HYPERELLIPTIC CURVES AND COVERINGS WITH RAMIFICATIONS

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1. Generalities

The objects of interest here are Riemann surfaces, or algebraic varieties over the field of complex numbers of (complex) dimension one. We first give a purely topological definition:

Definition 1.1. *A covering $\phi : X \to Y$ is a continuous surjective map from X to Y, such that for each point $p \in Y$, there is a neighborhood U containing p and the preimage of U is a disjoint union of open sets in X that are homeomorphically mapped to U.*

However, for algebraic varieties one uses the Zariski-topology in general. For coverings, we use the topology as a Riemann surface. Also, we allow for ramification points; at a ramification point there is no neighborhood such that the preimages are homeomorphic to the neighborhood. In complex algebraic geometry, when one speaks of an $n:1$ cover one means a regular (rational) map that is on an open dense subset an n-sheeted covering with the topology as a complex manifold, but at the ramification points the preimage is smaller. We hope that the following discussion makes clear what this means.

Given an $n:1$ cover $\phi : X \to Y$, and $p \in Y$ not a ramification point, let U be a neighborhood with disjoint preimages that are homeomorphically mapped to U. Let S be such a homeomorphic preimage of U and let $q \in S$ be the preimage of p in S and call φ the restriction of ϕ to S. Since the map φ is regular, it is locally analytic. If we take complex coordinates z around p, with $z = 0$ at p and w complex coordinates around q with $w = 0$ at q, then the derivative of φ does not vanish at $w = 0$, since φ is one-to-one;

$$\varphi(w) = \alpha w + \beta w^2 + \dots \quad \alpha \neq 0 \,. \tag{1}$$

Now let f be a function that is meromorphic in U, and suppose it has a zero or pole of order m at $z = 0$;

$$f(z) = z^m(a + bz + \dots)\,,\ a \neq 0\,. \tag{2}$$

We can pull-back f to a function f^* on S using φ and we see that

$$f^*(w) = w^m(\alpha^m a + cw + \dots)\,, \tag{3}$$

for some constant c, but note that in particular $\alpha^m a \neq 0$. That means that f^* has a pole or zero of the same order.

Now suppose p is a ramification point. Just try to draw a picture of a ramification point; some sheets that lie over Y come close to each other and touch each other above the ramification point. If r of those sheets come together, then pulling back a function f with a zero or pole of some order m, we get a function with a pole or zero of order rm it seems. This is best seen, by letting the zero or pole of the function f approach p (so, we have a family of functions). But also analytically this can be seen. If the map above p is not $n:1$ but $n-r:1$, then some sheets come together (not r sheets!!). Let us assume that q is a preimage of q where s sheets come together. Thus some neighborhood V of q is mapped to a neighborhood U of p such that all points in U except p have s preimages in V, thus when we call φ the restriction of ϕ to V. Choosing coordinates as before, we have:

$$\varphi(w) = w^s(\alpha + \beta w + \dots)\,,\ \alpha \neq 0\,. \tag{4}$$

Then pulling back a function with zero or pole of order m at p, we see that we obtain a function with a pole or zero of order ms ar q.

2. Hyperelliptic Curves

Definition 2.1. *A hyperelliptic curve is a curve in $\mathbb{C}^2$ of the form $y^2 = f(x)$, where f has no multiple zero's.*

Remember that if the degree of the defining polynomial of a Riemann surface is d, the genus g is given by

$$g = \tfrac{1}{2}(d-1)(d-2)\,. \tag{5}$$

If the degree of f is 2, we have a curve of genus 0. Therefore a curve of the form $y^2 = x(x-1)$ gives rise to a Riemann surface that is homeomorphic to the Riemann sphere. Such a curve is also called a rational curve.

If the degree of f is 3, there are 4 branch points of f; at the zero's of f and at infinity. Therefore there are two branch-cuts (one needs branch points with there branch cuts to get a well-defined analytic inverse). Draw the Riemann sphere with 4 branch points and two branch cuts, double the branch cuts (make it two lines), copy this picture and then glue the pictures together along the lines of the branch cuts, you obtain a donut. Thus, a curve of the form $y^2 = x(x-1)(x-2)$ is a torus $\mathbb{C}^2/\Lambda$ for some lattice Λ.

One can go on and generate more figures ...

We now present a new definition of hyperelliptic curve:

Definition 2.2. *A hyperelliptic curve is a $2:1$ cover of $\mathbb{CP}^1$.*

The latter definition is more general. Note first that any curve in $\mathbb{CP}^1$ of the form $F(x,y) = 0$ that is quadratic in y can be brought to the form $y^2 = f(x)$ after some tricky changes of coordinates. Second, try to homogenize the curve $y^2 = x(x-1)(x-2)$ to get a homogeneous equation in $\mathbb{CP}^2$; you get a singularity! Drawing the pictures as above already suggests that we have to resort to using two copies of $\mathbb{CP}^1$, and this is precisely what we will do.

We show how to get any $2:1$ cover of $\mathbb{CP}^1$ with the requested number of ramification points (always even though). Take two copies of $\mathbb{CP}^1$, one with homogeneous coordinates $(s:t)$ and the other with homogeneous coordinates $(u:v)$. Now consider the curves $\mathcal{C}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by the equation

$$s^2 p(u,v) = t^2 q(u,v)\,, \tag{6}$$

where p and q have no common zero's and also no multiple zero's. Of course, p and q have the same degree. The ingredients imply that if $p(u_0, v_0) = 0$ then the derivatives $\partial_u p$ and $\partial_v p$ do not both vanish at $(u_0 : v_0)$; similar for q. Call $F(s,t,u,v) = s^2 p(u,v) - t^2 q(u,v)$, then

$$\begin{aligned}
\frac{\partial F}{\partial s}(u,v) &= 2sp(u,v)\,,\\
\frac{\partial F}{\partial t}(u,v) &= 2tq(u,v)\,,\\
\frac{\partial F}{\partial u}(u,v) &= s^2\partial_u p(u,v) - t^2\partial_u q(u,v)\,,\\
\frac{\partial F}{\partial v}(u,v) &= s^2\partial_v p(u,v) - t^2\partial_v q(u,v)\,.
\end{aligned} \tag{7}$$

From the derivatives above it follows easily that $\mathcal{C}$ is a smooth curve: at a ramification point either $s = 0$ or $t = 0$, if $s = q = 0$, then $t \neq 0$ and $(\partial_u q, \partial_v q) \neq (0, 0)$, for a non-ramification point have that both all of the quartet (s, t, p, q) do not vanish.

We construct a map to $\mathbb{CP}^1$ by projecting to the u, v-coordinates;

$$\varphi((s:t),(u:v)) = (u:v)\,. \tag{8}$$

Restricting φ to $\mathcal{C}$ we get our wanted $2:1$ cover of $\mathbb{CP}^1$. The preimages are given by

$$\varphi^{-1}((u:v)) = (\pm\sqrt{q(u,v)}\ :\ \sqrt{p(u,v)}\ ) \times (u:v)\,. \tag{9}$$

So, outside the roots of p and q we have a two to one covering, and the ramification points are precisely the points where either p or q vanishes (they have no common zero's by assumption).

3. An Example

We use in this example that any two points are linearly equivalent in $\mathbb{CP}^1$, since the function

$$f = \frac{au - bv}{cu - dv} \tag{10}$$

relates the points $(b:a)$ and $(d:c)$. Also, we use the same notation as above.

Consider the curve $\mathcal{C}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by

$$\begin{aligned} &s^2 p(u,v) - t^2 q(u,v) = 0\,,\ \text{where} \\ &p(u,v) = uv(u-v)(u-2v)(u-3v)\,,\ \text{and} \\ &q(u,v) = (u+v)(u+2v)(u+3v)(u+4v)(u+5v)\,. \end{aligned} \tag{11}$$

We wish to see some linear equivalence between preimages of the map φ. Since knowing the most tricky case suffices for all, we take P a ramification point and Q not a ramification point. Explicitly we take

$$P = (1:1)\,,\ Q = (a:b)\,, \tag{12}$$

where $(a:b)$ is not a zero of p or q. We have

$$Q - P = \frac{av - bu}{u - v}\,. \tag{13}$$

Pulling this back to a function on $\mathbb{CP}^1 \times \mathbb{CP}^1$ we get a function that is zero at the preimages of Q and zero at the preimages of P, but there are two preimages of Q and just one of P. We thus suspect that the function

$$f^*(s,t,u,v) = \frac{av - bu}{u - v}\,, \tag{14}$$

has a double pole at $u = v$ on $\mathcal{C}$.

Going to the chart with local coordinates $x = u/v$ and $y = t/s$ we see that $\mathcal{C}$ is in this chart given by

$$F(x,y) = x(x-1)(x-2)(x-3) - y^2(x+1)\cdots(x+5) = 0\,. \tag{15}$$

Differentiating we see that $\partial_y F(a/b, 0) = 0$ and hence x is not a good local parameter (see for instance the book of Shafarevich, somewhere in the beginning: what it boils down to is that since $\mathcal{C}$ is a 1D curve it can be described by one coordinate locally, that coordinate is the local parameter. Then think of the circle $x^2 + y^2 = 1$; at $y = 0$, the coordinate x is not a good local parameter). Therefore we have to express the function f^* in y around the point $x = 1$. We do this by observing that on $\mathcal{C}$ around $x = 1$ we have

$$x - 1 = \frac{y^2(x+1)\cdots(x+5)}{x(x-2)(x-3)}\,. \tag{16}$$

Then we re-express f^* in the chosen chart

$$f^*(x,y) = \frac{(a+b)x(x-2)(x-3) - y^2(x+1)(x+2)(x+3)(x+4)(x+5)}{y^2(x+1)(x+2)(x+3)(x+4)(x+5)}\,, \tag{17}$$

which clearly has a pole of order two at $y = 0$.

Now around the point where $x = \frac{a}{b}$ (for simplicity and explicitness we assume that the preimages of Q lies in the same chart; for the general case (7) shows that x is a good local parameter). We have $y = \pm\frac{c}{d}$, where c and d are the values of p and q at Q, hence $y \neq 0$. We immediately see that $\partial_y F(\frac{a}{b}, \pm\frac{c}{d}) \neq 0$, and we can use x a local parameter. Thus f has a zero of order 1 at both preimages of Q.

We conclude that f^* makes $\varphi(Q)$ and $2\varphi(P)$ linearly equivalent;

$$(f^*) = \varphi^{-1}(Q) - 2\varphi^{-1}(P)\,. \tag{18}$$

In a similar fashion we see that if P and Q are both ramification points, then $2\varphi^{-1}(P) \sim 2\varphi^{-1}(Q)$. If P and Q are both not ramification points, then $\varphi^{-1}(Q) \sim \varphi^{-1}(P)$.