Linear realizations of conformal transformations

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Fall/Winter 2017

1 Introduction

Conformal transformations are those maps between manifolds that preserve a conformal structure. This comes down to the following: If $f : M \to N$ are manifolds with metrics g_M and g_N respectively, then the pull-back metric $f^*(g_N)$ differs from g_M only by a smooth function$\alpha : M \to \mathbb{R}$: $g_M = \alpha f^*(g_N)$. We will first restrict M and N to be $\mathbb{R}^n$ and consider only metrics that are multiples of the standard euclidean metric, which we denote by a dot: $a \cdot b$, the corresponding norm will be written with bars $|x|^2 = x \cdot x$. In this case the (local) conformal transformations are well-known and are given by compositions of the following basic transformations

$$\text{Translations:} \quad \tau_a(x) = x + a\,, \quad a \in \mathbb{R}^n\,,$$

$$\text{Orthogonal transformations:} \quad \omega_O(x) = Ox\,, \quad O \in O(n)\,,$$

$$\text{Dilations:} \quad D_\Delta(x) = e^\Delta x\,, \quad \Delta \in \mathbb{R}\,,$$

$$\text{Special Conformal Transformations:} \quad \varphi_b(x) = \frac{x + |x|^2 b}{1 + 2x \cdot b + |x|^2|b|^2}\,, \quad b \in \mathbb{R}\,.$$

Only two groups of these conformal tranformations types are linear and special conformal transformations are not even globally defined: for $x = -b/|b|^2$ the denominator vanishes. The special conformal transformations can be seen as the composition $I \circ \tau_b \circ I$ where $I(x) = x/|x|^2$ is the radial inversion. It is then clear that all these maps can be made globally defined by the standard projective map $f : \mathbb{R} \to S^n \subset \mathbb{R}^{n+1}$ given by

$$f_d(x) = \Big(\frac{2x_1}{1 + |x|^2}, \dots, \frac{2x_n}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1}\Big)\,, \quad x = (x_1, \dots, x_n)\,.$$

The map I then induces the map $I' : S^n \to S^n$ exchanging north and south: $I'(u_1, \dots, u_n, u_{n+1}) = (u_1, \dots, u_n, -u_{n+1})$. However, still, translations and dilations are not realized as linear maps on the sphere. We show in the sequel a way to find a linear realization of the conformal transformations. Then automatically all maps are globally defined.

The idea is not completely my own idea. I got inspiration from a lecture found on

`http://www.physics.usu.edu/Wheeler/GaugeTheory/09Mar27zNotes.pdf`

and in fact, it turns out, I just did something like James Wheeler did, just maybe from a slightly changed perspective. Therefore, this little not is to be seen as an alternative way to write down what James Wheeler explained already very well in his lecture on gauge theory.

2 The starting fibre bundle

We write $\pi : M = \mathbb{R}^n \times \mathbb{R}^{>0} \to \mathbb{R}^n$ for the bundle on $\mathbb{R}^n$ whose fibres contain the metrics that are conformally equivalent to the standard euclidean metric. Since any such metric is just a multiple of the standard euclidean metric, each fibre can be thought of as the set $\mathbb{R}^*$. Since however pull-back preserve the sign of the multiple, we even restrict to $\mathbb{R}^{>0}$. Any point (x, λ) in M can thus be thought as the metric on $T_x\mathbb{R}^n$ at x which is λ times the standard euclidean metric. Sections of this bundle can be seen as functions $\lambda : U \to \mathbb{R}^{>0}$ from an open set $U \subset \mathbb{R}^n$ to the set of positive real numbers.

If $f : U \to V$ is an invertible conformal transformation between two open sets $U, V \subset \mathbb{R}^n$, the pull-back f^* allows to define a map $\hat{f} : \pi^{-1}U \to \pi^{-1}V$ as follows: If we write η_x for the standard euclidean metric at x on $\mathbb{R}^n$, the pull-back along the inverse of f is a metric $(f^{-1})^*\eta$ on V and since f is conformal, we have $(f^{-1})^*\eta_x = \lambda(f^{-1})_{f(x)}\eta_{f(x)}$, where $\lambda(f^{-1})_{f(x)}$ is a positive real number. We have

$$\lambda(f^{-1})_{f(x)} = \frac{1}{\lambda(f)_x} \, ,$$

where $f^*(\eta_{f(x)}) = \lambda(f)_x\eta_x$ can be calculated from

$$\sum_k \frac{\partial f_k(x)}{\partial x_i}\frac{\partial f_k(x)}{\partial x_j} = \lambda(f)_x\delta_{ij}$$

where δ_{ij} are the components of the standard euclidean metric, i.e., 1 if $i = j$ and zero elsewise.

We find the following expressions for $\lambda(f)$ for conformal transformations:

- Translations: $\lambda(\tau_a) = 1$
- Orthogonal transfomations: $\lambda(\omega_O) = 1$
- Dilations: $\lambda(D_\Delta) = e^{2\Delta}$
- Special conformal transformations: $\lambda(\varphi_b) = \dfrac{1}{(1 + 2b \cdot x + |x|^2|b|^2)^2}$

The first three are proven rather easily. To find the fourth, either a tedious calculation will do the job, or use $\varphi_b = I \circ \tau_b \circ I$ and that $\lambda(I) = |x|^{-4}$; then, from

$$\lambda(\varphi_b)_x = \lambda(I)_x\lambda(\tau_b)_{I(x)}\lambda(I)_{I(x)+b} = \frac{1}{|x|^4}\frac{1}{|I(x) + b|^4}$$

and $|I(x)+b|^2 = |x|^{-2}(1+2b\cdot x+|x|^2|b|^2)$ the result follows. As the map I plays a prominent role, we will often include it; although it is not well-defined at the origin, it preserves the conformal structure.

Using the above found expressions λ, it is easy to find the induced maps $\hat{f} : M \to M$ for conformal transformations:

- Translations: $\widehat{\tau_a}(x,z) = (x+a,z)$.
- Orthogonal transformations: $\widehat{\omega_O}(x,z) = (Ox,z)$.
- Dilations: $\widehat{D_\Delta}(x,z) = (e^{\Delta}x, e^{-2\Delta}z)$.
- Inversion: $\widehat{I}(x,z) = (x/|x|^2, |x|^4 z)$.
- Special conformal transformations: $\widehat{\varphi_b}(x,z) = (\frac{x+|x|^2 b}{1+2b\cdot x+|x|^2|b|^2}, (1+2b\cdot x+|x|^2|b|^2)^2 z)$

We remark that the expression $x\sqrt{z}$ is invariant under dilations and inversions.

3 Mapping M to $\mathbb{R}^{n+2}$

We define a map $\Phi : M \to \mathbb{R}^{n+2}$ by the following recipe:

$$\Phi(x,z) = \left(\sqrt{z}x, \frac{1}{2}\sqrt{z}(1-|x|^2), \frac{1}{2}\sqrt{z}(1+|x|^2)\right).$$

We write the components of $\mathbb{R}^{n+2}$ as (Y,S,T) with $Y \in \mathbb{R}^n$, $S,T \in \mathbb{R}$. Then the points (Y,S,T) in the image of Φ satisfy $|Y|^2 + S^2 - T^2 = 0$. We see that the image of Φ lies in the light cone; the Lorentzian metric $\overline{\eta}$ on $\mathbb{R}^{n+2}$ is then of signature $(1,\ldots,1,-1)$.

We now want to prove the following statement:

All four kinds of conformal transformations $f : \mathbb{R}^n \to \mathbb{R}^n$ are realized in $\mathbb{R}^{n+2}$ as linear maps and preserve the Lorentzian metric, thus are elements of $O(n+1,1)$. In addition, any Weyl scaling on $\mathbb{R}^n$ is realized as a Wey rescaling on $\mathbb{R}^{n+2}$.

What we mean by this statement is the following. Any conformal transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ induces a map $\hat{f} : M \to M$. We show that we can find a map $\overline{f} : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ such that $\overline{f} \circ \Phi = \Phi \circ \hat{f}$. All these maps $\overline{f}$ are linear maps and take values in $O(n+1,1)$ and thus preserve the Lorentzian metric on $\mathbb{R}^{n+2}$. Finding $\overline{f}$ for the various f given above is not too difficult and hence we content with giving these results:

(t) For translations we find that $\overline{\tau_a}$ is given by the matrix

$$\begin{pmatrix} 1_n & a & a \\ -a^T & 1-|a|^2/2 & -|a|^2/2 \\ a^T & |a|^2/2 & 1+|a|^2/2 \end{pmatrix},$$

where a^T denotes the transpose in general, thus here it denotes the row vector associated to the column vector a, and 1_n denotes the $n \times n$ unit-matrix.

(o) For orthogonal transformations $\omega_O : x \mapsto Ox$ we find that $\overline{\omega_O}$ is represented by the matrix

$$\begin{pmatrix} O & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(d) For dilations $D_\Delta : x \mapsto e^\Delta x$ we find that $\overline{D_\Delta}$ is represented by the matrix

$$\begin{pmatrix} 1_n & 0 & 0 \\ 0 & \cosh(\Delta) & -\sinh(\Delta) \\ 0 & -\sinh(\Delta) & \cosh(\Delta) \end{pmatrix} .$$

Thus dilations induce Lorentz boosts.

(i) The inversion $I : x \mapsto \frac{x}{|x|^2}$ induces a map $\overline{I}$ represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(sct) The special conformation transformations φ_b induce mappings $\overline{\varphi_b}$ given by the matrix

$$\begin{pmatrix} 1 & -b & b \\ b^T & 1-|b|^2/2 & |b|^2/2 \\ b^T & -|b|^2/2 & 1+|b|^2/2 \end{pmatrix} .$$

By a direct calculation, we see that all of the above matrices satisfy the relation $\overline{f}^T \overline{\eta} \overline{f} = \overline{\eta}$ where $\overline{\eta}$ is the diagonal matrix with entries $(1, \dots, 1, -1)$. Thus all these matrices belong to $O(n+1, 1)$.

If w_σ is the Weyl-rescaling that multiplies the metric at x with $e^{2\sigma(x)}$. Then x is left fixed, but z is mapped to $e^{2w}z$. On $\mathbb{R}^{n+2}$ this induces a rescaling $(Y, S, T) \mapsto e^w(Y, S, T)$, which commutes with all the above. Although a Weyl rescaling is not realized as an $O(n+1, 1)$-transformation, it preserves the conformal class.

Thus, the statement has been proven.

4 Combining the conformal transformations

There are some useful relations between the various conformal transformations:

$$\omega_O D_\Delta = D_\Delta \omega_O , \quad D_{\Delta_1} \circ D_{\Delta_2} = D_{\Delta_2} \circ D_{\Delta_1} , \quad \tau_{a_1} \circ \tau_{a_2} = \tau_{a_2} \circ \tau_{a_1} ,$$

$$\omega_O \circ \tau_a = \tau_{Oa} \circ \omega_O , \quad \tau_a \circ D_\Delta = D_\Delta \circ \tau_{e^{-\Delta}a} , \quad I \circ O = O \circ I ,$$

$$I \circ D_\Delta = D_{-\Delta} \circ I , \quad I \circ \tau_a = \varphi_a \circ I , \quad I \circ I = \mathrm{id} .$$

A little inspection shows that all such relations also hold for the induced maps $\hat{f} : M \to M$. Furthermore, it can be checked by matrix calculations that all these relations also hold for the maps $\overline{f} : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$. The map $f \mapsto \overline{f}$ is then easily seen to have the property $\overline{f_1 \circ f_2} = \overline{f_1} \circ \overline{f_2}$ for all conformal transformations, translations, rotations, dilations, inversions and special conformal transformations. The maps $f \mapsto \overline{f}$ thus preserves the product structure.

By introducing a parameter t that takes values in a small interval around 0, one sees that the conformal transformations τ_{ta} and φ_{tb} can be connected to the identity map. Clearly, special orthogonal transformations and dilations can also be connected to the identity. But then it, using dimension counting, it is clear that the maps the images of τ_a, φ_b, ω_O and D_Δ under the map $f \mapsto \overline{f}$ generate a subgroup of $O(n+1,1)$ of dimension $n+n+\frac{1}{2}n(n-1)+1 = \frac{1}{2}(n+1)(n+2)$, which is the connected component of $O(n+1,1)$, which is contained in $SO(n+1,1)$.

5 The image of Φ and conclusion

As mentioned before, the image $\Phi(M)$ lies in the positive light-cone, $|Y|^2 + S^2 - T^2 = 0$ and $T > 0$. We furthermore have $S + T = \sqrt{z} > 0$ and $T - S = \sqrt{z}|x|^2 \geq 0$, with equality only for $x = 0$. For any point on the positive light cone (Y, S, T) with $S + T > 0$, we have $|Y|^2 + S^2 = T^2$ and we can put

$$(x, z) = \Big(\frac{Y}{S+T}, (S+T)^2\Big)$$

and then $\Phi(x, z) = (Y, S, T)$. This then shows that we have

The image of Φ is the open subset of the positive light-cone $\sqrt{|Y|^2 + S^2} = T > 0$ for which $T + S > 0$.

Thus $\Phi(M)$ can be seen as the positive light-cone with the line $l :\ S + T = 0$ cut out. The map $\overline{I}$ maps the line $l' :\ T - S = 0$ to the line l. Thus $\Phi(M) \cup \overline{I}(\Phi(M))$ is the complete positive light-cone.

The map $(Y, S, T) \mapsto (Y, S, -T)$, or equivalently $(Y, S, T) \mapsto (-Y, -S, -T)$, is not among the maps $\overline{f}$ for f running over the conformal transformations, including I. The other reflections $Y^i \mapsto -Y^i$ for some fixed i while keeping the other Y^j and S fixed are images of corresponding orthogonal transformations, and the reflection $(Y, S, T) \mapsto (Y, -S, T)$ is induced by I.

All in all we have shown that the conformal transformations on $\mathbb{R}^n$ can be realized as linear maps on $\mathbb{R}^{n+2}$ preserving a Lorentzian metric. Furthermore the conformal transformations generate a subgroup of $O(n+1,1)$ containing the connected component. The infinitesimal conformal transformations thus have a Lie algebra structure isomorphic to $so(n+1,1)$. The map $\mathbb{R} \to \mathbb{R}^{n+2}$ that is used can be seen as emerging from the line-bundle $\pi : M \to \mathbb{R}^n$ that attaches to each point of $\mathbb{R}^n$ a metric conformally equivalent to the standard euclidean metric.