# Caratheodory's extension theorem

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These notes are meant as introductory notes on Caratheodory's extension theorem. The presentation is not completely my own work; the presentation heavily relies on the presentation of Noel Vaillant on `http://www.probability.net/WEBcaratheodory.pdf`. To make the line of arguments as clear as possible, the starting point is the notion of a ring on a topological space and not the notion of a semi-ring.

## 1 Elementary definitions and properties

We fix a topological space $\Omega$. The power set of $\Omega$ is denoted $\mathcal{P}(\Omega)$ and consists of all subsets of $\Omega$.

**Definition 1.** *A ring on on $\Omega$ is a subset $\mathcal{R}$ of $\mathcal{P}(\Omega)$, such that*

*(i)* $\emptyset \in \mathcal{R}$

*(ii)* $A, B \in \mathcal{R} \quad \Rightarrow \quad A \cup B \in \mathcal{R}$

*(iii)* $A, B \in \mathcal{R} \quad \Rightarrow \quad A \setminus B \in \mathcal{R}$

**Definition 2.** *A $\sigma$-algebra on $\Omega$ is a subset $\Sigma$ of $\mathcal{P}(\Omega)$ such that*

*(i)* $\emptyset \in \Sigma$

*(ii)* $(A_n)_{n\in\mathbb{N}} \in \Sigma \quad \Rightarrow \quad \cup_n A_n \in \Sigma$

*(iii)* $A \in \Sigma \quad \Rightarrow \quad A^c \in \Sigma$

Since $A \cap B = A \setminus (A \setminus B)$ it follows that any ring on $\Omega$ is closed under finite intersections; hence any ring is also a semi-ring. Since $\cap_n A_n = (\cup_n A^c)^c$ it follows that any $\sigma$-algebra is closed under arbitrary intersections. And from $A \setminus B = A \cap B^c$ we deduce that any $\sigma$-algebra is also a ring.

If $(R_i)_{i\in I}$ is a set of rings on $\Omega$ then it is clear that $\cap_I R_i$ is also a ring on $\Omega$. Let $S$ be any subset of $\mathcal{P}(\Omega)$, then we call the intersection of all rings on $\Omega$ containing $S$ the ring generated by $S$.

**Definition 3.** *Let $\mathcal{A}$ be a subset of $\mathcal{P}(\Omega)$. A measure on $\mathcal{A}$ is a map $\mu : \mathcal{A} \to [0, +\inf]$ such that*

*(i)* $\mu(\emptyset) = 0$

*(ii) If $A_n \in \mathcal{A}$ are disjoint and $A = \uplus_n A_n \in \mathcal{A} \quad \Rightarrow \quad \mu(A) = \sum_n \mu(A_n)$.*

If $\mathcal{A}$ is a $\sigma$-algebra, we don't need to assume that in addition $\uplus_n A_n \in \mathcal{A}$. By taking all but finitely many $A_n$ to be the empty set one sees that $\mu(A_1 \uplus \cdots \uplus A_N) = \mu(A_1) + \ldots + \mu(A_n)$. If $A \subset B$ then $A \uplus (B \setminus A) = B$ and hence $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$.

**Definition 4.** *We call an outer measure on* $\Omega$ *a map* $\lambda : \mathcal{P}(\Omega) \to [0, +\infty]$ *with*

*(i)* $\lambda(\emptyset) = 0$

*(ii)* $A \subset B \quad \Rightarrow \quad \lambda(A) \leq \lambda(B)$

*(iii)* $(A_n)_{n\in\mathbb{N}} \in \mathcal{P}(\Omega)\,,\ \lambda(\cup_n A_n) \leq \sum_n \lambda(A_n)$

By taking all but finitely many $A_n$ to be the empty set one sees that an outer measure is subadditive; $\lambda(A \cup B) \leq \lambda(A) + \lambda(B)$.

# 2 The interplay between $\sigma$-algebras and (outer) measures

Let $\lambda$ be an outer measure on $\Omega$. We define $\Sigma_\lambda$ to be the set of all subsets $A \subset \Omega$ such that for any $X \subset \Omega$ we have

$$\lambda(X) = \lambda(X \cap A) + \lambda(X \cap A^c)\,.$$

In other words, $\Sigma_\lambda$ consists of all subsets $A \subset \Omega$ that cut $\Omega$ in two in a good way. Clearly, $\Omega \in \Sigma_\lambda$ and by the very form of the definition of $\Sigma_\lambda$, we have $A \in \Sigma_\lambda \Leftrightarrow A^c \in \Sigma_\lambda$. We can now present the following proposition, whose proof is a bit tedious, but which contains loads of information on the exact interplay.

**Proposition 5.** *Let* $\lambda$ *be an outer measure on* $\Omega$ *and let* $\Sigma_\lambda$ *be as defined above. Then* $\Sigma_\lambda$ *is a* $\sigma$*-algebra on* $\Omega$*.*

*Proof.* After the preliminary remarks preceding the proposition, it only remains to show that $\Sigma_\lambda$ is closed under countable unions. We will first prove that $\Sigma_\lambda$ is closed under finite intersections and unions.

Let $A, B \in \Sigma_\lambda$ and let $X$ be any subset of $\Omega$. We have $X \cap A^c = X \cap (A \cap B)^c \cap A^c$ since $(A \cap B)^c \supset A^c$. On the other hand we have $(A \cap B)^c = A^c \cup B^c$ and hence $X \cap (A \cap B)^c \cap A = (X \cap A \cap B^c) \cup (X \cap A \cap A^c) = X \cap A \cap B^c$. Therefore we have $\lambda(X \cap (A \cap B)^c) = \lambda(X \cap (A \cap B)^c \cap A) + \lambda(X \cap (A \cap B)^c \cap A^c = \lambda(X \cap A^c) + \lambda(X \cap A \cap B^c)$. Now adding $\lambda(X \cap A \cap B)$ and using that $\lambda(X \cap A) = \lambda(X \cap A \cap B) + \lambda(X \cap A \cap B^c)$ one obtains $\lambda(X \cap A \cap B) + \lambda(X \cap (A \cap B)^c) = \lambda(X)$. Hence $A \cap B \in \Sigma_\lambda$.

Since $A \cup B = (A^c \cap B^c)^c$ and $A \setminus B = A \cap B^c$ we see that $\Sigma_\lambda$ is closed under finite unions and the set-theoretic difference. Thus $\Sigma_\lambda$ is a ring on $\Omega$.

If $A, B \in \Sigma_\lambda$ are disjoint and $X \subset \Omega$ then $\lambda(X \cap (A \uplus B)) = \lambda(X) - \lambda(X \cap A^c \cap B^c) = \lambda(X) - \lambda(X \cap A^c) + \lambda(X \cap A^c \cap B) = \lambda(X \cap A) + \lambda(X \cap B)$ as $A^c \cap B = B$. Using induction we obtain $\lambda(X \cap \biguplus_{n=1}^N A_n) = \sum_{n=1}^N \lambda(X \cap A_n)$ whenever $A_n$ are in $\Sigma_\lambda$ and pairwise disjoint.

Now we fix a sequence $A_n$ in $\Sigma_\lambda$ which are pairwise disjoint and we denote the union $\cup_n A_n$ by $A$. Furthermore, we fix an arbitrary $X \in \Omega$ and an arbitrary large integer $N$.

Since $X \cap A^c \subset X \cap (\biguplus_{n=1}^N A_n)^c$ and $\Sigma_\lambda$ is closed under finite unions we have $\lambda(X \cap A^c) + \lambda(X \cap (\biguplus_{n=1}^N A_n)) \leq \lambda(X \cap (\biguplus_{n=1}^N A_n)^c) + \sum_n \lambda(X \cap A_n) = \lambda(X)$. But $N$ is arbitrary in this equation and we can safely let it go to $\infty$ and obtain

$$\lambda(X \cap A^c) + \sum_n \lambda(X \cap A_n) \leq \lambda(X)\,. \tag{1}$$

On the other hand we have $\lambda(X) \le \lambda(X \cap A^c) + \lambda(X \cap A)$, which again by the definition of an outer measure is less or equal $\lambda(X \cap A^c) + \sum_n \lambda(X \cap A_n)$. Hence using (1) we obtain

$$\lambda(X) \le \lambda(X \cap A^c) + \lambda(X \cap A) \le \lambda(X \cap A^c) + \sum_n \lambda(X \cap A_n) \le \lambda(X)\,.$$

It follows that we must equality. From this we conclude that $\Sigma_\lambda$ is indeed closed under countable unions and, by taking $X = A$, that $\lambda(A) = \sum_n \lambda(A_n)$. Therefore the restriction of $\lambda$ to $\Sigma_\lambda$ is a measure on $\Sigma_\lambda$.

□

We will call $\Sigma_\lambda$ the $\sigma$-algebra related to $\lambda$.

Now we come to a critical step; we want to associate an outer measure $\lambda_\mu$ to a given measure $\lambda$ on some ring $\mathcal{R}$. Of course, we want the restriction of the outer measure $\lambda_\mu$ to the ring to coincide with the measure $\mu$.

Let $\mathcal{R}$ be a ring on $\Omega$ and let $\mu$ be a measure on $\mathcal{R}$. If $X \subset \Omega$ is any subset we can cover $X$ with sets from $\mathcal{R}$ to approximate $X$ inside $\mathcal{R}$ - we call an $\mathcal{R}$-cover of $X$ a countable subset $(A_n)_n$ of $\mathcal{R}$ with $X \subset \cup_n A_n$. This leads to the following definition; for any $X \subset \Omega$ we $\lambda_\mu(X)$ to be the infimum of all sums $\sum_n \mu(A_n)$ where $(A_n)_{n\in\mathbb{N}}$ is any countable cover of $X$ with $A_n$ in $\mathcal{R}$. We need to check that this is an outer measure.

**Proposition 6.** *The map $\lambda_\mu : \mathcal{P}(\Omega) \to [0, +\infty]$ defined in the above paragraph defines an outer measure on $\Omega$.*

*Proof.* Since $\emptyset \in \mathcal{R}$ we have $\lambda_\mu(\emptyset) = 0$. If $X \subset Y$ are two subsets of $\Omega$, then any cover of $Y$ with sets from $\mathcal{R}$ also covers $X$ and hence $\lambda_\mu(X) \le \lambda_\mu(Y)$.

Now let $X_n$ be any sequence of subsets of subsets. By the definition of the infimum we can find for each $\epsilon > 0$ and for each $n$ an $\mathcal{R}$-cover $(A_{n,m})_m$ of $X_n$ such that $\sum_m \mu(A_{n,m}) < \lambda_\mu(X) + \frac{\epsilon}{2^n}$. The sets $A_{n,m}$ form a countable cover of $X = \cup_n X_n$; we can for example set $B_1 = A_{1,1}$, $B_2 = A_{2,1}$, $B_3 = A_{1,2}$, $B_4 = A_{3,1}$ and so on, similar to Cantor's proof of the countability of $\mathbb{Q}$. But then $\lambda_\mu(X) = \lambda_\mu(\cup_n X_n) \le \sum_{n,m} \mu(A_{n,m}) < \sum_n (\lambda_\mu(X_n) + \frac{\epsilon}{2^n}) = \sum_n \lambda_\mu(X_n) + \epsilon$. But $\epsilon$ was arbitrary and hence $\lambda_\mu(\cup_n X_n) \le \sum_n \lambda_\mu(X_n)$.

□

**Proposition 7.** *The restriction of $\lambda_\mu$ to $\mathcal{R}$ is $\mu$.*

*Proof.* For any $A \in \mathcal{R}$ the set $A$ itself forms a cover and hence $\lambda_\mu(A) \le \mu(A)$.

On the other hand, let $(A_n)$ be an $\mathcal{R}$-cover of $A$. We define $B_1 = A_1 \cap A$ and $B_{n+1} = (A_n \cap A) \setminus \cup_{k\ge n}(A_k \cap A)$ for $n \ge 1$. Then clearly $B_n \in \mathcal{R}$, the $B_n$ are disjoint, $\uplus_n B_n = A$ and $\mu(B_n) \le \mu(A_n)$. Since $\mu$ is a measure on $\mathcal{R}$ we have $\mu(A) = \sum_n \mu(B_n)$ which is less than or equal to $\sum_n \mu(A_n)$. Since this holds for any $\mathcal{R}$-cover of $A$ we have $\mu(A) \le \lambda_\mu(A)$. Therefore equality holds and the proposition is proved.

□

We will call $\lambda_\mu$ the outer measure associated to $\mu$.

So we now have two constructions; given a ring and a measure on it we can construct an outer measure. Given an outer measure we can construct a $\sigma$-algebra such that the restriction of the outer measure to the $\sigma$-algebra is a measure on the $\sigma$-algebra. So it seems feasible that we can construct a measure on a $\sigma$-algebra starting from a ring with a measure on it. That this really works and that all things work out nicely is the content of Caratheodory's theorem.

## 3 Caratheodory's theorem: Statement and Proof

**Lemma 8.** *Let $\mathcal{R}$ be a ring on $\Omega$ and let $\mu$ be a measure on $\mathcal{R}$. Let $\lambda$ be the outer measure associated to $\mu$. Let $\Sigma$ be the $\sigma$-algebra related to $\lambda$. Then $\mathcal{R} \in \Sigma$.*

*Proof.* Let $A$ be an element of $\mathcal{R}$ and let $X$ be any subset of $\Omega$. Since $\lambda$ is an outer measure on $\Omega$ we have $\lambda(X) = \lambda((X \cap A) \cup (X \cap A^c)) \leq \lambda(X \cap A) + \lambda(X \cap A^c)$.

Now let $(A_n)_{n \in \mathbb{N}}$ be any $\mathcal{R}$-cover of $X$. Then the $A_n \cap A$ form an $\mathcal{R}$-cover of $X \cap A$ and the $A_n \cap A^c$ form an $\mathcal{R}$-cover of $X \cap A^c$. Hence we have that $\lambda(X \cap A) + \lambda(X \cap A^c) \leq \sum_n \mu(A_n \cap A) + \sum_n (A_n \cap A^c) = \sum_n \mu(A_n)$, where the last step follows from the fact that $\mu$ is a measure and hence $\mu(C \uplus D) = \mu(C) + \mu(D)$. Since the inequality holds for any $\mathcal{R}$-cover of $X$ we need $\lambda(X \cap A) + \lambda(X \cap A^c) \leq \lambda(X)$. We thus need equality; for any $X \subset \Omega$ we have $\lambda(X) = \lambda(X \cap A) + \lambda(X \cap A^c)$, or in other words $A \in \Sigma$ and since $A$ was an arbitrary element of $\mathcal{R}$ the lemma is proved. □

**Remark 9.** *Any ring generates a $\sigma$-algebra; one at least has to enlarge the ring with countable unions and countable intersections. Put in a correct way, the $\sigma$-algeba generated by the ring $\mathcal{R}$ is the intersection of all $\sigma$-algebras that contain $\mathcal{R}$. Therefore the above lemma shows that the $\sigma$-algebra generated by $\mathcal{R}$ is contained in $\Sigma$.*

Now we come to Caratheodory's theorem:

**Theorem 10.** *Let $\mathcal{R}$ be a ring on $\Omega$ and let $\mu$ be a measure on $\mathcal{R}$. Then there exists a measure $\mu'$ on the $\sigma$-algebra generated by $\mathcal{R}$ such that the restriction of $\mu'$ to $\mathcal{R}$ coincides with $\mu$.*

*Proof.* Let $\lambda$ be the outer measure on $\Omega$ associated to $\mu$. Let $\Sigma$ be the $\sigma$-algebra associated to $\lambda$. Then by lemma 8 the $\sigma$-algebra generated by $\mathcal{R}$ is contained in $\Sigma$. Hence $\lambda$ restricts to a measure on the $\sigma$-algebra generated by $\mathcal{R}$. By proposition 7 this restriction of $\lambda$ to $\mathcal{R}$ coincides with $\mu$.

□

**Example 11.** *Let $\Omega$ be the real line. Then the open intervals generate a $\sigma$-algebra $\Sigma$. For any open interval $(a, b)$ with $a < b$ we can put $\mu((a, b)) = b - a$. Then there exists a measure $\mu'$ on $\Sigma$ such that $\mu'((a, b)) = b - a$. Indeed, for countable unions of disjoint intervals we can define $\mu(\cup_n (a_n, b_n)) = \sum_n (b_n - a_n)$. Hence $\mu$ does give rise to a measure on the ring generated by all intervals.*

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