Restricted partitions from a lottery game

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1 Introduction

A well-known lottery game goes as follows: One marks 6 (distinct) numbers on a 7×7-grid, ranging from 1 to 49. How many of the marked numbers are drawn in a drawing event determines your win. Suppose a good fairy tells you that the following drawing event will select six numbers $a_1,\ldots,a_6$ such that the sum $S = a_1 + \ldots + a_6$ times the number of ways this sum S can be achieved by marking six distinct numbers ranging from 1 to 49 precisely equals the product $P = a_1 \cdot \ldots \cdot a_6$ – do you know, which six numbers to select and win the jackpot? Below we analyse this problem and single out 4 possibilities that make us win the jackpot. Our method is that of brute force; we do not use very advanced mathematics, but reduce the problem to a calculation that can easily be performed on a computer.

From the information of the fairy it is clear that the sum of the six numbers divides their product. At first, this might not be happening too oft. However, the number of possibilities to select 6 numbers out of 49 is huge, 13983816. A computer analysis shows that at least several ten thousands selections of 6 numbers satisfy the condition that their sum divides their product.

We focus on how may ways a certain sum can be achieved by six different numbers ranging from 1 to 49. Clearly the lowest possible value is $1+2+3+4+5+6 = 21$, whereas the highest possible value is $49+48+47+46+45+44 = 279$. Since choosing (a,b,c,d,e,f) or $(50-a, 50-b, 50-c, 50-d, 50-e, 50-f)$ makes for the number of possibilities no difference, we see that the number of ways a sum S can be achieved is the same as the number of ways to obtain the sum $300-S$. Thus, we expect that the value 150 can be achieved in most ways.

2 Recursive formula

We define $W(a,b,c,)$ as the set of partitions of a into a sum of b different integers from the set $\{1,2,\ldots,c\}$. Each element λ of $W(a,b,c)$ is a b-tuble $\lambda = (\lambda_1,\ldots,\lambda_b)$ with $c \geq \lambda_1 > \lambda_2 > \ldots > \lambda_b \geq 1$.

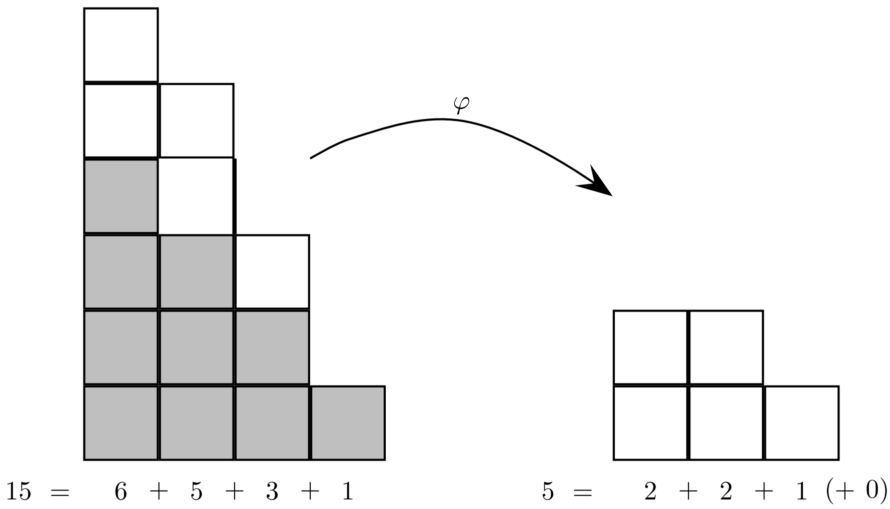

Figure 1: Pictorial description of the isomorphism $\varphi : W(a,b,c) \to W'(a - \frac{1}{2}b(b+1), b, c-b)$. The squares that are removed in the process are marked gray.

We define $W'(a,b,c)$ as the set of partitions of a into a sum of b nonnegative integers not exceeding c. Each element $\mu \in W'(a,b,c)$ is a b-tuple $\mu = (\mu_1, \dots, \mu_b)$ with $c \geq \mu_1 \geq \mu_2 \ldots \geq \mu_b \geq 0$.

The map $\varphi : W(a,b,c) \to W'(a - \frac{1}{2}b(b+1), b, c-b)$ defined by $\varphi : (\lambda_1, \lambda_2, \dots, \lambda_b) \mapsto (\lambda_1 - b, \lambda_2 - (b-1), \dots, \lambda_b - 1)$ substracts one from the lowest value, 2 from the one but lowest value and so on, thus substracting b from the highest value. The map φ is well-defined and sets up an isomorphism. An example: $16 = 6 + 5 + 3 + 1$ is mapped to $5 = 2 + 2 + 1$, depicted by means of (mirrored) Ferrer diagrams in figure 1.

If we let $I(a,b,c)$ and $J(a,b,c)$ denote the cardinalities of $W(a,b,c)$ and $W'(a,b,c)$ respectively, we have the following identity:

$$I(a,b,c) = J\Big(a - \frac{b(b+1)}{2}, b, c-b\Big)\,. \tag{1}$$

Although our object of interest is $I(a,b,c)$, we will focus on the $J(a,b,c)$.

Each element of $W'(a,b,c)$ can be assigned a coloring of a rectangle of $b \times c$ squares by which a squares are colored with one color (blue) and the remaining with another color (red), such that the number of blue boxes does not increase from left to right – see figure 2. If $a > bc$ we cannot do this, so $W'(a,b,c)$ is empty in this case. Exchanging rows and columns, we get a good coloring of a rectangle of $c \times b$ squares, thus

$$J(a,b,c) = J(a,c,b)\,. \tag{2}$$

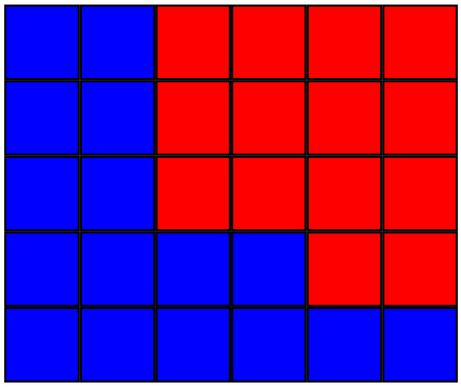

Figure 2: The number $J(a,b,c)$ counts the number of colorings of a rectangle with a total of $b \times c$ squares of which a are blue and $bc-a$ red, such that from left to right the number of blue squares is nonincreasing. We can thus pull all the blue boxes to the bottom line and the blue boxes make up a region below the graph of a nonincreasing step function.

If a boxes are colored blue, then $bc-a$ are colored red. Rotating the rectangle and exchanging colors leads to the identity

$$J(a,b,c) = J(bc-a,b,c)\,. \tag{3}$$

If $a < c$ all summands of a partition are not exceeding a, thus we can replace c by a:

$$J(a,b,c) = J(a,b,a) \quad \text{if } c > a\,. \tag{4}$$

And thus, if b and c both exceed a, we have

$$J(a,b,c) = J(a,a,a) \quad \text{if } b,c > a\,. \tag{5}$$

The set $W'(a,b,c)$ can be partioned into two disjoint sets: those μ for which $\mu_1 < c$ and those for which $\mu_1 = c$. Those of the first type are in one-to-one correspondence with the elements of $W'(a,b,c-1)$ and those of the second type are in one-to-one correspondence with the elements of $W'(a-c,b-1,c)$, which can be seen by removing μ_1 from μ to obtain $(\mu_2,\ldots,\mu_b)$. Therefore

$$J(a,b,c) = J(a,b,c-1) + J(a-c,b-1,c)\,. \tag{6}$$

By symmetry, we can find another partition of the set $W'(a,b,c)$: Those partitions for which the lower row is completely blue and those for which the lower row is not completely blue, i.e., those partitions $(\mu_1,\ldots,\mu_b)$ for which $\mu_b \neq 0$ and those for which $\mu_b = 0$. Reasoning as before, we find

$$J(a,b,c) = J(a,b-1,c) + J(a-b,b,c-1)\,. \tag{7}$$

The above equations (6) and (7) can be used as recursive relations to calculate the $J(a,b,c)$. We need some initial values:

$$\begin{aligned} J(0,b,c) &= 1 \quad b,c>0 \\ J(1,b,c) &= 1 \quad b,c>0 \\ J(a,1,c) &= 1 \quad c \geq a\,, \quad \text{and} \quad J(a,1,c) \quad = 0 \;\; \text{elsewise} \\ J(a,b,1) &= 1 \quad b \geq a\,, \quad \text{and} \quad J(a,b,1) \quad = 0 \;\; \text{elsewise}\,. \end{aligned} \tag{8}$$

Further conditions also can serve as initial values for calculating $J(a,b,c)$:

$$\begin{aligned} J(a,b,c) &= 0 \quad \text{if} \;\; bc < a\,, \\ J(bc,b,c) &= 1\,, \\ J(2,b,c) &= 2 \quad \text{if} \;\; b,c > 1\,. \end{aligned} \tag{9}$$

With the above identities it is simple to calculate $J(a,b,c)$ and thus $J(a,b,c)$ for any value of a, b and c. The calculations tend to be rather lengthy however. Therefore some simple code that is run on a computer can be of help; in the next section we will explain how this can be done.

3 Design of a computer aided computation

First, we describe how an algorithm could calculate $J(a,b,c)$ for $b=6$ and $c=43$ and where a ranges from 0 to 258. Then we show, how to find an algorithm that runs through all the possible ways to select 6 numbers from the set $\{1,2,\ldots,49\}$.

The triple (a,b,c) is inserted into an ordered list $L=\{(a,b,c)\}$ and a number x is set to 0. The algorithm takes the first triple (a,b,c) from the list and then considers following conditions and takes the corresponding actions:

1. If $bc<a$ then (a,b,c) is deleted from the list;
2. else, if $bc=a$, then (a,b,c) is deleted from the list and x is replaced by $x+1$;
3. else, if $a=1$, then (a,b,c) is deleted from the list and x is replaced by $x+1$;
4. else, if $a=0$, then (a,b,c) is deleted form the list and x is replaced by $x+1$;
5. else, if $b=1$, then (a,b,c) is deleted from the list and x is replaced by $x+1$;
6. else, if $c=1$, then (a,b,c) is deleted form the list and x is replaced by $x+1$;
7. else, if $a=2$ and $b>1$ and $c>1$, then (a,b,c) is deleted from the list and x is replaced by $x+2$;
8. else, if $a=3$ and $b>2$ and $c>2$, then (a,b,c) is deleted from the list and x is replaced by $x+3$;

9. else, if $c > a$, then (a, b, c) is deleted from the list, x is replaced by $x + 1$ and the triple $(a, b, a - 1)$ is inserted at the first place in L;

10. elsewise, then (a, b, c) is deleted from the list and then first $(a, b, c-1)$ is inserted at the first place in L, then $(a-c, b-1, c)$ is inserted at the first place in L (thus shifting $(a, b, c - 1)$ to the second place).

These steps are iterated until $L = \emptyset$, or equivalently, the length of L becomes zero, and then $x = J(a, b, c)$.

The idea behind first inserting $(a, b, c-1)$ and then $(a-c, b-1, c)$ is that the algorithm always takes the first element from L. Thus after (a, b, c) is replaced and x is not changed, the next triple to be considered is $(a - c, b - 1, c)$. By these steps the first number of the triple under consideration rapidly reaches 0, 1, 2 or 3 and therefore the size of L does not increase too rapid.

After the above one has found $J(a, b, c)$ and one proceeds to $J(a+1, b, c)$, thus obtaining $J(a, 6, 43)$ with a ranging from 0 to 258, or equivalently $I(a, 6, 49)$ with a ranging from 21 to 279. These numbers $J(a, 6, 43)$ are saved in list J ordered such that the first in the list is $J(0, 6, 43)$, the next is $J(1, 6, 43)$ and so on, and the last in the list J is thus $J(258, 6, 43)$.

Now, how to go through all the possible 6-tuples (a, b, c, d, e, f) with $1 \leq a < b < c < d < e < f \leq 49$. One can consider a lexicographical ordering on these 6-tuples: $(a, b, c, d, e, f) > (a', b', c', d', e', f')$ if the first (read from left to right) nonzero number of the list $(a - a', b - b', c - c', d - d', e - e', f - f')$ is positive. Then starting with an arbitrary 6-tuple (a, b, c, d, e, f), the next 6-tuple in the ordering is found as follows:

1. if $f < 49$, then $f \mapsto f + 1$;
2. else, if $e < 48$, then $(a, b, c, d, e, f) \mapsto (a, b, c, d, e + 1, e + 2)$;
3. else, if $d < 47$, then $(a, b, c, d, e, f) \mapsto (a, b, c, d + 1, d + 2, d + 3)$;;
4. else, if $c < 46$, then $(a, b, c, d, e, f) \mapsto (a, b, c + 1, c + 2, c + 3, c + 4)$;
5. else, if $b < 45$, then $(a, b, c, d, e, f) \mapsto (a, b + 1, b + 2, b + 3, b + 4, b + 5)$;;
6. else, if $a < 44$, then $(a, b, c, d, e, f) \mapsto (a + 1, a + 2, a + 3, a + 4, a + 5, a + 6)$;
7. elsewise (thus if $(a, b, c, d, e, f) = (44, 45, 46, 47, 48, 49)$) one is already at the last element in the lexicographical ordering.

With this algorithm and starting with $(1, 2, 3, 4, 5, 6)$ one goes through all the possible selections of 6 numbers from the set $\{1, 2, \ldots, 49\}$. For each selection (a, b, c, d, e, f) one can calculate $S = a + b + c + d + e + f$, multiply this number with the $(S - 21)th$ number from the list J. If this equals the product $abcdef$, then the result is printed.The output is a list of four possibilities:

(1) (1,2,3,4,11,15): sum 36; product: 3960; $I(36, 6, 49) = 110$;

(2) (1,4,5,7,9,19): sum 45; product: 23940; $I(45, 6, 49) = 532$;

(3) (2,8,11,14,19,22): sum 76; product: 1029952; $I(76, 6, 49) = 13552$;

(4) (5,8,11,26,37,43): sum 130; product: 18201040; $I(130, 6, 49) = 140008$.

4 Connection to standard partitions

If the size restriction, encoded by c, plays no role, we have

$$J(a, b, a + \Delta) = J(a, b, a) = p_b(a)\,, \qquad \Delta \geq 0\,, \tag{10}$$

where $p_k(n)$ is the number of partitions of n into k summands, which, as can be shown easily by reflecting the Ferrer diagramm, equals the number of partitions of n into summands of size at most k. One has to be careful not to confuse $p_k(n)$ with the number of partitions of n into exactly k summands. For examply, $5 = 3 + 2$ is an element of the set of partitions of 5 into at most 3 summands, but not element of the set of partitions of 5 into exactly 3 summands (summand 0 not allowed). The set of partitions of n into summands of size at most k can be divided into two disjoint sets; either at least one summand equals k, or not. If the largest summand is k, we can subtract it and find a partition of $n - k$ into summands of size at most k, if not, then the partition is an element of the set of partitions of n into summands of size at most $k - 1$. Hence we have

$$p_k(n) = p_{k-1}(n) + p_k(n - k)\,. \tag{11}$$

Since $J(a, b, c)$ satisfies $J(a, b, c) = J(a, b - 1, c) + J(a - b, b, c - 1)$, we have for $c > a$ and $b \geq 1$ that $a - b < a - 1$ and thus the relation

$$\begin{aligned} J(a, b, a) &= J(a, b - 1, a) + J(a - b, b, a - 1) \\ &= J(a, b - 1, a) + J(a - b, b, a - b) \end{aligned} \tag{12}$$

holds. Hence $J(a, b, a)$ satisfies the same recurrence relation as $p_b(a)$.

We let $\pi_k(n)$ denote the number of partitions of n into exactly k (nonzero) summands. Equivalently, $\pi_k(n)$ is the number of partitions of n with largest summand equal k. Since any partition of k whose summands do not exceed k must have a largest summand $l \leq k$ we have

$$p_k(n) = \sum_{l=1}^{n} \pi_l(n)\,,$$

and since any partition of n with summands not exceeding k must either have a largest summand k, or the summands do not exceed $k - 1$, we have

$$\pi_k(n) = p_k(n) - p_{k-1}(n)\,.$$

Consider a partition of n whose largest summand equals k. Then either there is exactly one summand equal k, or there are at least two summands equal k. Those partitions of the first kind are in one-to-one correspondence with the partitions of $n-1$, whose largest summand equals $k-1$ (as can be seen by subtracting one from the largest summand), or equivalently, by removing a block from the Ferrer diagram), whereas those of the second kind are in one-to-one correspondence with partitions of $n-k$ whose largest summand equals k (as can be seen by removing the largest summand completely, or equivalently, by removing the entire column in the Ferrer diamgram). From these considerations we obtain

$$\pi_k(n) = \pi_k(n-k) + \pi_{k-1}(n-1)\,. \tag{13}$$

A further relation between $p_k(n)$ and $\pi_k(n)$ exists, which connects the two recurrence relations (11) and (13). If $n = \lambda_1 + \ldots + \lambda_r$ is a partition of n, with $k = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$, then $n - k = \lambda_2 + \ldots + \lambda_r$ and no summand exceeds k. Removing the largest summand thus sets up an equivalence between partitions of n with largest summand k and partitions of $n-k$ with summands not exceeding k, hence $\pi_k(n) = p_k(n-k)$. Using equation (13) we thus have, $p_k(n) = \pi_k(n+k) = \pi_k(n) + \pi_{k-1}(n+k-1) = p_k(n-k) + p_{k-1}(n)$, which is eqn.(11).

If both the number of summands and the size of the summands plays no role, i.e., when $b > a$ and $c > a$, then $J(a,b,c)$ is just the number of partitions of a. We write $p(n)$ for the number of partitions of n. Clearly, we have

$$p(n) = p_n(n) = J(n,n,n) = \sum_{k=1}^{n} \pi_k(n) = \sum_{k=1}^{n} J(n-k,k,n-k)\,. \tag{14}$$

The numbers $p(n)$ can be thought as the diagonal of the matrix A with values $A_{ij} = p_i(j)$. In particular $p(n) = J(n,n,n)$. This finishes our discussion on the relation to the well-known partition numbers $p(n)$ and $p_k(n)$.