Line Bundles and Divisors: Some examples

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Abstract

In the text below you find some examples of how to connect Cartier divisors and line bundles. By means of these examples, we present the general material.

1 Some definitions

For convenience we work either over the field of complex numbers $\mathbb{C}$ or over the field of real numbers $\mathbb{R}$. When we say variety, we tacitly assume that the variety is an irreducible topological space.

Definition 1.1. *Let X be a variety over k. We call a Cartier divisor a set $\{(U_i, f_i)\}_i$ of pairs (U_i, f_i) where each U_i is an open set such that $\cup_i U_i = X$ and where the f_i are elements of the field $k(X)$ of rational functions on X such that on the intersection $U_i \cap U_j$ the quotients f_i/f_j and f_j/f_i are regular.*

Remark 1.2. *The reader that is familiar with some more advanced techniques as appear in Hartshorne's book will recognize that the above definition actually defines a representative of a Cartier divisor. The reader that is not familiar with these notions can just read on and take the definition as it is; for the examples it will certainly suffice to work with the above definition. For the group theoretical issues mentioned below it is however desirable to have at least once indicated how to be more precise: two Cartier divisors $(U_i, f_i)_i$ and $(V_j, g_j)_j$ are called equivalent if on all intersections $U_i \cap V_j$ the quotients f_i/g_j and g_j/f_i are regular.*

Definition 1.3. *Let X be a variety over k and let f be an element of $k(X)$, then for any open covering $(U_i)_i$ of X the set $\{U_i, f\}$ defines a Cartier divisor, which we call a principle Cartier divisor.*

The Cartier divisors can be multiplied to form an abelian group in a natural way: let $D = (U_i, f_i)_i$ and $D' = (V_k, g_k)_k$ be two Cartier divisors, then we define $D + D' = (U_i \cap V_k, f_i g_k)_{i,k}$. It is easily seen that the inverse of D is given by (the equivalence class of) $D^{-1} = (U_i, 1/f_i)_i$ and the unit element is $(X, 1)$. It is easily seen that the principle Cartier divisors form a subgroup. [Trivial exercise: show that this

subgroup is normal]

Definition 1.4. *Let X be a variety over k, then the quotient group formed by dividing the group of Cartier divisors by the subgroup defined by the principle Cartier divisors is called the Picard group and denoted* $\mathrm{Pic}(X)$.

For completeness we also mention Weil divisors, but actually we trust that the reader is familiar with Weil divisors on the level of section 1.1 of chapter *III* of the book "Basic algebraic geometry" by Shafarevich.

Definition 1.5. *Let X be a smooth variety over k. Then a Weil divisor is a finite abstract sum of irreducible smooth codimension* 1 *subvarieties. A principle Weil divisor is the formal sum of the zeros and poles, counted with multiplicities, of a rational function $f \in k(X)$ and is denoted (f). The group of Weil divisors is denoted* $\mathrm{Div}(X)$ *and the quotient of* $\mathrm{Div}(X)$ *divided out by the subgroup of the principle divisors is called the class group and denoted* $\mathrm{Cl}(X)$.

If X is smooth we can associate to each Cartier divisor $D = (U_i, f_i)_i$ the Weil divisor that on U_i is given by the zeros and poles of f_i, then taking the union of all these we obtain a set of closed codimension one subvarieties.

The following definition makes sense in the algebraic and in the C^∞-category:

Definition 1.6. *Let X be a variety over k. A vector bundle of rank r over X is a surjective morphism $\pi : E \to X$, such that*

(i) The preimage $\pi^{-1}(p)$ of each point $p \in X$ is a vector space of dimension r over k, thus $\pi^{-1}(p) \cong k^r$.

(ii) There exists an open covering $\{X_i\}$ of the base space X and ismorphism $f_i : \pi^{-1}(U_i) \to k^r \times U_i$, which are called local trivializations.

(iii) The local trivializations f_i don't change the fibres: that is $\mathrm{proj}_2 \circ f_i = \pi$, *where* proj_2 *is the projection onto the second factor (thus U_i).*

(iv) The local trivializations glue together in the sense that the maps $F_{ij} := f_i \circ f_j^{-1} : k^r \times U_i \cap U_j \to k^r \times U_i \cap U_j$ are isomorphisms.

Note that the maps F_{ij} map an element $(v, p) \in k^r \times U_i \cap U_j$ to an element $(w, p) \in k^r \times U_i \cap U_j$. The map $\Phi_{ij}(p)$ which is defined as that map that maps v to w has to be an element of $GL_r(k)$, and hence we have a set of maps $\Phi_{ij} : U_i \cap U_j \to GL_r(k)$, which are called the structure maps.

It is easily seen that the structure maps satisfy

$$\Phi_{ii} = 1\,, \tag{1}$$

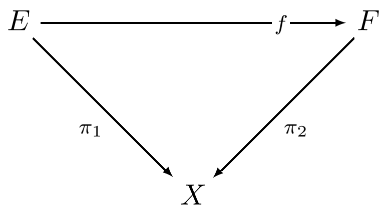
$$\Phi_{ij} \circ \Phi_{ji} = 1\,, \tag{2}$$

$$\Phi_{ij} \circ \Phi_{jk} \circ \Phi_{ki} = 1\,, \tag{3}$$

where the composition is defined. We call the equations (1)-(3) the Cech conditions.

Theorem 1.7. *Let X be a variety over k and let $\{U_i\}$ be an open covering of X and let $\Phi_{ij} : U_i \cap U_j \to GL_r(k)$ be morphisms satisfying the Cech conditions, then there is up to isomorphism of vector bundles a unique vector bundle $E \to X$ with the structure maps given by the Φ_{ij}.*

Of course then we have to say when two vector bundles are isomorphic. First, let $\pi_1 : E \to X$ and $\pi_2 : F \to X$ be two vector bundles over X, then a map $f : E \to X$ is said to be a morphism of vector bundles over X if the following diagram commutes



(4)

An isomorphism of vector bundles over X is then a morphism of vector bundles over X such that that there exists an inverse (left and right inverse at the same time).

Definition 1.8. *A line bundle over a variety X over k is a rank one vector bundle over X.*

Let $\pi : E \to X$ be a line bundle over a variety X over k with trivializing cover $\{U_i\}$. Then we can associate an element of the Picard group $\mathrm{Pic}(X)$ of X to E as follows: The structure maps Φ_{ij} of E are regular maps $U_i \cap U_j \to GL_k(1) \cong k^*$, where k^* are the invertible elements of k. Choose an arbitrary U_i of the covering, say U_0 and assign to U_0 the regular function $f_0 = 1$, then assign to U_i the regular function $f_i = \Phi_{i1}$ (or Φ_{1i}). In the algebraic sense this makes sense since $U_i \cap U_j$ is dense in X; in the C^∞-category one has to transport the regular functions in a way around X, but we won't bother about it. Then (U_i, f_i) defines a Cartier divisor, and hence an element of $\mathrm{Pic}(X)$. Conversely, given a Cartier divisor $D = \{U_i, f_i\}_i$, then we already have a covering of X and can put $\Phi_{ij} = f_i/f_j$ (or f_j/f_i). Then there is a up to isomorphism a unique line bundle with structure maps Φ_{ij}. Hence we have made plausible that

Theorem 1.9. *Let X be a variety over k (smooth?). Then there is a one-to-one correspondence between elements of* $\mathrm{Pic}(X)$ *and isomorphism classes of line bundles over X.*

Later we will see that we can give the set of isomorphism classes of line bundles over X the structure of an abelian group such that the above mentioned one-to-one correspondence is in fact a group isomorphism.

2 Real Möbiusband

2.1 Via a parametrization

For the readers that are not completely familiar with the notions from differential geometry we work this example out in quite some detail. We begin with describing the charts of the circle S^1.

Remark 2.1. *We urge the reader that is not familiar with the below mentioned concepts to make a sufficient amount of drawings of all open sets and maps between them.*

We define the circle as the set $S^1 = \left\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\right\}$. The circle is covered by two charts given by

$$A_1 = S^1 - \{(0, 1)\} = \{(\sin\phi, \cos\phi)|0 < \phi < 2\pi\} \quad (5)$$

$$A_2 = S^1 - \{(0, -1)\} = \{(\sin\psi, \cos\psi)| - \pi < \psi < \pi\} \, . \quad (6)$$

The ϕ and ψ are local coordinates.

Furthermore we define the open sets

$$B_+ = \left\{(x, y) \in S^2 | x > 0\right\} \, , \quad (7)$$

$$B_+ = \left\{(x, y) \in S^2 | x < 0\right\} \, , \quad (8)$$

so that $B_+ \cup B_- = A_1 \cap A_2$ and the $B_\pm$ are simply connected. On the open set $A_1 \cap A_2$ we have to describe how the ψ and ϕ are related. One finds:

$$B_+ : \quad \phi \in (0,\pi) \leftrightarrow \psi \in (0,\pi) \ \text{ and } \ \phi = \psi\,, \tag{9}$$
$$B_+ : \quad \phi \in (\pi,2\pi) \leftrightarrow \psi \in (-\pi,0) \ \text{ and } \ \psi = \phi - 2\pi\,, \tag{10}$$

and since $B_\pm$ are open and their intersection is empty this defines a C^∞ function on $A_1 \cap A_2$, mapping ϕ to ψ. Hence with this choice of charts we have made S^1 into a C^∞-manifold.

Now we come to the Möbius strip, which we will denote as M and describe as the union of two sets C_1 and C_2 inside $\mathbb{R}^4$. Inside $\mathbb{R}^4$ we use coordinates (x,y,u,v). The picture one should have in mind is that the M̈bius strip is obtained by letting a line, which is living in the (u,v)-plane, go around the circle while twisting it such that after having gone around the circle the line is standing upside down. The sets C_1 and C_2 are given by the parametrizations:

$$C_1 = \left\{(\sin\phi, \cos\phi, r\cos\frac{\phi}{2}, r\sin\frac{\phi}{2}) | 0 < \phi < 2\pi, r \in \mathbb{R}\right\} \tag{11}$$
$$C_2 = \left\{(\sin\psi, \cos\psi, t\cos\frac{\psi}{2}, t\sin\frac{\psi}{2}) | -\pi < \psi < \pi, t \in \mathbb{R}\right\} \tag{12}$$

We have $M = C_1 \cup C_2$. As for the circle we define

$$D_+ = \{(x,y,u,v) \in M | x > 0\}\ , \tag{13}$$
$$D_- = \{(x,y,u,v) \in M | x < 0\}\ , \tag{14}$$

so that $D_+ \cup D_- = C_1 \cap C_2$.

We make M to a line bundle over S^1 by defining the projection to the circle: we define $\pi : M \to S^1$ to be the restriction of the map $(x,y,u,v) \mapsto (x,y)$ to M. It is easy to check that

$$\pi^{-1}(B_\pm) = D_\pm\,, \quad \pi^{-1}(A_i) = C_i\,,\ i = 1,2\,. \tag{15}$$

Exercise 2.2. *It is tempting to use the trivializations $f_1 : C_1 \to A_1 \times \mathbb{R}$ and $f_2 : C_2 \to A_2 \times \mathbb{R}$ defined by*

$$f_1 : (\sin\phi, \cos\phi, r\cos\frac{\phi}{2}, r\sin\frac{\phi}{2}) \mapsto (\sin\phi, \cos\phi, r) \tag{16}$$
$$f_2 : (\sin\psi, \cos\psi, t\cos\frac{\psi}{2}, t\sin\frac{\psi}{2}) \mapsto (\sin\psi, \cos\psi, t)\,, \tag{17}$$

but for our algebraic setting this is not the thing to do. It is a good exercise however to calculate the structure maps in with the trivializations f_1 and f_2, that is, to calculate $f_1 \circ f_2^{-1}$ and $f_2 \circ f_1^{-1}$ and to see that they are C^∞ but not algebraic.

Important Remark 2.3. *Important to note is that* $\sin\frac{\phi}{2}$ *never vanishes for* $\phi \in (0, 2\pi)$ *and that* $\cos\frac{\psi}{2}$ *never vanishes for* $\psi \in (-\pi, \pi)$.

After this remark we give the trivializations that do the job:

$$\begin{aligned} &\varphi_1 : C_1 \to A_1 \times \mathbb{R}\,, \\ &\varphi_1 : (\sin\phi, \cos\phi, r\cos\frac{\phi}{2}, r\sin\frac{\phi}{2}) \mapsto (\sin\phi, \cos\phi, r\sin\frac{\phi}{2})\,, \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\varphi_2 : C_2 \to A_1 \times \mathbb{R}\,, \\ &\varphi_2 : (\sin\psi, \cos\psi, t\cos\frac{\psi}{2}, t\sin\frac{\psi}{2}) \mapsto (\sin\psi, \cos\psi, \frac{t}{2\cos\frac{\psi}{2}})\,, \end{aligned} \tag{19}$$

Note that φ_1 is the restriction of the map $(x, y, u, v) \mapsto (x, y, v)$ to C_1.

We now calculate (or actually indicate how the reader should calculate and find what we already found) the structure maps. Let (v, p) be an element in $\mathbb{R} \times A_1 \cap A_2$, then $\varphi_2 \circ \varphi_1^{-1}$ maps this element to $(\tilde{v}, p)$ and our taks is to find the relation between v and $\tilde{v}$.

First we take $(v, p) \in \mathbb{R} \times B_+$ where we now that the patrametrization coordinates ϕ, ψ, r and t are related by $\phi = \psi$ and $r = t$. It is not hard to find that

$$\tilde{v} = \frac{t}{2\cos\frac{\psi}{2}} = \frac{r\sin\frac{\psi}{2}}{2\cos\frac{\psi}{2}\sin\frac{\psi}{2}} = \frac{v}{x}\,, \tag{20}$$

as $x = \cos\psi = \cos\phi$. For $p \in B_-$ we know that the parametrizing coordinates are related by $\phi - 2\pi = \psi$ and $r = -t$. But since $r\sin\frac{\phi}{2} = -t\sin(\frac{\psi}{2} + \pi) = t\sin\frac{\psi}{2}$ we find again that $\varphi_2 \circ \varphi_1^{-1}(v, p) = (\frac{v}{x}, p)$. Hence the structure map $g : A_1 \cap A_2 \to GL_{\mathbb{R}}(1) \cong \mathbb{R}^*$ is given by

$$g : (x, y) \mapsto \frac{1}{x}\,, \tag{21}$$

and indeed on $B_- \cup B_+$ this describes a regular function such that $1/g$ also describes a regular function.

2.2 Algebraic description

Define the variety $X \in \mathbb{R}^4$ by the zero locus of the following equations:

$$x^2 + y^2 - 1 = 0\,, \tag{22}$$

$$2uv - x(u^2 + v^2) = 0\,, \tag{23}$$

$$u^2 - v^2 - y(u^2 + v^2) = 0\,. \tag{24}$$

From the parametric description of the Möbius strip in the previous subsection one verifies easily that $M \subset X$. But playing around with the equations, one sees that each point of X is of the form as given by the parametrization and hence $X = M$.

The structure map g given by eqn.(21) is a nice algebraic function. To conclude that all the manipulations of the previous subsection make M into an algebraic line bundle we must verify that the maps φ_1 and φ_2 are biregular maps. First we note the following identities that follow from eqns.(22)-(24):

$$u^2 = \frac{1+y}{1_y} v^2 \,, \tag{25}$$

$$u = \frac{x}{1-y} v \,, \tag{26}$$

$$\frac{v}{2x} = \frac{u^2+v^2}{2u} = \frac{u}{1+y} \,. \tag{27}$$

With these preparations we can write: φ_1 is a restriction of $(x, y, u, v) \mapsto (x, y, v)$ and φ_1^{-1} is the restriction to $A_1 \times \mathbb{R}$ of the map given by

$$(x, y, v) \mapsto (x, y, \frac{x}{1-y}, v) \,, \tag{28}$$

and $\frac{x}{1-y}$ is regular on A_1. We can write φ_2 as a restriction of the map

$$(x, y, u, v) \mapsto (x, y, \frac{u}{1+y}) \,, \tag{29}$$

and $\frac{u}{1+y}$ is regular on A_2. For φ_2^{-1} we find that we can write it as a restriction of the map

$$(x, y, u) \mapsto (x, y, (1+y)u, 2ux) \,, \tag{30}$$

which definitely is regular.

Now of course what we are after: Which Cartier divisor corresponds to the line bundle $M \to S^1$? It is an easy exercise to check that

$$D = \{(A_1, x), (A_2, 1)\} \tag{31}$$

does the job. But note that a convention enters the game here; either we can associate f_i/f_j with Φ_{ij} or f_j/f_i with Φ_{ij} and we let the reader sort out what is the most natural choice and what the conventions of different authors are.

3 Line bundles on $\mathbb{P}^n$

We have seen (during the seminar) that the Picard group of $\mathbb{P}^n$ is just $\mathbb{Z}$. This implies that we can find for each integer $m \in \mathbb{Z}$ a line bundle, which is then denoted $O(m)$. Looking at the global sections we will see that $O(m)$ corresponds for positive m to the homogeneous polynomials of degree m. Before we come to this remarkable correspondence we treat another piece of literature; the so-called "tautological bundle" aka "Serre's twisted sheaf".

3.1 Tautological bundle or Serre's twisted sheaf

The construction of $\mathbb{P}^n$ is via a projection $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$, sending a point outside the origin to the line through that point and the origin. This already sounds like a line bundle, however, the inverse image of a point in $\mathbb{P}^n$, which is a line in $\mathbb{C}^{n+1}$ through the origin, are all the points in $\mathbb{C}^{n+1}$ except for the origin. The obstruction to make the projection $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ is a line bundle is that all these lines in $\mathbb{C}^{n+1}$ that correspond to points in $\mathbb{P}^n$ go through the origin; they intersect. There is a classical tool of algebraic geometry to resolve this problem; we blow up the origin in $\mathbb{C}^{n+1}$.

From now we describe $\mathbb{C}^{n+1}$ with coordinates $w = (w_0, \dots, w_n)$ and we describe $\mathbb{P}^n$ with homogeneous coordinates $z = (z_0 : \dots, z_n)$. We consider the variety X inside $\mathbb{C}^{n+1} \times \mathbb{P}^n$ defined by the equations

$$w_i z_j - w_j z_i = 0\,, \quad 0 \le i, j \le n\,. \tag{32}$$

The experts might recognize (32) as the blowup of $\mathbb{C}^{n+1}$ in the point $z = 0$. We now have a nice well-defined projection $\pi : X \to \mathbb{P}^n$ given by

$$\pi : (w, z) \in X \mapsto z \in \mathbb{P}^n\,. \tag{33}$$

We remark that the solution set of (32) consists of the points with $w = 0$, which is called the exceptional divisor, and the points $w_i = \lambda z_i$ for some $\lambda \in \mathbb{C}$. In particular we see that for any point $z \in \mathbb{P}^n$ the preimage is

$$\pi^{-1}(z) = \{\lambda(z_0, \dots, z_n) | \lambda \in \mathbb{C}\} \cong \mathbb{C}\,. \tag{34}$$

We cover $\mathbb{P}^n$ with the usual open sets

$$U_i = \{(z_0 : \dots : z_n) | z_i \neq 0\}\ , \tag{35}$$

and we define the trivializations $\varphi_i : \pi^{-1}(U_i) \to \mathbb{C} \times U_i$ by

$$\varphi_i : (w, z) \mapsto (w_i, z)\,. \tag{36}$$

It is easily checked that for each point $z \in U_i$ the map φ_i provides an isomorphism betweeen the fibre $\pi^{-1}(z)$ and the complex line $\mathbb{C}$. The inverse maps $\varphi_i^{-1} : \mathbb{C} \times U_i \to \pi^{-1}(U_i)$ are given by

$$\varphi_i^{-1} : (\lambda, z) \mapsto (\lambda(\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}), z)\,, \tag{37}$$

which indeed is a regular map. It is easy to verify that $\varphi_i \circ \varphi_i^{-1} = \mathrm{id}_{\mathbb{C}\times U_i}$ and that $\varphi_i^{-1} \circ \varphi_i = \mathrm{id}_{\pi^{-1}(U_i)}$. It is furhermore an easy calculation to find that on the intersection $\mathbb{C} \times U_i \cap U_j$ we have

$$\varphi_i \circ \varphi_j^{-1}(\lambda, z) = (\frac{z_i}{z_j}\lambda, z)\,, \quad z \in U_i \cap U_j\,. \tag{38}$$

Hence we have the structure maps $\Phi_{ij} : U_i \cap U_j \to GL_{\mathbb{C}}(1) \cong \mathbb{C}^*$ given by

$$\Phi_{ij}(z) = \frac{z_i}{z_j}\,. \tag{39}$$

This finishes our description of the tautological line bundle, which is also denoted $O(-1)$. Later when we have discussed the line bundles $O(m)$ we calculate the global sections.

Then of course, what we are after is again the Cartier divisor the corresponding to the tautological line bundle. Of course, this is somewhat convention dependent, but we could look for a Cartier divisor $D = \{(U_i, f_i)\}$ such that $f_i/f_j = \Phi_{ij}$. It is easily checked that we could take

$$D = \left\{(U_0, 1), (U_1, \frac{z_1}{z_0}), \ldots, (U_n, \frac{z_n}{z_0})\right\}\,. \tag{40}$$

The Cartier divisor D corresponds to the Weil divisor $-(z_0 = 0)$. If we would have chosen the other convention, we would have found in the language of Weil divisors just the divisor $-D$, which is the zero locus of z_0, but since z_0 is not a regular function on $\mathbb{P}^n$ this is not a principal divisor. In terms of Cartier divisors the other convention corresponds to

$$-D = \left\{(U_0, 1), (U_1, \frac{z_0}{z_1}), \ldots, (U_n, \frac{z_0}{z_n})\right\}\,. \tag{41}$$

3.2 Intermezzo: Vector bundle constructions

We have already pointed out that a vector bundle is completely described upto isomorphism when the structure maps are given. We will use this fact to give two constructions of vector bundles: the direct sum and the tensor product.

Suppose we are given two vector bundles $\pi : E \to X$ and $\tilde{\pi} : \tilde{E} \to X$ of ranks r and s respectively over X. Let $\{U_i\}$ be an open covering of X that is sufficiently fine to give a trivialization for both E and $\tilde{E}$. Denote the structure map for E by $\Phi : U_i \cap U_j \to GL_{\mathbb{C}}(r)$ and for $\tilde{E}$ by $\tilde{\Phi}_{ij} : U_i \cap U_j \to GL_{\mathbb{C}}(s)$. Then the tensor product of E and $\tilde{E}$ is defined as the vector bundle over X that has the structure maps $\Phi^{\otimes}_{ij} : U_i \cap U_j \to GL_{\mathbb{C}}(rs)$ that act for $z \in U_i \cap U_j$ on an element $v \otimes w \in \mathbb{C}^r \otimes \mathbb{C}^s$ as

$$\Phi^{\otimes}_{ij}(z) : v \otimes w \mapsto \Phi_{ij}(z)(v) \otimes \tilde{\Phi}_{ij}(z)(w) \,. \tag{42}$$

The direct sum of E and $\tilde{E}$ is defined as the vector bundle over X that has the structure maps $\Phi^{\oplus}_{ij} : U_i \cap U_j \to GL_{\mathbb{C}}(r+s)$ that act for $z \in U_i \cap U_j$ on an element $(v, w) \in \mathbb{C}^r \oplus \mathbb{C}^s$ as

$$\Phi^{\otimes}_{ij}(z) : (v, w) \mapsto \Big(\Phi_{ij}(z)(v), \tilde{\Phi}_{ij}(z)(w)\Big) \,. \tag{43}$$

The dual bundle E^* to E is the vector bundle with structure functions $1/\Phi_{ij}$. Hence the bundle $E^* \otimes E$ has structure maps 1 and is thus trivial! Using the identification $(\mathbb{C}^r)^* \cong \mathbb{C}^r$ we can see that then E^* really represents the dual.

Now the remarkable feature of one-dimensional vector spaces is that the tensor product of two one-dimensional vector spaces is again one-dimensional. From this triviality it follows that one can multiply line bundles using the tensor product. But the dual of a line bundle is something like an inverse if we identify the isomorphism class of the trivial line bundle with the unit element. Thus, ... isomorphism classes of line bundles make a group!

3.3 What line bundle is $O(n)$?

One defines the line bundle $O(1)$ as the line bundle dual to the tautological bundle, which we previously already gave the name $O(-1)$. For $O(1)$ the structure maps Φ_{ij} are thus given by

$$\Phi_{ij}(z) = \frac{z_j}{z_i} \,. \tag{44}$$

Then one defines $O(m)$ for positive integers m as the line bundle that is formed by the m-fold tensor product of $O(1)$. For negative m one takes the m-fold tensor product of $O(-1)$. For any integer m we thus see that the structure maps $\Phi_{ij}^{(m)}(z)$ of $O(m)$ are given by

$$\Phi_{ij}^{(m)}(z) = \left(\frac{z_j}{z_i}\right)^m . \tag{45}$$

We see that we can associate to $O(m)$ the Cartier divisor

$$D^{(m)} = \left\{(U_0, 1), (U_1, \frac{z_0^m}{z_1^m}), \ldots, (U_n, \frac{z_0^m}{z_n^m})\right\} , \tag{46}$$

which corresponds to the Weil divisor $(z_0^m = 0) = m(z_0 = 0)$.

Let us calculate some global sections. First of $O(-1)$. Let σ be a global section. Then in the U_0-chart we find an expression $(\lambda_0(z), z)$ where $\lambda_0(z)$ is a regular function on U_0:

$$\lambda_)(z) = \frac{H_a(z)}{z_0^a} , \tag{47}$$

where $H_a(z)$ is a homogeneous polynomial of degree a. In the U_1-chart we find an expression $(\lambda_1(z), z)$, which needs to be given by

$$\lambda_1(z) = \frac{z_1}{z_0}\lambda_0(z) = \frac{H_a(z)z_1}{z_0^{a+1}} . \tag{48}$$

But then H_a needs to cancel an $a+1$th power of z_0, which is impossible and hence there are no global section of the tautological bundle. Note that one might counter that the above expression is only defined on the intersection $U_0 \cap U_1$ and there is no problem. However, on the whole of U_1 there must be an expression for σ which equals the expression (48) on $U_1 \cap U_0$. But $U_0 \cap U_1$ is dense in U_1 and two rational functions that are the same on a dense subset are everywhere the same.

Let us now analyse the global sections of $O(1)$. We take the same kind of expression for a global section σ as above. Then we find in the U_1 chart an expression

$$\lambda_1(z) = \frac{1}{z_1}\frac{H_a(z)}{z_0^{a-1}} . \tag{49}$$

In order to be regular over the whole of U_1 we need that $H_a(z) \sim z_0^{a-1}P(z)$ where then $P(z)$ is a homogeneous polynomial of degree 1. Therefore we conclude that there is a global section, which in the U_i-chart is given by the expression

$$\lambda_i(z) = \frac{p(z)}{z_i} , \tag{50}$$

where $p(z)$ is a homogeneous polynomial of degree one. Thus there is a one-to-one correspondence between the global sections of $O(1)$ and the linear forms. It follows that the complex vector space of global sections of $O(1)$ is $(n+1)$-dimensional.

Exercise 3.1. *Extend the above analysis to show that the global sections of $O(m)$ are in 1-to-1 correspondence with the homogeneous polynomials of degree m. Conclude that the complex vector space of global sections of $O(m)$ has dimension $\binom{n+m}{m}$. Verify that the divisor corresponding to $O(m)$ is related to the zero's of a generic global section. Conclude that projective varieties are defined not by the zero's of functions, but by the zero's of global sections of some line bundles.*

4 Tangent bundles

Although tangent bundles are in general not line bundles, we present some generalities on tangent bundles and give some examples. In the next section we discuss the tangent bundle of $\mathbb{P}^1$.

4.1 Generalities

4.2 Two-sphere

4.3 Three-sphere

4.4 Tangent bundle of $\mathbb{P}^1$

After all the preliminaries we allow ourselves to be short and mainly present the formulas.

Define

$$U = \{(x : y)|x \neq 0\} \ , \quad V = \{(x : y)|y \neq 0\} \ , \tag{51}$$

and use the usual trivializations

$$f_U : U \to , \quad f_U : (x : y) \mapsto \frac{y}{x} \, , \tag{52}$$

$$f_V : V \to , \quad f_V : (x : y) \mapsto \frac{x}{y} \, . \tag{53}$$

The inverse maps are given by

$$f_U^{-1} : u \mapsto (1 : u) \, , \tag{54}$$

$$f_V^{-1} : v \mapsto (v : 1) \, . \tag{55}$$

We calculate the structure map J^{UV}, which sends a vector in the U-chart to one in the V-chart, as follows: for $a \in \mathbb{R}$ a tangent vector in the U-chart we have

$$J^{UV}(a) = \frac{\partial v}{\partial u} a = -\frac{1}{u^2} a = -v^2 a \, . \tag{56}$$

To find a global section, let X be global section with expressions X^U and X^V in the U-chart and V-chart respectively. Then $X^U \in \mathbb{R}[u]$ and hence we may expand

$$X^U = a_0 + a_1 u + a_2 u^2 + \dots \,, \tag{57}$$

where it is understood that there are only finitely many terms nonzero. We find

$$X^V = -v^2 (a_0 + \frac{a_1}{v} + \frac{a_2}{v^2} + \dots) \,, \tag{58}$$

so that we conclude that $a_k = 0$ for all $k \geq 0$. We thus found

$$X^U = a + bu + cu^2 \,, \quad X^V = -av - bv - cv^2 \,. \tag{59}$$

Note that in the real case we find a global section $X^U = u^2 + 1$, $X^V = -v^2 - 1$ that vanishes nowhere. Hence in the real setting the tangent bundle of $\mathbb{P}^1$ is trivial. However, in the complex case it isn't; there are always two zero's.

Exercise 4.1. *Conclude that the tangent bundle of* $\mathbb{P}^1$ *corresponds to* $O(2)$.

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