An application of the Segre embedding

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Abstract

We pose and answer a question concerning rational functions on a surface in $\mathbb{P}^3$ once raised during an algebra class.

1 The question

Consider the surface X defined in $\mathbb{P}^3$, where we use coordinates $(z_0 : z_1 : z_2 : z_3)$, defined as the zero locus of the following equation:

$$X : \quad z_0 z_3 - z_1 z_2 = 0 \,. \tag{1}$$

Define the following open sets in $\mathbb{P}^3$:

$$U_i = \left\{ (z_0 : z_1 : z_2 : z_3) \in \mathbb{P}^3 | z_i \neq 0 \right\} \,, \quad i = 0, 1, 2, 3 \,. \tag{2}$$

We define the following open sets on X:

$$X_i = X \cap U_i \,, \quad i = 0, 1, 2, 3 \,. \tag{3}$$

We denote $\mathcal{O}_X$ the structure sheaf on X, that is, $\mathcal{O}_X(V)$ is the algebra of regular functions on V, where V is an open set on X.

Consider the following regular functions:

$$f \in \mathcal{O}_X(X_1) \,, \quad f = \frac{z_0}{z_1} \,, \quad \text{and} \quad g \in \mathcal{O}_X(X_3) \,, \quad g = \frac{z_2}{z_3} \,. \tag{4}$$

On the intersection $X_1 \cap X_3$ they define the same element:

$$f|_{X_1 \cap X_3} = g|_{X_1 \cap X_3} \in \mathcal{O}_X(X_1 \cap X_3) \,. \tag{5}$$

Hence f and g define a single regular function F on $X_1 \cup X_3$. The question is whether there exists a rational function of the z_i that equals f on X_1 and g on X_3. In other words, can't we get a single expression for F?

2 The answer

We define $V = X_1 \cup X_3$ and we will look for a description of the regular functions on V.

Consider the Segre map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ defined by

$$\varphi : (x_0 : x_1) \times (y_0 : y_1) \mapsto (x_0y_0 : x_1y_0 : x_0y_1 : x_1y_1) \,. \tag{6}$$

Then clearly the image of φ is contained in X. But in fact, the image of φ is precisely X. We can set up a biregular equivalence between X and $\mathbb{P}^1 \times \mathbb{P}^1$ by means of φ; the inverse of φ is on X_0 given by:

$$\varphi^{-1}|_{X_0} : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : z_1) \times (z_0 : z_2)$$

on X_1 the inverse is given by

$$\varphi^{-1}|_{X_1} : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : z_1) \times (z_1 : z_3)$$

on X_2 the inverse is given by

$$\varphi^{-1}|_{X_2} : (z_0 : z_1 : z_2 : z_3) \mapsto (z_2 : z_3) \times (z_0 : z_2)$$

and on X_3 the inverse is given by

$$\varphi^{-1}|_{X_3} : (z_0 : z_1 : z_2 : z_3) \mapsto (z_2 : z_3) \times (z_1 : z_3) \,.$$

Clearly X is the union of all the X_i and the complement of $X_1 \cup X_3$ consists of all the set where $z_1 = z_3 = 0$, the inverse image of which is the set where $x_1 = 0$. Hence V is biregular to $\mathbb{C}^1 \times \mathbb{P}^1$. A regular function on V thus pulls back to a regular function on $\mathbb{C}^1 \times \mathbb{P}^1$. The set of regular functions on $\mathbb{C}^1 \times \mathbb{P}^1$ are in the given setting the elements of $\mathbb{C}[\frac{x_0}{x_1}]$. Using the inverse maps of φ we see that any polynomial in $\frac{x_0}{x_1}$ pulls back to a polynomial function of $\frac{z_0}{z_1}$ on X_1 and to a polynomial function of $\frac{z_2}{z_3}$ on X_3. Hence we cannot meet both requirements of having neither powers of z_1 nor powers of z_3 in the denominator either of the two subsets X_1 or X_3. The answer to the question thus is: **NO**.