overline{W}$. Express $J^t$ in terms of the basis elements $a_i$ and $b_i$. Conclude that $J^t$ is in fact a real endomorphism, that is, maps $V^*$ to $V^*$.*

*(x) Define a map $J$ on $V$ as follows: $Jx$ is determined by $\varphi(Jx) = (J^t\varphi)(x)$ for all $\varphi \in V^*$. Show that $J^t$ is an almost complex structure.*

*(xi) Conclude that we can define an almost complex structure equivalently as a decomposition $W \oplus \overline{W}$ of the space of complex 1-forms.*

## 2.2 The complex structure on $S^2$

There are more or less two ways to get the complex structure on $S^2$; either we give the complex coordinates directly and verify that the transition functions are holomorphic, or we use some geometric insight. We first walk along the geometric way and then arrive at the complex coordinates.

Pick a point $p = (x, y, z)$ on the sphere. The tangent space at $p$ consists of those vectors in $\mathbb{R}^3$ that are perpendicular to the vector $(x, y, z)$. Thus the outer product of two tangent vectors at $p$ is proportional to $(x, y, z)$. Furthermore, taking the outer product with $(x, y, z)$ is a linear map from the tangent space at $p$ to the tangent space at $p$, which we will denote $J$. We thus have a well-defined map $J : T_pS^2 \to T_pS^2$ given by

$$J : \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} .$$

And by using some vector calculus we see $J^2 = -1$. We thus have an almost complex structure.

We want to know the expression of $J$ in the $U$-chart. We use the notation from section 1.8. Using the expressions for $x$, $y$ and $z$ we calculate

$$J(Tf_U^{-1}(e_1)) = \frac{2}{(1 + ||u||^2)^3} \begin{pmatrix} 2u_1 \\ 2u_2 \\ -1 + ||u||^2 \end{pmatrix} \times \begin{pmatrix} 1 + u_2^2 - u_1^2 \\ -2u_1u_2 \\ 2u_1 \end{pmatrix}$$

and find after some algebra that

$$J(Tf_U^{-1}(e_1)) = -Tf_U^{-1}(e_2) .$$

Similarly we find

$$J(Tf_U^{-1}(e_2)) = Tf_U^{-1}(e_1) .$$

We conclude that the almost complex structure is on $U$ represented by the constant matrix $J_U$ given by

$$J_U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

**Exercise 2.17.** *Do the same calculation for the $V$-chart to find that $J$ is represented in the $V$-chart by the matrix $J_V$, which equals*

$$J_V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -J_U .$$

In exercise 2.9 you were asked to use local coordinates $x^\mu$ and to calculate a local expression for the Nijenhuis tensor. If all went right you should have found: If we write $J(\partial_\mu) = \sum_\nu J_\mu{}^\nu \partial_\nu$ and $N_J(\partial_\mu, \partial_\nu) = \sum_\sigma N_{\mu\nu}{}^\sigma \partial_\sigma$ where $\partial_\mu = \partial/\partial x^\mu$ then

$$N_{\mu\nu}{}^\sigma = \sum_\rho J_\rho{}^\sigma(\partial_\mu J_\nu{}^\rho - \partial_\nu J_\mu{}^\rho) - \sum_\rho (J_\mu{}^\rho \partial_\rho J_\nu{}^\sigma - J_\nu{}^\rho \partial_\rho J_\mu{}^\sigma) .$$

Since the expression for $J$ is constant in each chart, we conclude that $N_J$ is zero for the two-sphere. Hence the sphere is a complex manifold. But of course, we already knew that. Indeed, the two-sphere is Riemannian, two-dimensional and orientable, hence must be complex.

We have seen that the expressions for $J$ in both charts differ by a minus sign. We now verify that this is compatible with the coordinate transition functions. We introduce the short-hand notation

$$\Delta = \frac{\partial(v_1, v_2)}{\partial(u_1, u_2)} .$$

Since the complex structure is a $(1,1)$-tensor we need that $J_V = \Delta J_U \Delta^{-1}$, that is $J_U\Delta + \Delta J_U = 0$. Using the explicit expression for $\Delta$ given in eqn.(49) we see that indeed we have $J_U\Delta+\Delta J_U = 0$. (We remark that one can rephrase the compatibility constraint $J_U\Delta + \Delta J_U = 0$ as $\Delta \in \mathfrak{sp}_2$, which equals $\mathfrak{sl}_2$ and indeed $\Delta$ is traceless.)

Let us now find the complex coordinates on the sphere. We complexify the tangent space first, that is, we allow complex valued vectors. From the form of $J_U$ we see that in $U$ the vector

$$t = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

satisfies $J_U t = it$ and $J_U \bar{t} = -i\bar{t}$. In $V$ the roles of $t$ and $\bar{t}$ are interchanged. But in $U$ we have

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \leftrightarrow \frac{\partial}{\partial u_1} + i\frac{\partial}{\partial u_2}\,,$$

which we thus define to be $2\frac{\partial}{\partial z}$. It then follows that $z = u_1 - iu_2$, indeed when we put

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial u_1} + i\frac{\partial}{\partial u_2}\,, \quad z = u_1 - iu_2\,, \tag{60}$$

$$2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial u_1} - i\frac{\partial}{\partial u_2}\,, \quad \bar{z} = u_1 + iu_2\,, \tag{61}$$

then

$$\frac{\partial}{\partial z}z = 1\,, \quad \frac{\partial}{\partial z}\bar{z} = 0\,, \quad \frac{\partial}{\partial \bar{z}}z = 0\,, \quad \frac{\partial}{\partial \bar{z}}\bar{z} = 1\,.$$

In the $V$-chart we can play the same game and get coordinates $w$ given by

$$2\frac{\partial}{\partial w} = \frac{\partial}{\partial v_1} - i\frac{\partial}{\partial v_2}\,, \quad w = v_1 + iv_2\,, \tag{62}$$

$$2\frac{\partial}{\partial \bar{w}} = \frac{\partial}{\partial v_1} + i\frac{\partial}{\partial v_2}\,, \quad \bar{w} = v_1 - iv_2\,. \tag{63}$$

We then find that the coordinate transition functions take the form

$$f_V \circ f_U^{-1} : z \mapsto w = \frac{1}{z}\,,$$

which indeed is holomorphic on the complex plane minus the origin (which is the Riemann sphere minus infinity and minus origin). For the metric we find (after some rescaling) that the matrix representations $G_U$ and $G_V$ in the $U$- and $V$-chart respectively are given by

$$G_U = \frac{1}{(1+||z||^2)^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad G_V = \frac{1}{(1+||w||^2)^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{64}$$

In other words $G_U = \frac{1}{(1+||z||^2)^2}(\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z)$ and similar for $G_V$. We have a globally defined volume-form $\omega$ defined in the $U$-chart by

$$\omega_U = \frac{i}{(1+||z||^2)^2}\mathrm{d}z \wedge \mathrm{d}\bar{z}\,,$$

and similarly in the $V$-chart. We can integrate $\omega$ over $S^2$ and get a volume. Since the $U$-chart covers the two-sphere up to a point it suffices to calculate the integral in the $U$-plane: we can write $z = re^{\imath\phi}$ so that $i\mathrm{d}z \wedge \mathrm{d}\bar{z} = 2ir\mathrm{d}r\mathrm{d}\varphi$ and hence

$$\int_{S^2} \omega = \int_0^\infty \frac{2r}{(1+r^2)^2} \int_0^{2\pi} \mathrm{d}\varphi = \pi\,. \tag{65}$$

# 3 Kähler manifolds

Let $M$ be a Riemannian manifold with an almost complex structure $J$ and metric $g$. We say that $g$ is Hermitian with respect to the almost complex structure $J$ if $g(X,Y) = g(JX, JY)$ for all smooth vector fields $X$ and $Y$. We often use the shorter expression that $g$ is Hermitian.

If $g$ is Hermitian then we have $g(X, JY) = -g(JX, Y) = -g(Y, JX)$. Hence the metric $g$ and the almost complex structure $J$ define a 2-form on $M$:

$$\omega(X,Y) = g(X, JY)\,.$$

We call $\omega$ the Hermitian form associated to $g$ and $J$. We complexify the tangent space and investigate the decomposition of $\omega$ into $T^*M^{(2,0)} \oplus T^*M^{(1,1)} \oplus T^*M^{(0,2)}$. We extend the action of $g$ and $J$ by $\mathbb{C}$-linearity. We note that if $g$ is any metric, then the metric $\tilde{g}$ defined by $\tilde{g}(X,Y) = g(X,Y) + g(JX, JY)$ is an Hermitian metric. We leave it as an exercise to verify that $\tilde{g}$ is indeed positive-definite.

**Lemma 3.1.** *The Hermitian form $\omega \in \bigwedge^2 T^*M$ lies in $T^*M^{(1,1)}$.*

*Proof.* Let $X$ and $Y$ be both holomorphic or antiholomorphic, then

$$\omega(X,Y) = g(X, JY) = \pm i g(X,Y)\,,$$

on the one hand, but

$$\omega(X,Y) = -g(JX,Y) = - \pm\, i g(X,Y)\,.$$

Hence $P^{(2,0)}\omega = P^{(0,2)}\omega = 0$. □

From the proof lemma 3.1 we see that an Hermitian metric must have a special form on the complexified tangent bundle. Indeed, let $X$ and $Y$ be both holomorphic or antiholomorphic, then $g(X,Y) = g(JX,JY) = -g(X,Y)$. Hence we get a sesquilinear form $h$ on the holomorphic vectorbundle $TM^{(1,0)}$ by defining $h(X,Y) = g(X,\bar{Y})$, where $X, Y$ are smooth holomorphic vector bundles and where $\bar{Y}$ denotes the complex conjugate of $Y$.

Now let $M$ be a complex Riemannian manifold of dimension $2n$ with almost complex structure $J$ (which is thus a complex structure) and with an Hermitian metric $g$. When we view $M$ as a real manifold with special transition functions, then we can say something more on the structure group of the (real) tangent bundle. Let $U$ be a coordinate chart with coordinates $x^\mu$ having a nonempty intersection with the coordinate chart $V$ with coordinates $y^\mu$. Since the metric is globally defined, we may choose the coordinates at any point $p \in U \cap V$ to be given in matrixform by the identity matrix. Thus for both $x$-coordinates and $y$-coordinates we may assume the metric is given by the identity matrix at $p$. If not, then we perform a coordinate transformation such that the metric at $p$ is given by the identity matrix, thereby making $U$ and $V$ maybe smaller but still containing $p$. But then the transition functions $\Phi_{UV}$ must preserve the standard inner product on $\mathbb{R}^{2n}$ and hence we may assume the charts are small enough such that $\Phi_{UV} \in O(2n)$. Such a refinement of the charts where the transition functions take values in a smaller group than $GL_{\mathbb{R}}(\dim M)$ is called a reduction of the structure group, already alluded at in section 1.1. At present, we have the structure group reduction $GL_{\mathbb{R}}(2n) \to O(2n)$, due to $M$ being Riemannian.

In the case of a complex manifold with an Hermitian metric, the structure group can be reduced even further. We fix a point $p$ and use that at $p$ we can take the metric to be given in matrixform by the identity matrix. Then $g$ being Hermitian implies that $J^tJ = 1$, where $^t$ denotes transpose. But then $J^t = -J$ and thus $J$ is antisymmetric. With some linear algebra (left as exercise; hint: every skew-Hermitian matrix admits an ortonormal basis with respect to the standard inner product $\langle x, y\rangle = \sum_i \bar{x}_i y_i$) one shows that one can bring $J$ by means of an $O(2n)$-transformation to the form

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{66}$$

Thus the transition functions of the tangent bundle can be chosen to preserve this form. Thus, the transition functions are matrices $H$ in $O(2n)$ satisfying the further constraint: $HJH^{-1} = J$, which is equivalent to $HJ = JH$. Writing $H$ in $n \times n$ blocks as

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we find that $H$ must take the form

$$H = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} . \tag{67}$$

The fact that $H \in O(2n)$ can then seen to be equivalent to the statement that $A+iB$ defines an element of $U(n)$. In fact, we have a morphism of groups $\iota : U(n) \to O(2n)$ mapping $A + iB$, with $A$ and $B$ real matrices, to the matrix $H$ in eqn.(67). As $\iota$ is clearly injective, $U(n)$ is a subgroup of $O(2n)$. All in all we conclude that the structure group of $M$ can be reduced to the $U(n)$-subgroup of $O(2n)$.

We now to come the definition of a Kähler manifold. Since there are a view equivalent definitions, we need to pave the way by means of some technical lemma's.

**Lemma 3.2.** *Let $(M, J, g)$ be a complex manifold with complex structure $J$, (pseudo-) Riemannian Hermitian metric $g$, $\omega$ the associated Hermitian form and $\nabla$ the Levi–Civitá connection. Then we have*

$$\mathrm{d}\omega(X, Y, Z) - \mathrm{d}\omega(X, JY, JZ) + 2g((\nabla_X J)Y, Z) = 0 . \tag{68}$$

*Proof.* The proof is a classical work-and-write proof and once one does the proof it is easy. To find the proof is another world. We therefore indicate what to do and leave the relatively easy, but nontrivial algebra to the reader.

First we apply Koszul's formula 46 to obtain:

$$\begin{aligned} 2g(\nabla_X(JY), Z) = X(g(JY, Z)) + JY(g(X, Z)) - Z(g(X, JY)) \\ + g([X, JY], Z) - g([X, Z], JY) - g([JY, Z], X) \end{aligned} \tag{69}$$

and similarly

$$\begin{aligned} 2g(\nabla_X Y, JZ) = X(g(Y, JZ)) + Y(g(X, JZ)) - JZ(g(X, Y)) \\ + g([X, Y], JZ) - g([X, JZ], Y) - g([Y, JZ], X) \end{aligned} \tag{70}$$

which equals $-2g(J\nabla_X Y, Z)$. The remaining part of the proof consists of using lemma 1.14 part $(ii)$ to write out $\mathrm{d}\omega(X, Y, Z)$ and $\mathrm{d}\omega(X, JY, JZ)$. Then one can calculate

$$\mathrm{d}\omega(X, Y, Z) - \mathrm{d}\omega(X, JY, JZ) + 2g((\nabla_X J)Y, Z)$$

by noting that

$$2g((\nabla_X J)Y, Z) = 2g(\nabla_X(JY), Z) - 2g(J\nabla_X Y, Z) .$$

By adding all the intermediate results and using that $g(JV, JW) = g(V, W)$ one sees most terms cancel except a term proportional to

$$g([Y, Z], JX) - g([JY, JZ], JX) + g(J[JY, Z], JX) + g(J[Y, JZ], JX)$$

which is nothing more than $g(N_J(Y, Z), JX)$, which vanishes as $N_J \equiv 0$. □

**Theorem 3.3.** *Let $(M, J, g)$ be a complex manifold with complex structure $J$, (pseudo-) Riemannian Hermitian metric $g$, $\omega$ the associated Hermitian form and $\nabla$ the Levi–Civit'a connection. Then the following conditions are equivalent:*

*(i)* $\nabla\omega = 0\,,$

*(ii)* $\nabla J = 0\,,$

*(iii)* $d\omega = 0\,.$

*Proof.* Note that $\nabla\omega = 0$ if and only if $\nabla_Z\omega = 0$ for all smooth vector fields $Z$, and that $\nabla J = 0$ if and only if $\nabla_Z J = 0$ for all smooth vector fields $Z$. Now we calculate

$$\begin{aligned}(\nabla_Z\omega)(X,Y) &= Z(\omega(X,Y)) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y)\\ &= Z(g(X,JY)) - g(\nabla_Z X, JY) - g(X, J\nabla_Z Y)\\ &= g(\nabla_Z X, JY) + g(X, \nabla_Z(JY)) - g(\nabla_Z X, JY) - g(X, J\nabla_Z Y)\\ &= g(X, (\nabla_Z J)Y)\,.\end{aligned}$$

Hence $\nabla\omega = 0$ if and only if $\nabla_Z JY = 0$ for all $Y$ (as $g$ is nondegenerate) and thus if and only if $\nabla_Z J = 0$.

To get the remaining equivalence (which is done wrong in [2]) we write

$$\begin{aligned}(\nabla_X\omega)(Y,Z) &= X(\omega(Y,Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z)\,,\\ (\nabla_Y\omega)(Z,X) &= Y(\omega(Z,X)) - \omega(\nabla_Y Z, X) - \omega(Z, \nabla_X Y)\,,\\ (\nabla_Z\omega)(X,Y) &= Z(\omega(X,Y)) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y)\,.\end{aligned}$$

Now we add the three equations above and using $[X,Y] = \nabla_X Y - \nabla_Y X$ and using the second identity of lemma 1.14we obtain

$$(\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(Z,X) + (\nabla_Z\omega)(X,Y) = \mathrm{d}\omega(X,Y,Z)\,. \tag{71}$$

It follows that if $\nabla_X\omega = 0$ for all vector fields $X$, then $\mathrm{d}\omega = 0$. For the converse we use lemma 3.2:

$$\mathrm{d}\omega(X,Y,Z) - \mathrm{d}\omega(X,JY,JZ) + 2g((\nabla_X J)Y, Z) = 0$$

for all vector fields $X, Y, Z$. If then $\mathrm{d}\omega = 0$, it follows that (as $g$ is nondegenerate) that $\nabla_X J = 0$. □

**Definition 3.4.** *Any Riemannian complex manifold $(M, J, g)$ that satisfies one of the equivalent conditions of theorem 3.3 is called Kähler manifold.*

The following lemma is quite useful in recognizing a Kähler manifold without too much effort:

**Lemma 3.5.** *Let $M$ be a manifold with a Riemannian metric and an almost complex structure $J$ that satisfies $\nabla J = 0$, where $\nabla$ is the Levi–Civitá connection. Then $M$ is a complex manifold and a Kähler manifold.*

*Proof.* We show that if $\nabla J = 0$, then the Nijenhuis tensor vanishes. Since the Levi–Civitá connection has no torsion we have $[X,Y] = \nabla_X Y - \nabla_Y X$. But then $J[X,JY] = J\nabla_X(JY) - J\nabla_{JY}X = -\nabla_X Y - J\nabla_{JY}X$ and we get similar expressions for the other terms. Hence

$$\begin{aligned}N_J(X,Y) &= \nabla_X Y - \nabla_Y X + J\nabla_{JX}Y + \nabla_Y X\\ &\quad - \nabla_X Y - J\nabla_{JY}X - J\nabla_{JX}Y + J\nabla_{JY}X\\ &= 0\,.\end{aligned}$$

□

**Exercise 3.6.** *Let $M$ be a Kähler manifold with almost complex structure $J$, and Riemannian Hermitian metric $g$. Let $U$ be a chart that can be described with complex coordinates $z^\mu$. Write down the general expression for the metric as $\sum_{\mu,\nu} g_{\mu\nu} \mathrm{d}z^\mu \otimes \mathrm{d}z^\nu$ for some smooth functions $g_{\mu\nu}$. Find the expression for the Hermitian form $\omega$. Write down the local expression for the differential equation for $\mathrm{d}\omega = 0$.*

Since on a Kähler manifold the complex structure $J$ is covariantly constant, that is, $\nabla J = 0$, the Levi–Civitá connection preserves the decomposition $TM_{\mathbb{C}} = TM^{(1,0)} \oplus TM^{(0,1)}$. Indeed, if $X$ is an holomorphic vector field, then $JX = iX$ and $J\nabla_Z X = \nabla_Z JX = i\nabla_Z X$ for any smooth vector field $Z$. Hence we can restrict the Levi–Civitá connection to the bundle of holomorphic vector fields: $\nabla : TM^{(1,0)} \to TM^{(1,0)} \otimes T^*M_{\mathbb{C}}$. This of course has consequences for the precise form of the Christoffel symbols. For the Riemann tensor it implies that if $Z_1, Z_2$ are two smooth vector fields and $X$ is holomorphic, then $R(Z_1, Z_2)X$ is holomorphic.

A peculiar property of the Hermitian form on a Kähler manifold is that it gives rise to a so-called Kähler potential, which has found its applications in particle physics. We have already seen that $\omega$ is a $(1,1)$-form, hence locally it can be written as

$$\omega = \sum_{\mu\nu} \omega_{\mu\nu} \mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\nu \,.$$

From $\mathrm{d}\omega = 0$ it follows that $\omega = \mathrm{d}\chi$ locally. We can write $\chi = \chi^+ + \chi^-$, where $chi^+$ ($\chi^-$) is the holomorphic (antiholomorphic) part of $\chi$. Since $\omega$ is an $(1,1)$-form we must have $\partial\chi^+ = 0$ and $\bar{\partial}\chi^- = 0$. Using an adapted version of Poincaré's Lemma we see that $\chi^+ = \partial f$ and $\chi^- = \bar{\partial}\tilde{f}$ locally for some functions $f, \tilde{f}$. Buth then $\omega = \partial\bar{\partial}\tilde{f} + \bar{\partial}\partial f = \partial\bar{\partial}(\tilde{f} - f)$ and we conclude that locally the components $g_{\mu\nu}$ of the metric can be written as

$$g_{\mu\nu} = \frac{\partial^2}{\partial z^\mu \partial \bar{z}^\nu} K \,, \tag{72}$$

where $K$ is some real-valued function, called the Kähler potential.

In definition 1.44 we defined the Ricci tensor. The Ricci tensor is not a two-form. For complex manifolds and especially for Kähler manifolds there is a nice way to embody the same information of the Ricci tensor into a two-form, which for cohomological reasons behaves nicer. The idea is to contract the complex form with the Ricci tensor; the result is a two-form. In exercise 1.45 you were asked to give a local expression for the Ricci tensor $Ric = Ric_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$. From this local expression and some properties of the Riemann tensor it is not too hard to show that $Ric_{\mu\nu} = Ric_{\nu\mu}$, i.e., $Ric(X,Y) = Ric(Y,X)$, see e.g. [13]. We define the Ricci-form associated to $Ric$ and $J$ to be the tensor $\rho$ that is given by

$$\rho(X,Y) = Ric(JX, Y) \,.$$

Without a proof we then state:

**Lemma 3.7.** *The Ricci-form $\rho$ is a two-form, that is, $\rho(X,Y) = -\rho(Y,X)$.*

To proof the lemma, we would really need to prove some additional properties of the Riemann tensor. The most important thing to remember is that the Ricci-form and the Ricci-tensor contain the same information: knowing the one, you know the other. In particular, a metric is Ricci-flat if and only if the Ricci-form vanishes.

## 3.1 The Kähler structure on $S^2$

When we use the form of the metric given in eqn.(53) we verify directly that the metric is Hermitian; what it boils down to is the verification that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Using the expression (64 for the metric we find that the Hermitian form is in the $U$-chart given by

$$\omega = \frac{i}{(1+|z|^2)} \mathrm{d}z \wedge \mathrm{d}\bar{z}\,.$$

**Exercise 3.8.** *Find the expression for $\omega$ in the $V$-chart.*

Since $S^2$ is two-dimensional, $\omega$ is automatically closed. Indeed, each two-dimensional complex Riemannian manifold with Hermitian metric is Kähler. We have seen that any almost complex Riemannian manifold admits a Hermitian metric. For two dimensions we thus see that any two-dimensional orientable Riemannian almost complex manifold is in fact already Kähler.

For the Kähler potential we find that we can take

$$K = \log(1+|z|^2)\,.$$

For the Levi–Civitá connection we find that its action on the complex derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ is given by

$$\nabla \frac{\partial}{\partial z} = \frac{-2\bar{z}}{1+|z|^2} \frac{\partial}{\partial z} \otimes \mathrm{d}z\,, \tag{73}$$

$$\nabla \frac{\partial}{\partial \bar{z}} = \frac{-2z}{1+|z|^2} \frac{\partial}{\partial \bar{z}} \otimes \mathrm{d}\bar{z}\,. \tag{74}$$

**Exercise 3.9.** *Find the action of the Levi–Civitá connection on the derivatives $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial \bar{w}}$ on the $V$-chart.*

For the Riemann tensor we find that its action is in the $U$-chart given by

$$R(\partial_z, \partial_{\bar{z}})\partial_z = \frac{2}{(1+|z|^2)^2}\partial_z\,, \quad R(\partial_z, \partial_{\bar{z}})\partial_{\bar{z}} = \frac{-2}{(1+|z|^2)^2}\partial_{\bar{z}}\,, \tag{75}$$

where $\partial_z = \frac{\partial}{\partial z}$ and $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}$. Thus the Riemann tensor can be represented by the matrix-valued two-form $R = \Omega(z,\bar{z})\mathrm{d}z \wedge \mathrm{d}\bar{z}$ where

$$\Omega(z,\bar{z}) = \frac{1}{(1+|z|^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,.$$

For the Ricci-tensor we find the following expression in the $U$-chart

$$Ric = \frac{2}{(1+|z|^2)^2} \Big( \mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z \Big)\,. \tag{76}$$

Whence the Ricci-form is given by

$$\rho = \frac{2}{(1+|z|^2)^2} \mathrm{d}z \wedge \mathrm{d}\bar{z}\,. \tag{77}$$

# 4 Intermezzo on Chern Classes

We give a short explanation on Chern classes. We cannot hope to be complete, but we try to provide enough such that the reader can understand the calculation of the first Chern class of the sphere.

Let $M$ be a manifold and let $E \to M$ be a vector bundle over $M$. Is there a way to see whether $E$ is isomorphic to another vector bundle $E' \to M$ over $M$? Is there some valuable information in some easy to calculate (thus algebraic) quantities? An algebraically calculable quantity that gives a partial answer to the question whether different vector bundles over the same base manifold are isomorphic, is the set of Chern classes.

The idea of Chern classes stems from the following idea: the characteristic polynomial is invariant under bases changes, hence we try to find a matrix associated to a bundle and then calculate the characteristic polynomial, the coefficients of which are called Chern classes. But there is a natural candidate for the matrix that we are looking for! Given a connection $\nabla$ on $E$, then the curvature $R^\nabla$ (field strength in physics language) associated to $\nabla$ is a matrix: $R^\nabla \in C^\infty(\bigwedge^2 T^*M \otimes \mathrm{End}(E))$. So, as a start we might try to work out the properties of the following function:

$$D(t,\nabla) = \mathrm{tr}\Big(1 + \frac{t}{2\pi}R^\nabla\Big) \in C^\infty(\bigwedge T^*M)[t]\,. \tag{78}$$

The trace is of course for each point $p \in M$ over a basis of the fibre $\pi^{-1}(p)$ and since each matrix element of $R^\nabla$ is a differential two-form, which commutes with any other two-form, we have a well-defined expression. A big drawback is that $D(t,\lambda)$ depends on $\nabla$. The big miracle is that as soon as pass to cohomology we are in better business:

**Theorem 4.1** (Chern–Weil)**.** *Let $E \to M$ be a vector bundle over $M$. Define $D(t,\nabla)$ as in eqn.(78). Then we have*

*(i)* $\mathrm{d}D(t,\nabla) = 0$.

*(ii) If $\nabla$ and $\nabla'$ are two connections on $E$ then $D(t,\nabla) - D(t,\nabla') = \mathrm{d}\Omega$ where $\Omega$ is a sum of $2k$-forms; $\Omega \in C^\infty(\bigoplus_k \bigwedge^{2k} T^*M)$.*

We won't treat the proof of this theorem, but refer the reader to for example [2] **and chern, and who else??**. The theorem allow us to make the following definition:

**Definition 4.2.** *Let $E \to M$ be a vector bundle over $M$. Then the total Chern class is defined as*

$$c(E,t) = [\mathrm{tr}\Big(1 + \frac{t}{2\pi}R^\nabla\Big)] \in \bigoplus_k H^{2k}(M,\mathbb{R})[t]\,,$$

*where the calculation of $c(E,t)$ can be done using any connection on $E$ and $[\ldots]$ denotes the cohomology class. We define the kth Chern class $c_k(E)$ as the kth coefficient in the expansion*

$$c(E,t) = \sum_{j=0}^{M} c_i(E)t^i\,,$$

*where $M \leq \max(\mathrm{dim}M, \mathrm{rk}E)$*

The language of Chern classes is developped for complex vector bundles and for complex vector bundles the definition is slightly altered:

**Definition 4.3.** *Let $E \to M$ be a complex vector bundle over $M$. Then the total Chern class is defined as*

$$c(E,t) = [\mathrm{tr}\Big(1 + \frac{it}{2\pi}R^\nabla\Big)] \in \bigoplus_k H^{2k}(M,\mathbb{C})[t]\,, \quad i = \sqrt{-1}\,,$$

*where the calculation of $c(E,t)$ can be done using any connection on $E$ and $[\ldots]$ denotes the cohomology class. We define the kth Chern class $c_k(E)$ as the kth coefficient in the expansion*

$$c(E,t) = \sum_{j=0}^{M} c_i(E)t^i\,,$$

*where $M \leq \max(\mathrm{dim}M, \mathrm{rk}E)$*

Expanding $c(E,t)$ upto first order in $t$ gives the very important formula for the first Chern class:

$$c_1(E) = \frac{i}{2\pi}\mathrm{tr}\Big(R^\nabla\Big)\,.$$

The reader is invited to get some other formulae for higher order Chern classes.

**Lemma 4.4.** *If $E \to M$ is trivial then the first Chern class vanishes.*

**Exercise 4.5.** *Do the proof of lemma 4.4 using lemma 1.29.*

Very often one speaks of the Chern classes of a manifold, without making any reference to a bundle. In the case of real manifolds, one then speaks of the tangent bundle and in the case of complex manifolds one refers to the vector bundle $TM^{(1,0)} \to M$ of holomorphic vector fields **correct nomenclature?**. Note that we then need a connection on $TM$ that respects the complex structure $\nabla(TM^{(1,0)}) \subset TM^{(1,0)} \otimes T^*M$, which is the case for the Levi–Civitá connection on a Kähler manifold.

Consider a Riemannian manifold, and choose on the tangent bundle the Levi–Civitá connection. Then the curvature defines an element of $\mathfrak{so}(n)$, which is semisimple for all $n$, except $n = 2$. If a Lie algebra $\mathfrak{g}$ is semisimple then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and thus any Lie algebra element $x \in \mathfrak{g}$ can be written as the sum of Lie brackets; $x = \sum_i [a_i, b_i]$. But it is a trivial exercise to verify that the trace of $ab - ba$ vanishes for any two matrices $a, b$. Hence for a Riemannian manifold, the first Chern class always vanishes. For a Kähler manifold that comes from a real Riemannian manifold with a complex structure, this need no longer the case, as we only look at the holomorphic vector part of the complexified tangent bundle. However we have $c_1(TM^{(1,0)}) = -c_1(TM^{(0,1)})$.

We present a lemma to get some feeling for

**Lemma 4.6.** *Let $E \to M$ and $E' \to M$ be vector bundles over $M$. Then $c(E \oplus E', t) = c(E, t)c(E', t)$ as a product in the commutative subalgebra $\bigoplus_k \bigwedge^{2k} T^*M$ of the differential algebra $\bigoplus_k \bigwedge^k T^*M$.*

*Proof.* The proof is just linear algebra: let $R$ be any commutative ring and let $F$ be a matrix with entries in $R$ of the form

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix},$$

then the characteristic polynomial of $F$ is the product of the characteristic polynomials of $F_1$ and $F_2$. □

The following lemma is a bit more useful for our purposes:

**Lemma 4.7.** *Let $E \to M$ be a vector bundle over $M$ and let $F \to M$ be the associated determinant bundle (see section 1.2). Then $c_1(F) = c_1(E)$.*

*Proof.* As $F$ is a line bundle, to get the first Chern class, we only need to calculate the curvature of any connection. Let $\nabla$ be any connection on $E$, then we get an induced connection on $F$. Let $R_E$ be the curvature on $E$ associated to $\nabla$ and let $R_F$ be the associated induced curvature on $F$. Let $U$ be a small open set and let $e_1, \ldots, e_r$ be a set of linearly independent sections that span $\Gamma(E, U)$. We then calculate

$$\begin{aligned} R_F(e_1 \wedge \ldots \wedge e_r) =& (R_E e_1) \wedge e_2 \wedge \ldots \wedge e_r + \\ & + e_1 \wedge (R_E e_2) \wedge e_3 \wedge \ldots \wedge e_r + \ldots \\ =& \mathrm{trR_E} \left( e_1 \wedge \ldots \wedge e_r \right), \end{aligned} \tag{79}$$

and hence $-2\pi i c_1(F) = [R_F] = [\mathrm{tr} R_E] = -2\pi i c_1(E)$. □

A warning on nomenclature: For complex manifolds the name Chern class is reserved not for the tangent bundle, but for the holomorphic tangent bundle $TM^{(1,0)} \to M$. Also the determinant bundle for complex manifolds is understood to be $TM^{(n,0)}$, where $n$ is the real dimension of $M$. Thus the canonical divisor is the divisor associated to the line bundle of holomorphic $n$-forms.

We conclude the section on Chern classes by remarking the following identity, the proof of which is not presented.

**Lemma 4.8.** *Let $M$ be a Kähler manifold with Hermitian metric $g$, complex structure $J$ and Ricci-form $\rho$, then $\rho$ is related to the curvature of the holomorphic tangent bundle $TM^{(1,0)} \to M$ as follows*

$$[\rho] = \kappa c_1(M)\,,$$

*where $c_1(M)$ denotes the first Chern class of the holomorphic tangent bundle $TM^{(1,0)} \to M$ and $\kappa$ is some constant that depends on the conventions chosen in order to define curvature, Chern classes and so on. In particular $\rho$ is exact if and only if the first Chern class of the the holomorphic tangent bundle $TM^{(1,0)} \to M$ vanishes.*

# 5 Finally: Calabi–Yau's

Now we finally come to the subject of the title of these notes. Unfortunately then, this section will be quite short. The main thing will be the following theorem:

**Theorem 5.1.** *Let $M$ be a compact Kähler manifold of (real) dimension $2n$, then the following are equivalent:*

*(i) $M$ has a trivial canonical class.*

*(ii) The first Chern class of the holomorphic vector bundle $TM^{(1,0)} \to M$ vanishes.*

*(iii) $M$ has holonomy $SU(n)$.*

*(iv) $M$ is Ricci-flat.*

We postpone the proof of theorem 5.1. First we define a Calabi–Yau manifold as a compact Kähler manifold satisfying any of the equivalent conditions of theorem 5.1. The proof will rely heavily on the following (difficult to prove) theorem due to Yau:

**Theorem 5.2.** *[Yau] Let $M$ be a compact Kähler manifold with complex structure $J$, Hermitian metric $g$ and Hermitian form $\omega$. Suppose $\rho'$ is a real closed $(1,1)$-form on $M$ with $[\rho'] = 2\pi c_1(M)$. Then there exists a unique Hermitian metric $g'$ with closed Hermitian $\omega'$, such that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$ and the Ricci-form of $g'$ is $\rho'$.*

We immediately obtain:

**Corollary 5.3.** *Let $M$ be a compact Kähler manifold with Hermitian metric $g$ and closed Hermitian form $\omega$. Then if $c_1(M)$ we can find another metric $g'$ such that $g'$ is Ricci flat and the Hermitian form $\omega'$ associated to $g'$ differs from $\omega$ by an exact form.*

*Proof.* **of theorem 5.1**. To be done later. □

## 5.1 Why $S^2$ is not a Calabi–Yau

There are now two straightforward methods to see why the two-sphere is not a Calabi–Yau manifold. First of all we see that the Ricci tensor given in 76 does not vanish, or equivalently, the Ricci form, given in 77 does not vanish. The second method is to calculate the first Chern class of the holomorphic vector bundle. The total Chern class is given by the cohomology class of $d(t)$:

$$\begin{aligned} d(t) =& \det\Big(1 + \frac{it}{2\pi}\Omega(z,\bar{z})\mathrm{d}z \wedge \mathrm{d}\bar{z}\Big) \\ =& 1 + \frac{it}{2\pi}\Omega(z,\bar{z})\mathrm{d}z \wedge \mathrm{d}\bar{z} \\ =& 1 + \frac{i}{\pi}\frac{t}{(1+|z|^2)^2}\mathrm{d}z \wedge \mathrm{d}\bar{z}\,, \end{aligned}$$

The first Chern class is then read off to be given in the $U$-chart by

$$c_1 = \left[ \frac{i}{\pi} \frac{1}{(1+|z|^2)^2} \mathrm{d}z \wedge \mathrm{d}\bar{z} \right] .$$

The expression of the first Chern class is simply obtained by replacing $z$ by $w$ and $\bar{z}$ by $\bar{w}$. To show that the first Chern class is not trivial, we integrate it over the sphere, a calculation which was actually done already in section 2.2 in eqn.(65). If $c_1$ we exact, then its integral would vanish due to Stokes' theorem. Hence $S^2$ is not a Calabi–Yau manifold.

A third approach would be to calculate the holonomy. We have seen that the holonomy was the whole $SO(2)$, which is isomorphic to $U(1)$. The group $SU(1)$ is trivial and the Riemann tensor of the two-sphere is not trivial, hence the holonomy Lie algebra is not trivial and thus the holonomy group neither. Hence, the two-sphere is not Calabi–Yau.

# 6 Calabi–Yau's in supergravity and superstring theories

In this section we give a short explanation on superstrings, supergravities and supersymmetry and why Calabi–Yau's are interesting for physicists. We start with some general remarks (the general blabla, or nonsense) and then later go deeper in some apsects that might be useful to understand the reasons why Calabi–Yau varieties are interesting in physics. In the appendix we have made a short physics dictionary.

## 6.1 Some words on Superstrings and supergravities

String theory starts with the basic assumption that elementary particles are not pointlike objects but one-dimensional objects, called strings. A first attempt of a string theory is then obtained by applying some principles of classical dynamics of the vibrating string in this setting. After combining with quantum mechanics one sees some interesing features, one of them is that the number of spacetime dimensions has to be chosen carefully in order to preserve Lorentz symmetry. The second peculiarity is that there seems to be a particle with imaginary mass, the so-called tachyon. A third bad property of this naive string theory is that it only describes bosons, which means that the particles all come in tensorial representations of the Lorentz group, and thus no spinorial representations. Nature needs spinorial representations, as we can observe them. So, naive string theory is bad and we need a successor. The best successor seems to be superstring theory. Superstring theory solves at once the problem of the tachyon and of the spinors; fermions, the particles that take values in some spinorial bundle, are put in by hand in a certain sense and the connection with the bosons, the tensorial representations, is done with supersymmetry. The number of dimensions has to be chosen to be 10 in superstring theory in order to preserve Lorentz symmetry. Superstring theory is a difficult theory, therefore one is interested in a sophisticated approximation with which we can work more easily. Such an approximation is supergravity theory. As there are 5 different consistent formulations of superstring theory (type I, type IIA, type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$), there are 5 different supergravity theories.

After this general nonsense, let us take some time and explain some aspects in more (mathematical) detail. One of the most efficient tools in describing elementary particles is quantum field theory. The standard model of elementary particles is based on quantum field theory. If we want to get some understanding of the theory it is often sufficient to treat the theory as a classical field theory. A classical field theory is characterized by a view ingredients: (1) a manifold $M$, which very often plays the role of spacetime, (2) the particle content, which in mathematical terms means that one chooses a number of bundles $E_a \to M$ and identifies certain fields with elements of the different $C^\infty(E_a)$, and (3) a bunch of physical principles that are to give differential equations, which are called the equations of motion and the solutions of which are giving the behavior of the fields. The three ingredients (more or less) determine a (classical) field theory. We have seen that not all combinations of bundles and manifolds are compatible; if we want a spin bundle the second Stiefel–Whitney class has to vanish, for example. Further restrictions on $M$ might come from compatibility with the equations of motion. A particle corresponds to a field and a solution to the equations of motion describes the dynamics of the particles.

Among the many possible solutions to the equations of motion[3], there are a view solutions that are most interesting; the so-called vacuum solutions. The vacuum solutions are characterized by having a large symmetry and being stable against small perturbations (which is really in the sense of differential equations). Often the two requirements come together; a solution that is invariant under a large symmetry group is almost always stable (an explanation would be again to much blabla for a mathematical audience, see [12] for an explanation in the context of compactifications). The vacuum solutions correspond precisely to the solutions where many fields are turned off, which means that most fields are given by trivial sections (that is, vanishing everywhere). An interesting idea is to search for nontrivial vacuum solutions: stability combined with a large symmetry group.

In view of the success of the standard model and quantum field theory in general, the vacuum solutions of any theory of elementary particles should be compatible with the Minkowski spacetime structure we see around us. Spacetime around seems to be locally flat (which means Ricci-flat) with a metric that can be represented by the diagonal matrix with diagonal entries $(-1, +1, +1, +1)$; such a metric is called a Minkowski metric. The symmetry group of the Minkowski is $SO(3, 1)$, the connected component of which is called the Lorentz group. We say that a theory has Lorentz symmetry if the theory is invariant under the action of the Lorentz group, meaning that if we are given a solution, then letting the Lorentz group act on it, we obtain another solution with equivalent properties. For theories living in higher spacetime dimensions, we still require that the theory has local Lorentz symmetry, in order that quantum field theory still works. In ten spacetime dimensions, where superstring theory seems to live, the Lorentz group takes the form $SO(1, 9)$.

Since superstring theory lives in more dimensions that 4, we need to find an explanation for the reason why we cannot see the 6 extra dimensions of superstring theory. The way to do so is called compactification. The main idea of compactification is that if certain dimensions of a compact manifold become too small, we cannot see them as we would need too high energies to see them. For example, suppose that instead of ten dimensions, string theory needed 5 dimensions. Then with compactification we look for solutions of superstring theory where spacetime takes locally the form $M \times S^1$, where $M$ is a four-dimensional manifold. In more fancy terms, spacetime would be a fibre bundle $\hat{M} \to M$ where the fibre would be a Riemannian space diffeomorphic to a circle. Since we have a metric we can measure distances, and hence the circumference of the circle above each point of $M$; let us denote the circumference by $L$. In order to see the local direct product structure we need wavelengths that can detect the length $L$, that is, we need wavelengths that are not bigger than $L$. But the smaller the wavelength of a beam of light (or electrons, or protons, or ...), the more energy is need to produce the beam. Hence if the circumference of the $S^1$ is too small, we cannot see it with the energies that we can reach today. For superstring theory we thus are especially interested in solutions to equations of motion where the geometry of spacetime takes the form of a fibre bundle $\hat{M} \to M$ where the fibre of

[3] In almost all cases the differential equations are nonlinear, hence solutions abound.

each point above $M$ is a six-dimensional Riemannian space.

The prefix super- might have wondered the reader. The prefix means in this context that the theory is supersymmetric. Supersymmetry is a peculiar symmetry that intertwines the spin bundles and the tensor bundles. The concept of supersymmetry results rather difficult to explain to a mathematical audience and therefore we will be very sketchy about the true nature of supersymmetry - in order to be more explicit we would have to introduce either heavy machinery or to be more explicit so that the reader sees what we mean but then the formulae get rather involved. In order for a theory to be supersymmetric there is an additional spin bundle that parametrizes the supersymmetry and constitutes a connection on the total bundle of fields, which combines spin and tensorial bundles in one bundle. The total bundle of spinorial and tensorial bundles is a bundle with as fibres a representation of a Lie superalgebra; for any vector bundle the fibre is a vector space, for the total bundle of a supersymmetric theory the fibre is the sum of all the vector spaces and comes with the structure of a module of a Lie superalgebra.

We say that a solution is supersymmetric if the supersymmetry transformations leave the solution invariant. A solution of a supersymmetric theory need not be supersymmetric; more generally, if a differential equation has some symmetry, then there might exist solutions that do not have this symmetry. Nature (a very particular solution) however, seems to be nonsupersymmetric; if it were so, then fermions and bosons would have to appear on equal footing. But then there would have to be a fermionic particle for each bosonic particle, and they should share some properties, one of which would be that they have equal mass. Looking at any table with elementary particles and their masses, one concludes that supersymmetry is not reflected in the empirically observable world.

We finish this section by explaining a little on the relation between superstring theory and supergravity theory. There are 5 different consistent formulations of superstring theory, but we will just pretend like there is only one (the reader that knows more can assume we focus on type I). Superstring theory is not a field theory and is in fact a difficult theory if one wants to do explicit calculations. First of all, there is a parameter, called the string coupling constant, which dominates the theory. If the string coupling constant is zero the theory becomes rather uninteresting, but if the coupling constant is nonzero, the theory is highly nonlinear. One is therefore interested in a perturbative analysis. One can compare this situation with the case of the differential equation

$$(D - m^2)\phi = -\lambda\phi^3\,,$$

where $D$ is the Laplacian and $m, \lambda$ are real positive constants. If $\lambda = 0$ one knows all the solutions, but if $\lambda$ is just any constant one has no clue what the solutions might look like. In order to say something qualitative one assumes that $\lambda$ is relatively small and tries a perturbative analysis. Then the solutions are given in terms of power series in the constant $\lambda$. In string theory the situation is the same with the string coupling constant; one hopes the constant is small and does a perturbative analysis. In a perturbative analysis one obtains solutions that are power series in the string coupling constant. An approximation is obtained by cutting off at a certain power of the coupling constant. This is one aspect in which supergravity theories are an approximation of superstring theory. The other aspect is a rather difficult one; from the cut-off superstring perturbation theory one takes a low-energy limit. The procedure to obtain a low-energy limit is rather delicate and hence we state that supergravity theory is a theory which at low energies is a good approximation to the first view orders of perturbative superstring theory. The attractive feature of supergravity is that it is a field theory. Therefore much more managable then superstring theory, which, even in perturbative form, is not a field theory.

## 6.2 The '85 paper 'Vacuum configurations for superstrings'

In this section we shortly discuss the main aspects of the influential paper [1], where Calabi–Yau varieties were introduced into elementary particle physics. Also, the paper [1] seems to be the one that coined the name Calabi–Yau varieties in the literature.

The intention of [1] is to investigate the possible vacuum solutions in superstring theory. The focus is on vacuum solutions that are physically realistic and could possibly describe the actual empirically observable world. Under that restriction it is plausible to assume that instead of working directly with superstring theory one can work in the setting of supergravity theory, where one disposes over the powerful tools of field theory (thus, vector bundles). Hence the problem of finding realistic vacuum solutions of superstring theory is reduced to finding realistic vacuum solutions of supergravity theory. The paper then first focusses on the classical field theory.

In the setting of classical (thus not quantum) supergravity theory the paper [1] gives three conditions that is demanded from a solution. Two of them are related to being 'realistic' and the third is a technical one, but one which eventually leads to Calabi–Yau's. The first two requirements are

(i) The vacuum solution should have a local geometry $M \times K$, where $M$ is four-dimensional Minkowski spacetime and where $K$ is a compact six-dimensional manifold. In more mathematical terms, one requires that the geometry of space-time is determined by a fibre bundle $\hat{M} \to M$ where $\hat{M}$ is ten-dimensional, $M$ is four-dimensional spacetime and where the fibres are compact six-dimensional manifolds.

(ii) The particle content should resemble the particle content of the standard model. This gives certain constraints on some topological numbers of the compact manifold $K$. We will not go deeper into this point.

The third requirement is that

(iii) The vacuum solution should allow supersymmetry. This means that not all supersymmetry should be broken by the solution. It is however possible (and actually preferable) that supersymmetry is partially broken, but not completely.

A few explanations are at place. The concept of symmetry-breaking is an ever-returning item in physics. To give a mathematical and elementary example, consider the vector space $\mathbb{R}^3$ equipped with the standard inner product. Since we view it as a vector space we have fixed the origin. The symmetry transformations are the linear transformations that leave the standard inner product invariant; hence the symmetry group is $O(3)$, the group of all orthogonal transformations. Suppose that we know fix an orientation (in three dimensions this is called the handedness) so that we can integrate three-forms. Then the symmetry group is reduced from $O(3)$ to $SO(3)$. Now suppose we draw a circle with centre the origin. Then the symmetry group is the set of all $SO(3)$-transformations that leave the circle invariant. But then the normal vector to the plane in which the circle lies must be left invariant. Hence the symmetry group is $SO(2)$. Thus we see that we can partially break the symmetry down in a few steps. This is more or less how partial symmetry breaking can be viewed in physics; one just has to replace the orientation and the circle by 'solutions'.

One might wonder why we need the third requirement. Nature is not supersymmetric so supersymmetry is not realistic! That is all true, but the question is whether we want the supersymmetry to be broken already in the classical field theory or later, in the quantum field theory. The field theories of supergravity theory come with certain parameters that determine the energies at which certain processes take place. For supergravity theories these energies are rather high, whereas many quantum field theory processes take place at lower energies. From phenomenological considerations it is preferable that supersymmetry is still present (or at least partially) down to energies where supergravity would break supersymmetry. In other words, suppose that in our desired vacuum solution supersymmetry is broken, then at any experiment up to the high energies of typical processes in supergravity one cannot observe supersymmetry; this is not desirable. To break supersymmetry completely, one can resort to quantum field theory; some processes that cannot be explained by a classical field theory but can be explained with quantum field theory do indeed break supersymmetry. In short, in order to get a phenomenologically interesting theory we need to break supersymmetry in a particular way and such that at the level of where classical supergravity theory has its validity regime, nature should admit supersymmetry.

**Remark 6.1.** *It is a very general feature of physical theories that symmetries become observable at high energies. One can compare this with the behavior of water. When it is ice, a lot of symmetries are broken as the molecules have to take their place in the lattice. The symmetry is reduced to a discrete group. In the fluid phase the molecules don't feel the strong bonds that keep them together in the solid phase and as a result the orientation of the molecules is random; the full $SO(3)$ symmetry is recovered.*

With the preliminaries we can now show why the Calabi–Yau varieties made their appearance in physics: Combining the equations of motion and the equations that say that a solution is supersymmetric one arrives with some algebra (to be found in [1]) at the following conclusions:

(i) The compact manifold $K$ must be Ricci-flat.

(ii) The compact manifold $K$ must admit a global nowhere vanishing section $\eta$ of a spin bundle that is covariantly constant, i.e. $\nabla\eta = 0$, where $\nabla$ is the connection on the spin bundle associated to the Levi–Civitá connection on the tangent bundle.

If a covariantly constant spinor exists then it is invariant under the holonomy group. Hence the holonomy group $G$ must be a subgroup of $SO(6)$ such that the spinor representation splits into two parts one of which must contain the trivial representation. The double cover of $SO(6)$ is $SU(4)$ and the spinor representation of $SO(6)$ corresponds to the complex four-dimensional vector representation of $SU(4)$. Hence the biggest subgroup of $SU(4)$ that admits an invariant vector in the is $SU(3)$, the little group of the vector representation of $SU(4)$. Furthermore, from the covariantly constant spinor one can construct a $(1,1)$-tensor that squares to $-1$ and that is covariantly constant. For those that are familiar with Clifford algebras; one chooses a multiple of $\eta^+\Gamma^a b\eta$ where $\eta^+$ is the Hermitian conjugate. As a result, we obtain that the manifold $K$ is a compact six-dimensional manifold with a complex structure, $K$ is a Kähler manifold and the holonomy lies in $SU(3)$. Hence $K$ is a Calabi–Yau manifold.

# 7 Historical note

The attentive reader might have noticed that in the whole story around Calabi–Yau varieties the name Calabi never appeared. The name Calabi is attached to the name as it was Calabi who made a conjecture about Kähler manifolds. The content of the Calabi Conjecture is precisely the content of the theorem of Yau 5.2 and was made by Calabi in 1954. The proof had to wait for 22 years, when Yau proved the conjecture to be true in 1976. The reason why the theorem is so powerful is that in general it is quite hard to construct Calabi–Yau manifolds by looking for Ricci-flat Kähler manifolds. Up to now, only from a very few Calabi–Yau manifolds the Kähler metric is known. However, it turns out to be relatively easy to construct a Calabi–Yau by considering complete intersections in $\mathbb{P}^n$. Then it is a matter of arithmetic with Chern classes to calculate the first Chern class; for example one can via this strategy see that all quintics in $\mathbb{P}^4$ are Calabi–Yau varieties. It is also precisely for this reason that the Calabi–Yau conjecture is important for physics; in general physicists try to find solutions by being very explicit. Hence a physicists first approach to finding a Calabi–Yau would precisely be by trying to find a Kähler manifold with vanishing Ricci tensor. With the theorem of Yau, the problem of finding nice Calabi–Yau manifolds is then reduced to arithmetic with Chern classes, which - once one knows the rules - is just elementary addition and multiplication of integers.

The name Calabi–Yau seems to have appeared first in the paper [1] in 1985. In that paper there is a relatively large section on what a Calabi–Yau is and how to manage them. Back then, only a few families of Calabi–Yau manifolds were known. Nowadays with the tools of toric geometry, the number of known Calabi–Yau manifolds has increased enormously. However, explicit metrics remain a rather unknown and oscure item.

# 8 Literature

For a gentle introduction into differential manifolds, but with an eye towards physics and hence once and a while physics notation, can be found in [3]. A bit more advanced and with some tough material but still very readable (and as noticed containing a few mistakes that are not typo's) and to be recommended is [2]. For a more mathematical but still elementary text we recommend [4], which stresses the use of the bundle language. The more serious mathematical work on vector bundles and differential geometry is of course [6], which starts with a serious but still readable account on vector bundles. As it seems, nowadays the language of bundles is the one to use for (differential) geometry.

# A A small physics dictionary

**Anti de Sitter spacetime**: A maximally symmetric spacetime with negative (Ricci) curvature.
**Boson**: A particle described in field theory by a field that takes values in a bundle of the form $TM^{\otimes r} \otimes T^*M^{\otimes s}$ and subbundles thereof over the space-time manifold $M$.
**Compactification**: Making a space compact. More general, converting a theory living in $n + d$ dimensions to a theory living in $d$ dimensions. In string theory spacetime needs 10 dimensions. One way to ensure the inobservability of the 6 dimensions that seem to be too much is to impose a local geometry of the form $M_4 \times K$ with $M_4$ four-dimensional and $K$ six-dimensional. Making $K$ small ensures that we need large energies to see the direct product structure.
**Configuration**: A time-independent solution to the equations of motion.
**De Sitter spacetime**: A maximally symmetric spacetime with positive (Ricci) curvature.
**Equations of Motion**: Equations that describe the dynamics of quantities of interest. Mostly, they are differential equations.
**Field Theory**: Formalism in which quantities of interest are described by functions and sections of some bundles over the (smooth) manifold of space or spacetime.
**Field strength**: The curvature of a connection.
**Fermion**: A particle described in field theory by a field that takes values in a bundle of the form $TM^{\otimes r} \otimes T^*M^{\otimes s} \otimes Spin(g)$, where $Spin(g)$ denotes the spin bundle of $g$, and subbundles thereof over the space-time manifold $M$ with metric.
**Fermionic index**: Index used to denote components of a spinor (=fermion) field with respect to some local basis for the sections of the spin bundle.
**Gauge field**: A connection $\nabla$ on a bundle $E \to M$ can be locally written as $d + A$ where $A \in \mathrm{End}(E) \otimes T^*M$. The object $A$ is called the gauge field.
**Gauge symmetry**: In field theory, any symmetry that is related to a vector bundle. More specific, the structure group of a vector bundle. Thus relates to principle bundle. NB: symmetry means a group leaving some structure invariant. More classically, a gauge symmetry is a symmetry that arises when there observers can make a certain choice; the symmetry that describes the transitions to different (gauge) choices is the gauge symmetry.
**Infinitesimal**: Working with first order variations. Thus, for group actions, the infinitesimal transformations describe the action of the Lie group.
**Maximally symmetric spacetime**: spacetime with the maximal number of isometries, which equals the maximal number of Killing vectors, which in dimension $n$ is $n(n-1)/2$.
**Metric**: Just like a metric in mathematics, but we allow non-positive definite metrics. Thus a metric in physics is a nondegenerate symmmetric bilinear form.
**Minkowski spacetime**: A maximally symmetric spacetime with zero (Ricci) curvature.
**Multiplet**: Representation content for some (Lie) group, Lie algebra and sometimes even for a Lie superalgebra. It is thus a module.
**Singlet**: A one-dimensional multiplet.
**Solution**: Solution to the equations of motion.
**Spacetime**: The smooth manifold that parameterizes the locations and moments that we observe or can observe.
**Supergravity**: Field theory which contains gravity and supersymmetry. Low energy limits of superstring theory.
**Superstring theory**: Theory build on the assumptions: (1) Special relativity is valid, (2) quantum mechanics should work, (3) elementary particles are one-dimensional objects.
**Supersymmetry**: Physical concept in which some fields are grouped together to form a multiplet of some Lie superalgebra, which in the literature are denoted by some $\mathcal{N}$-number: $\mathcal{N} = 1, 2, 4, 8$. A vacuum state is supersymmetric if it is a singlet of the Lie superalgebra.
**Vacuum**: Solutions to the equations of motion in a field theory such that the solution has a large symmetrygroup and such that the solution is stable against small perturbations. In many cases the stability can be obtained by minimizing the energy. Some other words to denote the vacuum are *vacuum solution*, *vacuum state* and *vacuum configuration*.

# References

[1] Candelas, Horowitz, Strominger and Witten, Vacuum configurations for superstrings.

[2] Michio Nakahara, Geometry, Topology and Physics.

[3] Theodore Frankel, The geometry of physics.

[4] Serge Lang, Differential and Riemannian Manifolds.

[5] Shiing-Shen Chern, Complex manifolds without potential theory.

[6] Kobayashi and Nomizu, Foundations of differential geometry I & II.

[7] Dominic Joyce, Lectures on Calabi–Yau and special Lagrangian geometry, math/0108088.

[8] W. Ambrose, I.M. Singer, A theorem on holonomy, 1953, Trans. Amer. Math. Soc. 75: 428443.

[9] N. Jacobsen, Lie algebras, Dover, 1963.

[10] Stefan Vandoren, Lectures on Riemannian geometry, Part II: Complex manifolds. MRI Masterclass, Utrecht, Fall 2008: http://www.math.uu.nl/people/ban/riemgeom2008/riemgeom2008.html.

[11] Peter West, Supergravity, brane dynamics and string duality, hep-th/9811101.

[12] D.B. Westra, Symmetries in $\mathcal{N}=4$ Supergravities, Ph.D.-Thesis University of Groningen, http://dissertations.ub.rug.nl/faculties/science/2006/d.b.westra/

[13] R. d'Inverno, Introducing Einstein's Relativity.