Forces in a rotating reference frame DBW

June 15, 2017

Many derivations of the formulae for the centrifugal and coriolis forces exist. In this writeup I present my way of deriving the centrifugal and coriolis forces, and of course, my way can be criticized in many ways. It might not be too intuitive, but is straightforward. It does use some notions like group representations, which might not be well-known to all first-semester students. In the appendix I present some mathematical details that are left out in the main argument. The words in italics are explained very shortly in the appendix - the reader not acquainted with the mathematical machinery is urged to first skim the appendix.

The set-up. We consider to coordinate frames K and K'. The coordinate frame K is an inertial frame, whereas K' is a rotating frame. We choose the coordinate centers to coincide, so that both frames are linked by a *rotation matrix* $R(t) \in SO(3)$ at all times t. The coordinates in K are denoted x, y, z and the position vector is $r = (x, y, z)$. The coordinates in K' are primed, x', y' and z'; the position vector is denoted $r' = (x', y', z')$. We assume we have some small mass moving around; in K its position at time t is denoted $r(t)$, in K' its position is denoted $r'(t)$. These position vectors are related by a time-dependent $SO(3)$-transformation $r'(t) = R(t)r(t)$. We use a dot above a symbol for time derivatives.

If we differentiate the expression $r'(t) = R(t)r(t)$ twice we get the relations:

$$\dot{r}'(t) = \dot{R}(t)r(t) + R(t)\dot{r}(t) = \dot{R}(t)R^{-1}(t)r'(t) + R(t)\dot{r}(t)\,, \tag{1}$$

and

$$\ddot{r}'(t) = \ddot{R}(t)r(t) + 2\dot{R}(t)\dot{r}(t) + R(t)\ddot{r}(t)\,. \tag{2}$$

In fact, all information is already in these expressions. The rest of this writeup consists in making sense of these expressions.

Translating to angular velocity. In general, if we have a rotation around an axis, we have a angular velocity vector ω such that for any fixed point in K' we have $\dot{r}(t) = \omega \times r(t)$. Differentiating the relation defining ω another time, assuming it does not depend on time, we have

$$\ddot{r} = \omega \times (\omega \times r) = (\omega \cdot r)\omega - \omega^2 r = -\omega^2 r_\perp\,. \tag{3}$$

The term $r_\perp = r - \frac{\omega\cdot r}{\omega^2}\omega$ is nothing more than the *component of* r *perpendicular* to ω. Eqn.(3) gives the centripetal acceleration, which is directed inwards; the point P needs to be accelerated to the rotation axis in order to rotate at all.

The vector ω is the angular velocity as indicated above; its direction indicates the axis of rotation and the magnitude gives the speed of rotation – the direction of rotation is dictated by the right-hand rule. This can be seen from the relation $\dot{r}(t) = \omega \times r(t)$. The vector ω is the angular velocity as measured in K. The vector $\omega' = R\omega$ is the angular velocity in K'. Since R is an orthogonal matrix, ω and ω' have the same magnitude. The speed of rotation is the same in K and K'.

Consider a point P which is comoving in frame K', that is, the point P has fixed coordinates in K'. Thus if r' denotes its position in K', it does not depend on time. We thus have for this point

$$0 = \dot{R}(t)r(t) + R(t)\dot{r}(t)$$

which leads to

$$\dot{r}(t) = -R^{-1}(t)\dot{R}(t)r(t)\,. \tag{4}$$

We therefore define ω by the relation $-R^{-1}(t)\dot{R}(t)v = \omega \times v$ for any vector v.

The matrix $T = -R^{-1}\dot{R}$ is an antisymmetric matrix, so that T is an element of the *Lie algebra* $so(3)$, and the group $SO(3)$ acts on T by *conjugation.* Besides T we introduce the second $so(3)$-element $S = -\dot{R}R^{-1}$. We have $S = RTR^{-1}$. The following relations are easily satisfied

$$\dot{S} = S^2 - \ddot{R}R^{-1}\,, \qquad \dot{T} = T^2 - R^{-1}\ddot{R}\,. \tag{5}$$

We also find, using the definitions of S and T, that $\dot{S} = R\dot{T}R^{-1}$, thus $\dot{S} = 0$ if and only if $\dot{T} = 0$.

The relation $-R^{-1}(t)\dot{R}(t)v = \omega \times v$ implies $T_{ij} = -\epsilon_{ijk}\omega_k$. For the *indices* on matrices we use the *Einstein convention.* In fact, switching from T to ω is a realization of a map from the *adjoint representation* of $SO(3)$ to the *vector representation* of $SO(3)$, as is indicated in the appendix. Since the map $T \mapsto \omega$ commutes with the group action, as is proven in the appendix, and since $S = RTR^{-1}$, we know we can write $S_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega'_k$, where $\omega' = R\omega$. From the definitions of S and T we have $Tv = \omega \times v$ and $Sv = \omega' \times v$. Since $\dot{S} = 0$ if and only if $\dot{T} = 0$, the notion of an accelerated rotation is frame-independent. We also find that $\dot{T}v = \dot{\omega} \times v$ and similar for S and ω'.

Interpreting the terms in eqn.(2). Now we consider the right-hand side of equation (2) and express the right-hand side in terms of r' and $\dot{r}'$ and $\ddot{r}$. We obtain

$$\ddot{r}' = (\ddot{R}R^{-1})r' + 2\dot{R}R^{-1}\dot{r}' - 2\dot{R}R^{-1}\dot{R}R^{-1}r' + R\ddot{r} \tag{6}$$

We can express $\ddot{R}R^{-1}$ and $\dot{R}R^{-1}$ in terms of S and $\dot{S}$ giving rise to

$$\ddot{r}' = R\ddot{r} - \dot{S}r' - 2S\dot{r}' - S^2 r = R\ddot{r} - \dot{\omega}' \times r' - 2\omega' \times \dot{r}' - \omega' \times (\omega' \times r')\,. \tag{7}$$

The first term $R\ddot{r}$ has the interpretation of the non-inertial forces (modulo mass); as $\ddot{r}$ is the acceleration in an inertial frame, we can write it as $\frac{F}{m}$, where F is the force. Multiplying with R does no more than transforming the vector quantity $\frac{F}{m}$ to the frame K'. This term thus denotes the real forces.

The second term $-\dot{\omega}' \times r'$ denotes the acceleration of the frame K'. If an object is at a fixed position r' and the rotation is accelerated, then the object has the acceleration $\dot{\omega}' \times r'$. In the system K' this is noted as a tendency to accelerate in the opposite direction as the acceleration of the frame, hence the minus sign.

The third term $-2\omega' \times \dot{r}'$ is the Coriolis force. It can be seen to play a role when people throw a ball at each other on a rotating merry-go-round. It is also the force that lets the wind rotate around low-pressure areas and high-pressure areas.

The fourth term $-\omega' \times (\omega' \times r')$ can be simplified to $\omega'^2 r - (\omega' \cdot r)\omega' = \omega^2 r_\perp$. Thus the fourth term is the centrifugal force; it points outwards and its magnitude is $\omega'^2 = \omega^2$ times the orthogonal distance to the axis of rotation.

Appendix: Some mathematical details

(1). The Einstein convention is a convention often used in physics when group theoretic arguments are used. The convention is as follows: On does in general not write summation symbols, since only contracted indices are group invariants, which implies that only indices that are repeated are summed over. Thus $x_i y_i$ is $\sum_i x_i y_i$ and in an expression like $x_{ijk} y_{klm}$ only the index k is summed over; the others are free and hence not summed over. Although it might take some time to get used to this convention, often it comes handy. The first index on a matrix indicates the row, the second the column. Multiplying matrices A and B results in a matrix with ij-entry $A_{ik}B_{kj}$.

(2). Let $U : t \mapsto U(t)$ be a matrix-valued function, i.e., a matrix whose entries depend on the parameter t. We assume that for all values of t the inverse of $U(t)$ exists. We wish to have an expression for the derivative of $U^{-1}(t) = U(t)^{-1}$. From $U(t)U^{-1}(t) = 1$ we get $\dot{U}(t)U^{-1}(t) + U(t)\frac{\mathrm{d}}{\mathrm{d}t}U^{-1}(t) = 0$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}U^{-1}(t) = -U^{-1}(t)\dot{U}(t)U^{-1}(t)\,. \tag{8}$$

(3). The group $SO(3)$ can be described as those 3×3-matrices G with determinant 1 satisfying $G^{-1} = G^T$, i.e., $GG^T = 1$. We call the elements of $SO(3)$ rotation matrices. Any matrix satisfying $GG^T = 1$ is called orthogonal; any matrix with unit determinant is called special – the abbreviation SO stands for special orthogonal. If a matrix is orthogonal its determinant can only equal ± 1. Since the determinant is a continuous function of the entries of the matrix, the group $SO(3)$ can be considered as an algebraic set in some abstract higher-dimensional space consisting of two disconnected pieces. $SO(3)$ is a group because the identity matrix is orthogonal and has determinant 1, the product is associative and the product of orthogonal matrices is again orthogonal and furthermore, the determinant of a product is the product of the determinants. The group $SO(3)$ has a natural action on $\mathbb{R}^3$ whereby the $SO(3)$-element G maps any vector v to Gv. Such an action is called a representation of $SO(3)$. This particular representation is called the vector representation.

For some conventions it is useful to use the indices explicitly. We can write the identity $GG^T = 1 = G^TG$ as $G_{ia}G_{ja} = \delta_{ij} = G_{ai}G_{aj}$. For orthogonal matrices G we thus have $(G^{-1})_{ij} = G_{ji}$.

(4). Assume we have a trajectory $G : t \mapsto G(t)$ from some real interval to $SO(3)$. We thus have $G(t)G(t)^T = 1G(t)^TG(t)$. Differentiating this relation, and leaving the argument t away, we get

$$(G\dot{G}^T)^T = \dot{G}G^T = -G\dot{G}^T\,, \qquad (G^T\dot{G})^T = \dot{G}^TG = -G^T\dot{G}\,. \tag{9}$$

Thus the matrices $G^{-1}\dot{G} = G^T\dot{G}$ and $\dot{G}G^{-1} = \dot{G}G^T$ are antisymmetric matrices. For more general matrix groups, in this way one obtains the tangent space at the identity element, i.e., the Lie algebra. Indeed, $\dot{G}$ is a tangent vector at $G(t)$, and by multiplying with G^{-1} on maps it to the identity. For the group $SO(3)$ we see that the tangent space at the identity consists of antisymmetric matrices.

The Lie algebra $so(3)$ can be described as those 3×3-matrices M satisfying $M + M^T = 0$, i.e. they are antisymmetric. The commutator $[M, N] = MN - NM$ of two antisymmetric is again antisymmetric and the commutator is precisely the Lie product. Any Lie algebra is a vector space. The group $SO(3)$ acts on the Lie algebra by conjugation as follows: If $G \in SO(3)$ and $M \in so(3)$, then GMG^{-1} is again an element of $so(3)$. Such an action also defines a representation of $SO(3)$, called the adjoint representation.

(5). There are only three linearly independent antisymmetric 3×3-matrices. One can for example take as a basis

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{10}$$

The matrices can easily described by the completely antisymmetric symbol on three indices ϵ_{ijk}, which only takes the values ± 1 and 0 and is defined as follows: If ijk is a cyclic permutation of 123, then $\epsilon_{ijk} = 1$, if ijk is a cyclic permutation of 213 then $\epsilon_{ijk} = -1$ and for any other values of ijk it is zero. With this definition we have

$$(X_i)_{jk} = \epsilon_{ijk} .$$

(6). Any element $M \in so(3)$ can be written as $M = m_k X_k$, thus $M_{ij} = \epsilon_{ijk} m_k$. This equation can be solved for the $,_k$ as follows: $m_k = \frac{1}{2}\epsilon_{ijk} M_{ij}$. We have the identity $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$. The symbol δ_{kl} is like the identity matrix, only if $k = l$ it takes the value 1, elsewise it is 0.

(7). The determinant of a 3×3-matrix M can be calculated by $\det M = \epsilon_{ijk} M_{1i} M_{2j} M_{3j} = \epsilon_{ijk} M_{i1} M_{j2} M_{j3}$. By permuting the indices 1, 2 and 3 one obtains the useful formula $\epsilon_{ijk} M_{ai} M_{bj} M_{cj} = \epsilon_{abc} \det M$. If G is an element of $SO(3)$ we thus have $\epsilon_{ijk} G_{ai} G_{bj} G_{cj} = \epsilon_{abc}$.

(8). Now we can prove the following fact: Mapping an antisymmetric matrix M to a vector m with components $m_i = \frac{1}{2}\epsilon_{ijk} M_{jk}$ defines a map from the adjoint representation of $SO(3)$ to the vector representation that commutes with the group action.

To prove this statement we assume $G \in SO(3)$, and let it act on both the $M \in so(3)$ and on $m \in \mathbb{R}^3$, where m is defined by $m_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$. On M, G acts by GMG^{-1}. On m the element G acts by Gm. We have to check that GMG^{-1} is mapped to Gm. This is not too hard: We write m' for the vector with indices $m'_i = \frac{1}{2}\epsilon_{ijk}(GMG^{-1})_{jk}$ and then calculate, inserting the identity matrix

$$m'_i = \frac{1}{2}\epsilon_{ijk}G_{ja}M_{ab}G_{kb} = \frac{1}{2}\epsilon_{mjk}G_{ja}G_{kb}\delta_{im}M_{ab}\,, \tag{11}$$

which can be simplified by $\delta_{im} = G_{mc}G_{ic}$ and using the determinant identity $\epsilon_{ijk}G_{ai}G_{bj}G_{cj} = \epsilon_{abc}$. We find

$$m'_i = \frac{1}{2}\epsilon_{mjk}G_{ja}G_{kb}G_{mc}G_{ic}M_{ab} = \frac{1}{2}\epsilon_{abc}G_{ic}M_{ab} = G_{ic}m_c\,. \tag{12}$$

This was to be proven.

(9). If n is some given vector and v is a second, then $v_{||} = \frac{v\cdot n}{n\cdot n}n$ is the component of v parallel to n. Hence $v_\perp = v - \frac{v\cdot n}{n\cdot n}n$ is the component of v perpendicular to n. Indeed, $v_\perp \cdot n = 0$.

(10). Let a, b and c be three vectors, then $b \times c$ has components $(b \times c)_i = \epsilon_{ijk}b_jc_k$. Thus the ith component of $a \times (b \times c)$ equals

$$\epsilon_{ijk}a_j(b\times c)_k = \epsilon_{ijk}\epsilon_{kmn}a_jb_mc_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})a_jb_mc_n = (a\cdot c)b - (a\cdot b)c\,. \tag{13}$$

In this calculation we used $\epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$, which can be seen to be true as follows: ϵ_{ijk} is only zero if all three i, j and k are different. Thus the summation only includes one nonzero term. For this term, we need thus that either $i = m$ and $j = n$ or the other way around $i = n$ and $j = m$.