Nice Orbifolds: $\mathbb{P}^1 \times \mathbb{P}^1/\mathbb{Z}_2 \;\cong\; \mathbb{P}^2$

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Abstract

I was pointed out by a friend of mine that modding $\mathbb{P}^1 \times \mathbb{P}^1$ out by the symmetry that exchanges the two factors seemed to have a rather simple structure. He explained me how to construct such orbifolds and the result is this write-up.

The projective space $\mathbb{P}^1 \times \mathbb{P}^1$ can be described by two pairs of homogeneous coordinates on $\mathbb{P}^1$; we use homogeneous coordinates $(x_0 : x_1)$ for the first factor and $(y_0 : y_1)$ for the second factor. We write $\bar{0}$ and $\bar{1}$ for the elements of the group $\mathbb{Z}_2$, where $\bar{0}$ is the identity element. Consider the following $\mathbb{Z}_2$ =-action on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\bar{1} : (x_0 : x_1) \times (y_0 : y_1) \mapsto (y_0 : y_1)) \times (x_0 : x_1)\,. \tag{1}$$

We map $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ by the Segre map:

$$(x_0 : x_1) \times (y_0 : y_1) \mapsto (x_0 y_0 : x_1 y_1 : x_0 y_1 + x_1 y_0 : x_0 y_1 - x_1 y_0)\,, \tag{2}$$

which is slightly different from the standard Segre map. It is easily checked that the Segre map is an embedding. We call Z the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre map. It is a standard fact that Z is an algebraic variety. The homogeneous coordinates for $\mathbb{P}^3$ are denoted $(z_0 : z_1 : z_2 : z_3)$. The variety Z is given by the equation $Z : z_0 z_1 = z_2^2 - z_3^2$.
The action of $\mathbb{Z}_2$ on Z can be seen as the restriction to Z of the following action of $\mathbb{Z}_2$ on $\mathbb{P}^3$:

$$\bar{1} : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : z_1 : z_2 : -z_3)\,. \tag{3}$$

The fixed points consist of the point $(0 : 0 : 0 : 1)$, which does not lie on Z, and the divisor $D = (z_3 = 0)$. The restriction of D to Z we call E. One might think that modding out by $\mathbb{Z}_2$ gives rise to singularities along D. As D has codimension one and invariant subrings of polynomial rings are integrally closed (when G is a finite group), D cannot be a singularity of $\mathbb{P}^3/\mathbb{Z}_2$. The point $(0 : 0 : 0 : 1)$ does not worry us as it lies off Z. We now give coordinate patch describtions for Z.
a-Chart: U_a. This is where $z_0 \neq 0$. We consider the map

$$U_a \to \mathbb{C}^3\,, \quad (z_0 : z_1 : z_2 : z_3) \mapsto (a_1, a_2, a_3) = \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_3}{z_0}\right).$$

The equation for Z in this chart is

$$Z_a : \quad a_1 = a_2^2 - a_3^2\,.$$

In this chart, and in the two that follow, the action of $\mathbb{Z}_2$ flips the sign of the third coordinate. The invariant ring $\mathbb{C}[a_1, a_2, a_3]^{\mathbb{Z}}_2$ is given by $\mathbb{C}[a_1, a_2, a_3^2]$. Therefore choosing new generators of the $\mathbb{C}$-algebra $\mathbb{C}[a_1, a_2, a_3]^{\mathbb{Z}}_2$ as follows

$$\bar{a}_1 = a_1\,, \quad \bar{a}_2 = a_2\,, \quad \bar{a}_3 = a_3^2\,,$$

we see that the coordinate ring of Z_a is given by $\mathbb{C}[\bar{a}_1, \bar{a}_2]$. Thus, as in the modern language of algebraic geometry, $Z_a \cong \mathrm{Spec}(\mathbb{C}[\bar{a}_1, \bar{a}_2]) = \mathbb{A}^2_{\mathbb{C}}$: the maximal ideals then make up $\mathbb{C}^2$.

b-Chart: U_b. This is where $z_0 \neq 0$. We consider the map

$$U_b \to \mathbb{C}^3 , \quad (z_0 : z_1 : z_2 : z_3) \mapsto (b_1, b_2, b_3) = \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}, \frac{z_3}{z_1} \right) .$$

The equation for Z in this chart is

$$Z_b : \quad b_1 = b_2^2 - b_3^2 .$$

We use the same trick as in the a-chart to obtain that Z_b has the coordinate ring $\mathbb{C}[\bar{b}_1, \bar{b}_2]$; we take as above $\bar{b}_1 = b_1$, $\bar{b}_2 = b_2$ and $\bar{b}_3 = b_3^2$ corresponding to the generators of the invariant ring $\mathbb{C}[b_1, b_2, b_3]^{\mathbb{Z}}_2 = \mathbb{C}[b_1, b_2, b_3^2]$.

c-Chart: U_c. This is where $z_0 \neq 0$. We consider the map

$$U_c \to \mathbb{C}^3 , \quad (z_0 : z_1 : z_2 : z_3) \mapsto (c_1, c_2, c_3) = \left(\frac{z_0}{z_2}, \frac{z_1}{z_2}, \frac{z_3}{z_2} \right) .$$

The equation for Z in this chart is

$$Z_c : \quad c_1 c_2 = 1 - c_3^2 .$$

In the coordinates corresponding to the generators of the invariant ring, we have $\mathbb{C}[c_1, c_2, c_3]^{\mathbb{Z}}_2 = \mathbb{C}[c_1, c_2, c_3^2]$ we see that the equation defining Z_c is $\bar{c}_1 \bar{c}_2 = 1 - \bar{c}_3$. Hence the coordinate ring of Z_c is $K[\bar{c}_1, \bar{c}_2]$.

The attentive reader might be thinking that we are forgetting a chart, namely, the chart where $z_3 \neq 0$. The only point in this chart that is not already covered by the other charts is $(0 : 0 : 0 : 1)$, which does not lie on Z.

Now we need to glue the different patches together: the transformations are the usual one for projective space

$$\bar{b}_1 = \frac{1}{\bar{a}_1} , \quad \bar{b}_2 = \frac{\bar{a}_2}{\bar{a}_1}$$

and

$$\bar{c}_1 = \frac{1}{\bar{a}_2} , \quad \bar{c}_2 = \frac{\bar{a}_1}{\bar{a}_2} .$$

But these transformation rules are no more than the transformation rules between the three standard patches of $\mathbb{P}^2$ and all our charts are isomorphcic to these coordinates charts of $\mathbb{P}^2$, namely, affine two-space. This concludes the proof of the claim that

$$\mathbb{P}^1 \times \mathbb{P}^1 / \mathbb{Z}_2 \ \cong \ \mathbb{P}^2 .$$