Constrained dynamics by example

DBW

summer 2016

Abstract

By means of an example we provide a small lesson on constrained dynamics. This lesson has its natural place in a course on classical dynamics, lagrangean and hamiltonian dynamics.

1 The example

Suppose we have an object with mass m that can slide without friction along a surface that has the form of a parabola $P: \quad y = \frac{x^2}{2b}$ with $b > 0$. We assume the presence of a constant gravitational force field $\mathbf{g} = (0, -g)$. Our goal will be to motivate the equations of motion and to find a method to identify the forces that hold the object on the parabola.

Before we tackle the problem on a more formal level, let us go through some basic physical considerations. The parabola will not exert an amount of work; if this were so, we could invent a trick to extract in infinite amount from the parabola by letting the mass slide a bit along the parabola absorbing the energy from the parabola and move the mass back up, which only requires the gravitational energy. This would give us a surplus of the energy given by the parabola. Also, the assumption that the mass can slide frictionless means that the parabola will not absorb an amount of work. If F is the force exerted by the parabola and δx is an infinitesimal displacement vector between two positions of the mass, then $F\delta x = 0$. As F and δx are vectors and both are nonzero, this means that F is normal to δx, and thus, the force exerted by the parabola is a normal force. The only force doing work is the gravitational field. Hence $\Delta \frac{1}{2} m v^2 = mg\Delta h$. We can thus give the velocity as a function of the y-coordinate.

Suppose the mass is initially at $t = 0$ at rest at the point (x_0, y_0). The total energy is thus mgy_0. The kinetic energy is given by

$$\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m\Big(1 + \frac{x^2}{b^2}\Big)\dot{x}^2 \,.$$

We thus arrive at an energy balance equation

$$\frac{1}{2}\Big(1 + \frac{x^2}{b^2}\Big)\dot{x}^2 + gy = gy_0 \,.$$

Substituting $y = \frac{x^2}{2b}$ we arrive at a differential equation for x. In principle such an equation can be solved, approximately, numerically, or even exactly with more patience (maybe). However, we did not obtain an equation that describes the normal forces of the parabola. In the appendix we give an expression for the period of the oscillation of the object in the parabola.

We now combine some physical concepts with the geometry of the problem. Since the mass is to follow the parabola, we insist that for the path $(x(t), y(t))$ we have $y(t) = \frac{x(t)^2}{2b}$ and thus

$$\dot{y}(t) = \frac{x(t)\dot{x}(t)}{b}, \quad \ddot{y}(t) = \frac{\dot{x}(t)^2 + x(t)\ddot{x}(t)}{b}.$$

From this expression we can read off the total force $\mathbf{F} = (F_x, F_y)$

$$F_x = m\ddot{x}(t), \quad F_y = \frac{m}{b}(\dot{x}(t)^2 + x(t)\ddot{x}(t)).$$

We thus have a relation

$$-xF_x + bF_y = m\dot{x}(t)^2.$$

For the normal force $\mathbf{N} = (N_x, N_y)$ we thus find

$$-xN_x + bN_y = m\dot{x}(t)^2 + mg.$$

The vector $\mathbf{n} = (-x, b)$ is normal to the parabola. Thus we can solve the last equation for $\mathbf{N}$. The result is

$$\mathbf{N} = \frac{m\dot{x}(t)^2 + mg}{x^2 + b^2}(-x, b),$$

which contains a term $m\dot{x}(t)^2$, which reminds us of the centripetal force.

Let us show how to slightly generalize the above way to derive the normal forces. We consider the movement of an object attached to a hypersurface N with a normal vector $\mathbf{n}$. The gradient of the function $C(x, y) = y - \frac{x^2}{2b}$ is a normal vector $\mathbf{n}' = (-\frac{x}{b}, 1)$ to the parabola; for convenience, we might as well work with $\mathbf{n} = (-x, b)$. For more general submanifolds $N \subset M$, one has to consider a set of normal vectors that are a complement to TN in TM.

Since the velocity $\mathbf{v}$ is an element of TN, we obtain

$$\mathbf{n} \cdot \mathbf{v} = 0,$$

which for the parabola leads to $-xv_x + bv_y = 0$. Differentiating $\mathbf{n} \cdot \mathbf{v} = 0$ we obtain

$$-\mathbf{v} \cdot \frac{d}{dt}\mathbf{n} = \mathbf{n} \cdot \mathbf{a}.$$

We can write the acceleration due to the normal forces as $\alpha\mathbf{n}$ for some real number α, since the surface is not to do any work. If one then writes $\mathbf{a} = \mathbf{a}' + \alpha\mathbf{n}$, where $\mathbf{a}'$ contains the known forces, not due to the normal forces, one can use the above equations to solve for α. This then solves the problem of finding the normal forces.

For the parabola we differentiate $-xv_x + bv_y = 0$ and find

$$v_x^2 = -xa_x + ba_y.$$

and writing the total acceleration as $\alpha(-x, b) - \mathbf{g}$, we get

$$v_x^2 = \alpha(x^2 + b^2) - gb$$

and hence

$$\alpha = \frac{\dot{x}^2 + gb}{x^2 + b^2}\,.$$

For the normal force we then find again

$$\mathbf{N} = m\alpha\mathbf{n} = m\frac{\dot{x}^2 + gb}{x^2 + b^2}(-x, b) = \frac{(-x, b)}{x^2 + b^2}m(\dot{x}^2 + gb)\,.$$

The norm of the normal force is thus $\frac{m(\dot{x}^2+gb)}{\sqrt{x^2+b^2}}$. If the mass is at rest at the bottom of the parabola we recover that the magnitude of the normal force is mg.

2 The formalism

We are interested in the case, where the degrees of freedom $(q^i, \dot{q}^i)$ satisfy a set of constraints C_a of the type $C_a(q^i) = 0$. The constraints are thus independent of the velocities and only depend on time through $q^i(t)$. Such constraints are called holonomic. The most general holonomic constraint is by definition on that involves constraint equations that depend on the coordinates and possibly of time, but not of the velocities.

2.1 Elimination of coordinates

The most straightforward approach would be to use the constraints to eliminate some of the coordinates. We thus actually try to perform a coordinate transformation

$$u^i = f^i(q^1, \ldots, q^n)$$

such that the constraints take the form $u^i = 0$ for some of the i. Let us assume we have m constraints and that we find a coordinate transformation so that the constraints read $u^i = 0$ for $i = n - m + 1, \ldots, n$. Then the Lagrangean can be translated to a Lagrangean depending only on $n - m$ coordinates $u^1, \ldots, u^{n-m}$, which are unconstrained. For this to work, one needs to assume certain smoothness conditions on the constraints.

In order that a local coordinate transformation works, the constraints need to define a smooth submanifold. A necessary and sufficient condition is that the functions $C_a(q^i)$ have linearly independent derivative one-forms at each point, where the constraint is satisfied. Alternatively and equivalently $\mathrm{d}C_1 \wedge \mathrm{d}C_2 \wedge \ldots \wedge \mathrm{d}C_m(p)$ is nonzero for each p with $C_a(p) = 0$ for all a. In this case, the constraints define new coordinates orthogonal to the subset defined by the constraints.

Let us see how this works for our example. There is only one constraint $C = y - \frac{x^2}{2b} = 0$ and its differential is $\mathrm{d}C = \mathrm{d}y - \frac{x}{b}\mathrm{d}x$. The one-form $\mathrm{d}C$ vanishes nowhere. Consider the coordinate transformation

$$u(x, y) = x\,, \qquad v(x, y) = y - \frac{x^2}{2b}\,.$$

The Jacobian matrix of this transformation is

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} 1 & 0 \\ -\frac{x}{b} & 1 \end{pmatrix}\,,$$

which is invertible everywhere. Thus at least locally, we can always invert the coordinate transformation and express x and y in terms of u und v. It is however not too difficult to show that it can also be inverted globally:

$$x(u,v) = u\,, \qquad y(u,v) = v + \frac{u^2}{2b}\,.$$

In the new coordinates the constraint reduces to $v = 0$. Let us try and express the old Lagrangean in the new coordinates. The kinetic terms transform to

$$m(\dot{x}^2 + \dot{y}^2) = m(\dot{u}^2 + \dot{v}^2 + 2\frac{u}{b}\dot{u}\dot{v} + \frac{u^2}{b^2}\dot{u}^2)$$

which reduces by the constraint $v = 0$ to

$$m(\dot{u}^2 + \frac{u^2}{b^2}\dot{u}^2)\,.$$

For the potential term we simply get $mgy = mg\frac{u^2}{2b}$. Thus the new Lagrangean is

$$L(u) = L(u, v = 0) = \frac{m}{2}(1 + \frac{u^2}{b^2})\dot{u}^2 - mg\frac{u^2}{2b}\,.$$

As to be expected, we obtain the same result when we use the equation $y = \frac{x^2}{2b}$ to eliminate y from the Lagrangean. Using the Euler–Lagrange equations the new equations of motion are now

$$(1 + \frac{u^2}{b^2})\ddot{u} = -\frac{\dot{u}^2 u}{b^2} - \frac{gu}{b}\,.$$

Getting the expression for the normal force from this expression is not an easy issue! The interpretation is somewhat blurred by the kinetic term, which now contains a position dependent "mass matrix" $\frac{m}{2}(1 + \frac{u^2}{b^2})$.

One can try to manipulate the mass matrix to get a standard form and then read off the normal forces. Let us consider the variable

$$z = \frac{1}{2}u\sqrt{1 + \frac{u^2}{b^2}} + \frac{b}{2}\sinh^{-1}\Big(\frac{u}{b}\Big)\,.$$

We then find that

$$\frac{dz}{du} = \sqrt{1 + \frac{u^2}{b^2}}\,.$$

Now the kinetic term becomes $\frac{1}{2}m\dot{z}^2$, which should come as no surprise. z is just the arc length of the parabola between $x = 0$ and $x = u$. $\dot{z}$ is thus nothing more than $\sqrt{\dot{x}^2 + \dot{y}^2}$, the real velocity of the mass. There is however little hope to express u in z to get a tractable expression for the potential term.

2.2 Lagrangean approach

There is a convenient way to incorporate constraints into a lagrangean scheme. For each constraint C_a one introduces a Lagrange-multiplier λ^a. By varying the action, one also varies the multipliers; this then reduces to the constraint. Let us see how this works in general, and then we apply it to the example of the parabola.

The action with the constraints added is

$$S[q,\lambda] = \int (L(q,\dot{q}) - \lambda^a C_a(q))\mathrm{d}t = \int L^*(q,\dot{q})\mathrm{d}t\,, \qquad L^* = L - \lambda^a C_a\,,$$

and thus variing the action we get the following equations of motion

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}^i} = \frac{\partial L^*}{\partial q^i}\,, \qquad -\frac{\partial L^*}{\partial \lambda^a} = C_a = 0\,.$$

Working out, we get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = -\lambda^a \frac{\partial C_a}{\partial q^i}$$

together with the constraints $C_a = 0$. The ordinary equations of motion, those without constraints, are thus no longer zero, but are a linear combination of gradients ∇C_a. The term $\lambda^a \nabla C_a$ can be interpreted as a force that arises due to the constraint. Indeed, by dimensional analysis, L has units of energy and thus so does $\lambda^a C_a$ and hence $\lambda^a \nabla C_a$ thus has units of force. The normal forces thus arise almost immediately.

Let us check this with our example. We have just one constraint, $y - \frac{x^2}{2b}$, which has units of length. The parameter λ in $\lambda(y - \frac{x^2}{2b})$ which we add to the Lagrangean thus has units of force. Our new Lagrangean is

$$L^* = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy - \lambda\Big(y - \frac{x^2}{2b}\Big)\,.$$

The Euler–Langrange equations now become

$$m\ddot{x} = \lambda\frac{x}{b}\,, \qquad m\ddot{y} = -\lambda - mg \qquad y = \frac{x^2}{2b}\,.$$

Whereas $-mg$ represents the gravitational force, the normal force is found by looking at the remaining terms of $(m\ddot{x}, m\ddot{y})$. We then find $\mathbf{N} = (\lambda\frac{x}{b}, -\lambda) = -\frac{\lambda}{b}(-x,b)$. In order to find λ we have to work with the constraint.

Differentiating the constraint, we have $b\ddot{y} = \dot{x}^2 + x\ddot{x}$. Inserting the other Euler–Lagrange equations we get

$$mb\ddot{y} = -\lambda b - mbg = m\dot{x}^2 + \lambda\frac{x^2}{b}\,.$$

Thus we find

$$\lambda = -\frac{mb^2 g + mb\dot{x}^2}{b^2 + x^2}$$

and thus for the normal force we find

$$\mathbf{N} = -\frac{\lambda}{b}(-x,b) = \frac{mbg + m\dot{x}^2}{b^2 + x^2}(-x,b)\,.$$

2.3 Restricted variations

When one has a Lagrangean, one obtains the equations of motion by varying the action. However, in a constrained situation, not all variations are allowed. Let us try to formalize this.

Suppose we have a Lagrangean $L(q, \dot{q})$ and an action $S[q] = \int_{t_1}^{t_2} L(q, \dot{q})\mathrm{d}t$. Applying a variation δq to q and thus a variation $\frac{d}{dt}\delta q = \delta\dot{q}$ to $\dot{q}$ we obtain

$$\delta S = \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\Big)\delta q + \Big[\delta q \frac{\partial L}{\partial \dot{q}}\Big]_{t_1}^{t_2} .$$

The variations are chosen to be zero at the endpoints so the boundary term vanishes. We stress, that δq vanishes at the endpoints, but that $\delta\dot{q}$ is free; the variation keeps the endposition fixed but the endvelocity may vary.

The general conclusion is that the term in the brackets inside the integral has to vanish, and thus the equations of motion are obtained. This argument brakes down when constraints need to be considered. So suppose we have a finite set of constraints $C_a(q)$, where we again assume that the constraints involve explicitly only the coordinates. Also, we assume that the constraints are regular. They must define a smooth subvariety. Constraints of the form $q_1^2 = 0$ are not allows and these can be reduced to $q_1 = 0$. However, constraints of the form $q_1 q_2 = 0$ define a singularity at the point $q_1 = q_2 = 0$. We thus exclude these kind of situations.

Let us try to sketch the bigger picture. In the unconstrained situation the coordinates q parametrize a manifold M. For the sake of simplicity we may as well take $M = \mathbb{R}^n$. The constraints are to define a smooth submanifold $N \subset M$. At each point $n \in N$ we can write the tangent space at n as a direct sum $T_n M = T_n N \oplus T_n N^\perp$. Since the variations are to lie inside $T_n N$, their inner product with $T_n N^\perp$ vanishes. Now we look again at the variation of the action and recognize that the term inside the brackets must lie in $T_n N^\perp$ at each point $n \in N$.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \in T_n N\,, \quad \forall n \in N\,.$$

Our next goal is thus to identify $TN^\perp$, the collection of all $T_n N^\perp$ for all $n \in N$. This is rather easy, since the constraints $C_a(q) = 0$ define N, and hence the gradients $\nabla C_a = (\partial_1 C_a, \ldots, \partial_n C_a)$ are perpendicular to N. Thus TN is spanned by the gradients ∇C_a. But this implies nothing more than that there exist functions $\lambda_a(q)$, for each constraint one, at each point $n \in N$ such that

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \sum_a \lambda_a(q)\partial_i C_a(q)\,.$$

We thus see that the restricted variations lead directly to the method of lagrange multipliers.

2.4 The Hamiltonian

From the unconstrained Lagrangean $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$ we obtain the momenta

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}\,, \qquad p_x = \frac{\partial L}{\partial \dot{y}} = m\dot{y}\,.$$

Therefore we obtain the unconstrained Hamiltonian $H = \frac{1}{2m}(p_x^2 + p_y^2) + mgy$. To deal with the constraint, we could eliminate y of course. Since this will lead to nothing new, we leave this exercise to the reader and go on to another method. A suggestion would be to add a term to the Hamiltonian of the form $\lambda^a C_a$, similar to Lagrange multipliers. But then what is the role of the λ^a? Are they going to be dynamical variables? If so, then phase space might become odd-dimensional, not allowing for a dymplectic form. In order to do so, we thus need a guiding principle, a good motivation. To reach this goal we first go over some general aspects about Lagrangeans and Hamiltonians.

A dynamical system is often formulated in terms of a Lagrangean L on the tangent bundle $\mathcal{P} = TM$ of some appropriate configuration space M. Then one defines a map $\mathcal{L} : \mathcal{P} \to \mathcal{Q} = T^*M$ to the phase space by putting locally

$$(q^i, \dot{q}^i) \mapsto (q^i, p_i), \qquad p_i = \frac{\partial L}{\partial \dot{q}^i}.$$

This map is not always an isomorphism. The image might be a lower-dimensional subset of $\mathcal{Q}$. Often this is a hint that there are some symmetries and not all variables are physical. In the above (unconstrained) example however, the map is one-to-one and invertible. However, in the example not the whole space $\mathcal{P}$ is allowed, hence also the image of $\mathcal{L}$ will be a proper subset of $\mathcal{Q}$. We are thus in the situation where we have subsets $\mathcal{P}' \subset \mathcal{P}$ and $\mathcal{Q}' \subset \mathcal{Q}$ and a map $\mathcal{L}$ that is to map $\mathcal{P}'$ to $\mathcal{Q}'$. Since in the unconstrained case the map $\mathcal{L}$ is invertible, one can directly translate the constraints C_a on $\mathcal{P}$ to constraints D_a on $\mathcal{Q}$. This then will define the subset $\mathcal{Q}'$ and again, as above, we require that the functions D_a define a regular submanifold.

Having defined the map $\mathcal{L}$ and two appropriate submanifolds, we obtain a restricted map $\mathcal{L}' : \mathcal{P}' \to \mathcal{Q}'$. We define the Hamiltonian in the usual way

$$H = \sum_i \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L(q, \dot{q}) = q^i p_i - L(q, \dot{q}).$$

As it stands, H seems to be a function of q and $\dot{q}$. Indeed, given q and $\dot{q}$ it is unambiguous how to calculated H. We can by the third form $H = \dot{q}p - L$ also view H as a function of q, $\dot{q}$ and p. Then under infinitesimal variations

$$\delta H = -\frac{\partial L}{\partial q}\delta q + \Big(p - \frac{\partial L}{\partial \dot{q}}\Big)\delta\dot{q} + \dot{q}\delta p.$$

Thus, if we consider a points $(q,p) \in \mathcal{Q}$ and go along its fibre $\mathcal{L}^{-1}(q,p)$ the function H does not change! We might thus view H as a function of p and q alone. In order to calculate H we can take any point in the fibre $\mathcal{L}^{-1}(q,p)$ since H is constant along this fibre.

To get the temporal dynamics we write the last equation as

$$\Big(\frac{\partial H}{\partial q} + \dot{p}\Big)\delta q + \Big(\frac{\partial H}{\partial p} - \dot{q}\Big)\delta p = 0.$$

But we should remember that not all variations are allowed! The variations are parallel to $\mathcal{Q}'$ and thus the above equation means that the vector

$$\Big(\frac{\partial H}{\partial q} + \dot{p},\ \frac{\partial H}{\partial p} - \dot{q}\Big)$$

is normal to $\mathcal{Q}'$ and thus a linear combination of the gradients $(\partial_q D_a, \partial_p D_a)$. We thus arrive at the following equations fo motion

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + \mu^a \frac{\partial D_a}{\partial q^i}\,, \quad \dot{q}_i = \frac{\partial H}{\partial p^i} + \mu^a \frac{\partial D_a}{\partial p^i}\,.$$

The dynamics thus seems to be governed by the Hamiltonian $H' = H + \mu^a D_a$ and the Poisson-bracket $\{,\}$ defined by

$$\{A, B\} = \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}$$

which allows a recasting of the equation of motion into the form

$$\dot{p} = \{p, H'\}\,, \quad \dot{q} = \{q, H'\}\,, \quad \dot{A} = \{A, H'\}\,.$$

One might wonder how to treat the undetermined μ^a in these Poisson-brackets. First we remark that if A is any function of p and q, then

$$\{A, \mu^a D_a\} = \{A, \mu^a\} D_a + \mu^a \{A, D_a\}$$

and the first term vanishes on $\mathcal{Q}'$ thus not giving any problems. The second term precisely appears in the equation for $\dot{A}$. So terms of the form $\{A, \mu^a\}$ seem to be excluded.

In order to get consistent dynamics we need to ensure that the constraints are preserved in time, we need to check that $\dot{D}_a = \{H, D_a\}$ vanishes. This might lead to new constraints. The whole programm of Dirac and Bargmann is developped to deal with these cases. To return to simplicity, we focus on our example.

The following Poisson brackets are easily deduced using the definitions

$$\{x, p_x\} = \{y, p_y\} = 1\,, \quad \{x, p_y\} = \{y, p_x\} = \{x, y\} = \{p_x, p_y\} = 0\,.$$

Also the following identites are easily obtained

$$\{A, B\} = -\{B, A\}\,, \quad \{A, BC\} = \{A, B\}C + B\{A, C\}\,.$$

With a bit more labor one also obtains the Jacobi identity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0\,.$$

From these identities all other Poisson brackets can now be calculated. One finds for example

$$\{x, H\} = \frac{p_x}{m}\,, \quad \{y, H\} = \frac{p_y}{m}\,, \quad \{p_x, H\} = 0\,, \quad \{p_y, H\} = -mg\,.$$

These are the usual definitions; $m\dot{x} = p_x$ and $m\dot{y} = p_y$.

First we consider the dynamics of the constraint $D = y - \frac{x^2}{2b}$ itself

$$\dot{D} = \{D, H'\} = \{D, H\} = \{y, H\} - 2\frac{x}{2b}\{x, H\} = \frac{p_y}{m} - \frac{x p_x}{bm}\,.$$

In order that the dynamics stays on the surface defined by the parabola, we need the following (secondary) constraint

$$D' = b p_y - x p_x = 0\,.$$

This equation is in fact nothing more than the requirement $\dot{y} = \frac{x\dot{x}}{b}$, which we have already encountered before. This constraint says that the velocity is parallel to the parabola.

Second, we check whether the new constraint gives rise to new constraints, since we need $\dot{D}' = 0$. We find

$$\{D', H'\} = \{bp_y - xp_x, H\} + \mu\{bp_y - xp_x, y - \frac{x^2}{2b}\} = -mgb - \frac{p_x^2}{m} - \mu(b + \frac{x^2}{b}) \,.$$

This equation lets us solve for μ without giving rise to new constraints. This then stops the Dirac–Bargmann algorithm, and we end up with the following value for μ

$$\mu = -b\frac{mgb + \frac{p_x^2}{m}}{b^2 + x^2} \,.$$

This precisely corresponds to λ from section 2.2. The was to be expected sind the Lagrange multipliers added to the Lagrangean arise through $L' = K - U - \lambda_a C^a$, where K is the kinetic energy and U the potential energy. The Hamiltonian obtained from L' is then of the form $H' = 2K - L' = K + U + \lambda_a C^a$. We can thus identify μ_a with λ_a.

We thus find the following expression for the total Hamiltonian

$$H' = H + \mu D = \frac{p_x^2 + p_y^2}{2m} + mgy - b\frac{mgb + \frac{p_x^2}{m}}{b^2 + x^2}(y - \frac{x^2}{2b}) \,.$$

This Hamiltonian is to be supplemented with the following constraints $D = y - \frac{x^2}{2b} = 0$ and $D' = bp_y - xp_x = 0$. The dynamics thus takes place on a two-dimensional surface Σ in the four-dimensional space $\mathbb{R}^4$ with coordinates (x, y, p_x, p_y), where Σ is defined by $D = 0$ and $D' = 0$. The time-evolution of any function $A = A(x, y, p_x, p_y)$ on Σ is determined by $\dot{A} = \{A, H'\}$.

One word about the interpretation of the additional term in the Hamiltonian μD. On Σ the Poisson brackets $\{x, \mu D\}$ and $\{y, \mu D\}$ vanish and the nontrivial brackets are

$$\{p_x, \mu D\} = -x\frac{mgb + \frac{p_x^2}{m}}{b^2 + x^2} \,, \qquad \{p_y, \mu D\} = b\frac{mgb + \frac{p_x^2}{m}}{b^2 + x^2}$$

where we restrict to Σ in order to only need to calculate $\mu\{p_x, D\}$ since $D\{p_x, \mu\}$ is zero by $D = 0$ on Σ, for example. Since $\{\mathbf{p}, H'\} = \dot{\mathbf{p}}$ is a force, we see that the additional term in the Hamiltonian gives rise to a force $\frac{mgb+\frac{p_x^2}{m}}{b^2+x^2}(-x, b)$. This precisely corresponds to the normal force $\mathbf{N}$. This then ends our discussion on the Hamiltonian.

3 Conclusion

From this simple example one can learn that constraining the dynamics on a submanifold can be done by Lagrange multipliers, which have an interpretation as the forces that restrict the objects on the submanifold. In the example the map from the tangent bundle to the cotangent bundle is not singular and the constraints are essentially the same. Whereas in the lagrangeanapproach further constraints are obtained by differentiating the constraint, in the Hamiltonian approach this happens by virtue of the Poisson bracket.

From a pedagogical point of view one might first give such an example before one teaches the singular nature of the map $\mathcal{L} : TM \to TM^*$ when one treats electromagnetism. In this example of the parabola the role of the constraint is clear and does not arise due to maps with singular nature. The procedure for obtaining an appropriate Hamiltonian is the same however. Therefore the conceptual difficulties disappear in the background.

Appendix: Period of oscillations

From conservation of momentum we find

$$\dot{x} = \sqrt{2gb^2}\sqrt{\frac{y_0 - y}{x^2 + b^2}}\,.$$

We substitute $y = \frac{x^2}{2b}$ and $y_0 = \frac{x_0^2}{2b}$ and find

$$\dot{x} = \sqrt{gb}\sqrt{\frac{x_0^2 - x^2}{x^2 + b^2}}\,.$$

$$T = 4\int_{-x_0}^{x_0} \mathrm{d}t = 4\int_0^{x_0} \frac{\mathrm{d}x}{\dot{x}} = \frac{4}{\sqrt{gb}}\int_0^{x_0} \sqrt{\frac{x^2+b^2}{x_0^2 - x^2}}\mathrm{d}x$$

Writing $x = x_0\cos(\theta)$ we obtain

$$T = \frac{4}{\sqrt{gb}}\int_0^{\pi/2} \sqrt{x_0^2\cos^2(\theta) + b^2}\ \ \mathrm{d}\theta = 4\sqrt{\frac{x_0^2+b^2}{gb}}\int_0^{\pi/2} \sqrt{1 - k^2\sin^2(\theta)}\ \ \mathrm{d}\theta$$

where $k^2 = \frac{x_0^2}{x_0^2+b^2}$. The integral on the right-hand side is an elliptic integral of the second kind:

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2(\theta)}\ \ \mathrm{d}\theta\,, \qquad E(\phi|k) = \int_0^{\phi} \sqrt{1 - k^2\sin^2(\theta)}\ \ \mathrm{d}\theta$$

so that $E(k) = E(\frac{\pi}{2}|k)$. Elliptic integrals of the second kind $E(k)$ cannot be expressed in terms of other elementary functions unless k takes trivial values, e.g. $k = 0$ or $k = 1$. With these definitions we obtain

$$T = 4\sqrt{\frac{x_0^2+b^2}{gb}}E\Big(\sqrt{\frac{x_0^2}{x_0^2+b^2}}\Big)\,.$$

We can obtain a more intuitive form if we first define the characteristic time of the parabola $\tau = \sqrt{\frac{b}{g}}$. Then we find an angle $0 \le \alpha \le \pi/2$ such that $\sin(\alpha) = \frac{x_0}{\sqrt{x_0^2+b^2}}$ and $\cos(\alpha) = \frac{b}{\sqrt{x_0^2+b^2}}$ and then

$$T = 4\tau\frac{E(\sin(\alpha))}{\cos(\alpha)}\,.$$