Special relativity

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This is a small essay on special relativity, aimed at familiarizing an interested audience at the level of first-year students of exact sciences with the main ideas from special relativity. For a more trained audience this essay hopes to bring some ideas under one title so that these notes can be used for own purposes or even to base a lecture upon. The goal of writing this essay was to find a most lucid presentation of some items concerning special relativity, such as time dilation, length contraction and so on. The goal was not too present one single text about special relativity with the most lucid presentation, but to present single items as clear possible, without much ado. As a consequence, different approaches are tried in order to explain the same aspects.

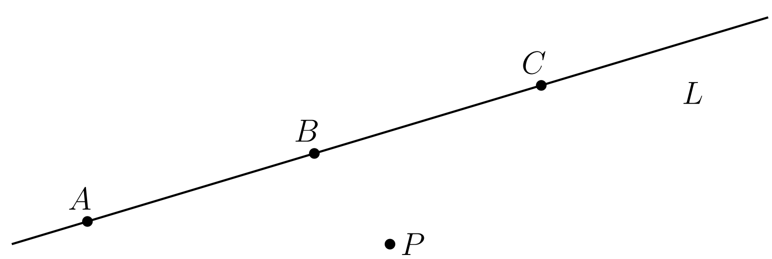
The main premises. The major breakthrough that initiated the theory of special relativity comes from the measured invariance of the speed of light by Michelson and Morley. This essay is neither an historical overview nor an essay on experiment physics, so that we will not dwell on the exact peculiarities that accompanied these experiments and this discovery. Another important ingredient, which in fact seemed to have put Einstein on the right track, was the invariance of the Maxwell under Lorentz transformations; the Maxwell equations are not compatible Galilean coordinate transformations. However, again, we will not expand in this direction.

What we will use however is the following; we will assume invariance of the velocity of light for any observer. Of course, when we speak of the velocity of light, we mean the velocity of light in vacuum. Secondly, we will in fact assume that the passage from one observer to another, moving with respect to each other at a constant velocity, leaves the spatial directions perpendicular to the velocity inert. This second assumption seems natural and one can wonder if it can be deduced from other assumptions like rotation invariance; the author is however not aware of any rigorous proof.

On some convections let us mention the following: An observer together with a set of coordinates used by this observer, such that the spatial coordinates vanish at the point where the observer sits, is called a frame. Two different frames are often indicated with capital roman letters. The relative velocity between two frames is denoted v, whereas the speed of light is denoted c, unless we use units such that $c = 1$. The Lorentz factor γ is defined by $\gamma^{-1} = \sqrt{1 - \frac{v^2}{c^2}}$.

Maximal velocity. If the velocity of light takes the same numerical value for each observer, it is impossible to travel at the speed of light. If some observer B is traveling with the speed of light with respect to some other observer J, then J can send a light signal in the direction in which B is moving at the moment B passes J. For B, the light signal is moving away from him and hence arriving earlier at some distant point X. If we put some switch at point X, one could point a gun at B and shoot B, so he will never reach X. For J the light signal and B arrive at the same time at X; the gun shoots in a void - eventually killing J, maybe. Thus for the one observer B is alive, whereas for the other observer B is killed. Thus, traveling at light speed seems to contradict the observed world.

But then, we must put the maximum attainable velocity at the speed of light. Let some object travel at a speed exceeding that of light. If this object moves along a line L, we can place an observer at a point P not contained in L.



We let the observer measure the velocity of the traveling object. If the points A, B and C are chosen well enough, we will see that the traveling object must have some velocity greater than the speed of light at A and C but some velocity smaller than the speed of light at B. But by continuity we should be able to measure that the object attains the speed of light at two points; one between A and B and another between B and C. This seems to contradict the previous conclusion that the speed of light cannot be attained.

The way out seems to be that the speed of light cannot be obtained in a uniform way; in the above example the object does not have a constant velocity with respect to P. However, the observer at P could perform a rotation such that the velocity of the object in one coordinate direction of the observer is constant. Another point is that if no inertial frame can move with a velocity exceeding that of light it seems impossible to write down laws of physics for the objects moving at these ultra-high velocities. Also, it is impossible to break such an object down to zero velocity or to accelerate an object to a velocity exceeding that of light. The only way out seems to assume that these ultra-fast object cannot be seen or cannot interact with ordinary observers. But if the latter is the case, we might as well assume they are not there. Hence from now on, we assume that relative velocities are always smaller than the speed of light.

Time dilation and length contraction. Imagine two observers A and B traveling with relative velocity v. At the time of crossing, one of the observers, say A, sends a light signal in a direction perpendicular to the relative motion. The figure on the left shows the path of the light ray from A's perspective and the figure on the right shows the path of the light ray from the perspective of B.

As we assumed that perpendicular directions are not altered under coordinate transformations, we may conclude that if the light signal hits some object at a time t measured in frame A, both observers will agree that the separation of this object from the light source has the numerical value ct. However, in the second frame the light signal has traveled a distance $\sqrt{(ct)^2 + (vt')^2}$, which ought to be ct'. Hence we find $t' = t\gamma > t$, and thus time intervals transform linearly. But then space intervals must also transform linearly. Indeed suppose that a bar of length l lies in frame B. We suppose that the one end passes A at time $t = 0$ and that the other end passes A at a time T. Hence the length in frame A is vT. But the event where the second end passed A occurred in frame B at a time $T' = T\gamma$, where $T'v = L$. Hence A measures a length $vT'/\gamma = L/\gamma < L$.

A warning is at place; one cannot simply consider two events P and Q in different frames and infer that the spatial interval gets dilated and that the temporal interval between P and Q gets contracted. More specifically, if P has coordinates (t_P, x_P) in frame A and (t'_P, x'_P) in frame B and if Q has coordinates (t_Q, x_Q) in frame A and (t'_Q, x'_Q) in frame B, then one cannot conclude that the temporal intervals $|t_P - t_Q|$ and $|t'_P - t'_Q|$ are related by a time dilation and that the spatial intervals $|x_P - x_Q|$ and $|x'_P - x'_Q|$ are related by a length contractions. The question arises, in which frame the length should be shortest and in which frame the time should be longest. Time dilation takes place if two events take place at the same spatial point in one frame and then the time interval measured in this frame is shortest. Length contraction takes place if two events occur at the same time in one frame and then the length in this frame is longest.

In the above examples, the light ray emanated by A stayed at coordinates $x = 0$ in frame A; the observer B measured the length of the bar by remaining at the same place whereas observer A can measure the rod by measuring at the same time (connecting the points by a ruler and looking at the indicated values.

How coordinates transform. First of all, linear space-time transformations appear to be so natural that they are usually more or less postulated. But as we have seen, temporal and spatial intervals are changed in a linear fashion by going from one observer to another observer. Now suppose that observers A and B synchronized their clocks such that the two frames coincided at $t = t' = 0$, where the primed coordinates are used by B. In this case, the two frames are related by a linear transformation.

This can be seen as follows: We may assume the velocity takes place in the x-direction for A and the x'-direction for B, such that B moves in the positive direction for A - this assumption is justified by performing a rotation such that the spatial axes are alined. The other spatial coordinates we can then leave out as they are assumed to be inert. We also put $c = 1$.

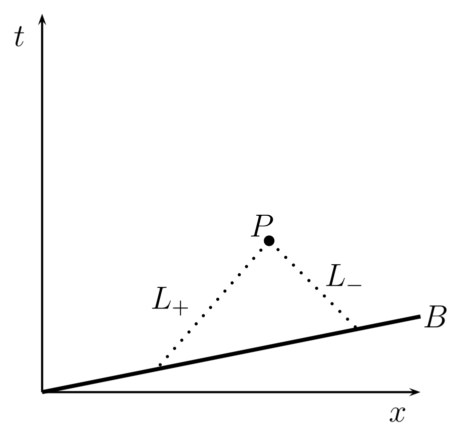
Consider some event P having coordinates (t, x) in frame A and coordinates (t', x') in frame B. Then P takes place a point a distance x from A; indeed, at time t A can measure where P took place. Hence in the B-frame, in which A at time t' is at the point with x'-coordinate $-vt'$, this event took place at the point (t', x') such that $x' + vt' = \gamma^{-1}x$, as $x' + vt'$ is the spatial distance between A and the event measured in the B-frame, which becomes contracted. This length is contracted in the B-frame, since we have two events: in frame A these events are (t, x) and $(t, 0)$, the latter being A measuring the distance to P.

However, as the case is symmetric, we can interchange v with $-v$, x with x' and t with t' to obtain $x-vt = \gamma^{-1}x'$. Thus $x' = \gamma(x-vt)$. Combining with $\gamma(x'+vt') = x$ we obtain

$$x' = \gamma(x - vt)\,, \qquad t' = \gamma(t - vx)\,.$$

In particular, coordinates transform linearly.

Macdonald's argument[1]. From the fact that a clock running in a moving frame runs slower than the clocks of a frame it passes, we can deduce rather easily the Lorentz transformations. Let us consider two frames A and B, where B moves with a velocity in the positive x-direction with respect to B. Again, we only consider one spatial and one temporal coordinate. We consider some event P marked by coordinates (T, X) in frame A and designated with coordinates (T', X') in frame B. We additionally consider two light waves both passing through P, but where one, let's call it L_+, is moving in the positive x-, and hence also in the positive x'-direction, and where the other, which we'll call L_- is moving in the negative x- and x'-direction.



Since the speed of light is constant and $P \in L_+$, in frame A all points along L_+ satisfy the equation $x - t = X - T$, and similarly on L_- we have $x + t = X + T$. Similarly, $x' - t' = X' - T'$ on L_+ and $x' + t' = X' + T'$ for B.

Along the worldline of B, we have $x' = 0$, $x = vt$, $t = \gamma t'$. The latter is, since we can consider a fixed clock at the origin of B; due to time dilation, this clock

[1]This paragraph is based on Macdonald, A., *Derivation of Lorentz transformation*, Am. J. Phys. 49, 493 (1981) and on Macdonald, arXiv 0606046.

is running slower than the clocks in A that this clock in B is passing by. We can conclude that on B's worldline we have

$$x + t = \gamma(1+v)(x' + t')\,, \qquad x - t = \gamma(1-v)(x' - t')\,.$$

However, the light rays $L_\pm$ cross B's worldline and go through P. Hence these equations are also true in P. Therefore, we must have the following relation between (T, X) and (T', X')

$$X + T = \gamma(1+v)(X' + T')\,, \qquad X - T = \gamma(1-v)(X' - T')\,.$$

Solving for X and T we find the usual Lorentz transformations.

Time-dilation and Doppler. Suppose two observers A and B both equipped with a clock travel at a constant speed relative to each other and for simplicity we assume the motion takes place in one dimension. We assume that at the time where the observers cross, both clocks are set to zero. The idea now is to introduce the time dilation factor k as follows: If one of the observers, say A, sends a light signal to B at a time t, then B will receive this signal when his time indicates that an amount kt of time has passed, since time zero.

We can assume there is a linear relation between the two events. We have seen that under Lorentz transformations coordinates transform linearly. Hence if t is enlarged by a factor of two, B is a distance twice as large away and thus kt will also grow by a factor of two.

Now suppose that at the moment where the second observer receives the light signal, it immediately sends a light signal back - maybe using a mirror. As the situation is symmetric, A will receive this signal at a time $k(kt) = k^2 t$. But now A can use this information to know where and when B received its signal; indeed, the light signal has traveled a distance twice the separation between the two observers at the time the light signal was reflected by B. The elapsed time between sending and receiving for A is $k^2 t - t$. The distance between the two observers is then $\frac{1}{2}(k^2 - 1)t$. The light signal was received by B at a time $\frac{1}{2}(k^2-1)t + t = \frac{1}{2}(k^2+1)t$ - measured in the frame of the first observer. But then we must have

$$v = \frac{\frac{1}{2}(k^2-1)t}{\frac{1}{2}(k^2+1)t} = \frac{k^2-1}{k^2+1}$$

and solving for k we find

$$k = \sqrt{\frac{1+v}{1-v}}\,.$$

This gives an immediate derivation of the Doppler effect, since we can imagine that some signal (crest of light wave) was sent at time $t = 0$ and later at a time t, $2t$, $3t$, ... The received signals arrive at times kt, $2kt$, $3kt$ and so on. Hence the time intervals between two signals is altered by a factor of k, so that the frequency gets altered by a factor k^{-1}.

The runner's naive arguing. Suppose we have measured a track and concluded it was a hundred meters. Now we let a runner run along the track and record an official its time; 10 seconds. Its velocity was therefore 36 kilometers per hour. The runner disagrees; there is a length contraction and a time dilation, and hence his time is more that 10 seconds and the length of the track is less than hundred meters. Therefore his velocity is less than 36 km/h.

The error of the runner lies in the fact that, although length contraction and time dilation exist, the question remains, who sees the length contraction. The own time of an object, running between points A and B, is always slower than the time an observer measures as the objects runs from A to B. Hence the time of the official measuring the time of the runner is the time which is dilated. Hence the time of the runner is less than 10 seconds, and the velocity is agreed upon by both, official and runner.

Doppler - using time dilation. A given source X sends out a signal at specific intervals, with a frequency ν. An observer Y moves away from X with a velocity v. We want to know the frequency ν' observed by the observer. All primed quantities are the observed values for the observer Y that are the analogs of the same observables for X.

We will focus on the time interval between two signals and use units where $c = 1$. Let $t_E(1)$ and $t_E(2)$ be the times where signal 1 respectively signal 2 is emitted. Let $t_A(1)$ and $t_A(2)$ be the times where signal 1 respectively signal 2 is absorbed by the observer as measured in the frame of the source. We denote $\Delta_E t = t_E(2) - t_E(2)$ and $\Delta_A t = t_A(2) - t_A(1)$, which are the same for the source; $\Delta_E t = \Delta_A t$.

In frame Y the second signal has to travel a distance $v\Delta_E t'$ more than the first signal;

this enforces an additional time difference to the emission time difference $\Delta_E t'$. Hence $t_A(2)' = t_A(1)' + \Delta_E t' + v\Delta_E t'$. Therefore we have $\Delta_A t' = (1+v)\Delta_E t'$. But the time differences $\Delta_E t$ and $\Delta_E t'$ are related by $\Delta_E t' = \gamma \Delta_E t$, where $\gamma^{-1} = \sqrt{1-v^2}$ since the process of emission takes place at rest in the frame of X and therefore is dilated in the frame Y. Hence we obtain

$$\Delta_A t' = \sqrt{\frac{1+v}{1-v}}\,\Delta_E t$$

and from these the frequency ν' as the reciprocal value of $\Delta_A t'$

$$\nu' = \sqrt{\frac{1-v}{1+v}}\,\nu\,.$$

Twin paradox - using Doppler effect. Suppose two persons A and B of the same age - so let's take them to be twins - take on the following experiment. Person A stays at home for the next few years, whereas B undertakes a trip at constant speed v to a distant place (let's call it station ω) and then immediately returns to home-sitting A. They agree upon the following; each day at 7:00 in the morning A sends a text message with light speed containing the weather forecast to the traveling twin B. Suppose that with traveling at speed v it takes a time interval of t days to travel to station ω. Then on its way the traveling twin receives

$$\sqrt{\frac{1-v}{1+v}}\,t$$

weather forecast messages from his twin A. On the way back B receives

$$\sqrt{\frac{1+v}{1-v}}\,t\,.$$

Hence the total number of received messages is

$$\sqrt{\frac{1-v}{1+v}}\,t + \sqrt{\frac{1+v}{1-v}}\,t = \frac{2t}{\sqrt{1-v^2}} > 2t\,.$$

For the traveling twin $2t$ days have passed by, but for twin A the number of days passed by is $2t\gamma$. Thus sitting home makes you older.

Chasing twins. Suppose that the traveling brother B stays at station ω and the other twin A comes over - with the same velocity and sending the same messages each day and departing at the moment, which in his frame coincides with the arrival of B at ω. During its travel to station ω the twin B receives again

$$\sqrt{\frac{1-v}{1+v}}t \equiv \kappa t$$

messages.

Let us first focus on the total number of signals twin A is sending. Twin B needs a time t in his own frame, hence the time it takes in the frame of A is γt. Hence A has sent γt messages until B reaches station ω. We have agreed that a traveling person needs t days to reach station ω. Hence A will send t messages on journey. Therefore A sends in total $(1+\gamma)t$ messages. On logical grounds, this must also be the number of messages B receives in total.

Let us now secondly focus on the number of messages B receives during the time twin A takes to travel to him. In the frame of B already at station ω the traveling time is dilated; in his frame the other twin A needs a time $t\gamma$. Hence B receives during this time

$$t\gamma\sqrt{\frac{1+v}{1-v}} = \frac{t}{1-v^2}$$

messages of his brother in flight. The twin A sent γt messages during the flight of B, from these only κt were received. Hence $\gamma t - \kappa t$ are received at a frequency of one day.

It is now tempting to add up the results so far obtained, but then one thinks that B receives $\gamma t + \frac{t}{1-v^2} > (1+\gamma)t$ messages.

The error lies in the fact, that when A was flying over, B first received at a frequency 1 a day and only later at a frequency $1/\kappa$ a day. The signals received at a frequency 1 a day arrive at B during a time $(\gamma - \kappa)t$. The remaining time the signals are received at a frequency $1/\kappa$. Hence only

$$(\gamma t - (\gamma - \kappa)t)\kappa^{-1} = t$$

are received at a frequency of κ^{-1} a day. This makes the addition work again.

Invariant line element - Minkowski metric. Let us consider the case where a particle travels between two points A and B. Suppose that in a frame of some observer O the distance between A and B is Δx. Let τ be the time coordinate in the frame of the traveling particle. If the particle measures a time $\Delta\tau$ to travel from A to B, then in the frame of O the time interval Δ, in which the particle moves from A to B, is dilated and given by $c\Delta\tau = \Delta t\sqrt{c^2 - v^2}$, where v is the velocity of the particle as measured in frame O. Squaring the latter relation and using that

$\Delta x = v\Delta t$, we obtain

$$c^2(\Delta\tau)^2 = c^2(\Delta t)^2 - (\Delta x)^2 .$$

Any other observer O' will have a similar relation as the one above;

$$c^2(\Delta\tau)^2 = c^2(\Delta t')^2 - (\Delta x')^2 ,$$

where the primes indicate the quantities related to observer O'. Hence we find that for two events that can possibly be connected by a traveling particle, we have

$$c^2(\Delta t)^2 - (\Delta x)^2 = c^2(\Delta t')^2 - (\Delta x')^2 .$$

If two events can be connected by a traveling particle, the quantity $s^2 = c^2(\Delta t)^2 - (\Delta x)^2$ is positive. Hence we conclude that if $s^2 > 0$ in one frame O, then $s'^2 > 0$ in all frames O' and then they are equal. Our next goal is to prove that $s^2 = s'^2$ irrespective if it is negative or positive. As a first step, we can extend equality between s^2 and s'^2 if one of them is nonnegative; the case $s^2 = 0$ corresponds to a light ray, which has a constant velocity in all frames.

Consider two general events A and B. In comparing the quantity $s^2_{AB} = c^2(t_A - t_B)^2 - (x_A - x_B)^2 - (y_A - y_B)^2 - (z_A - z_B)^2$ in two frames O and O' we may simplify the matter a bit. First, we may assume that B is in fact at the origin for O and for O'; this is just a shift and s^2 is translation invariant. In other words, B is coinciding with the event that the clocks in both frames have $t = t' = 0$ at the moment the two origins of the two frames pass each other. In this case we can consider s^2 as a quadratic function σ on the vector space $\mathbb{R}^4$. We now set $c = 1$, so that $\sigma(t, x, y, z) = t^2 - x^2 - y^2 - z^2$.

Let e_1, e_2, e_3 and e_4 be a basis of $\mathbb{R}^4$ as seen in frame O and we choose these vectors such that $\sigma(e_i) \geq 0$ for all i. For example, we can take $e_1 = (1, 0, 0, 0)$, $e_2 = (1, 1, 0, 0)$, $e_3 = (1, 0, 1, 0)$ and $e_4 = (1, 0, 0, 1)$. Suppose the map $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ relates the coordinate frames O and O'. Then $\sigma(t, x, y, z) = \sigma(\varphi(t, x, y, z))$ whenever one of both sides is nonnegative. Any vector in $\mathbb{R}^4$ can be written as $\sum_i \alpha_i e_i$ for some real coefficients α_i. The coefficients α_i form a four-vector, thus $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$. Now we consider the quotient

$$F(\alpha) = \frac{\sigma(\sum_i \alpha_i e_i)}{\sigma(\sum_i \alpha_i \varphi(e_i))} ,$$

which is a rational function on the four-dimensional real space in which α takes its values. The set where $\sigma(\sum_i \alpha_i e_i) \geq 0$ is a set of dimension 4 and on this set $F(\alpha) = 1$. Writing $F = f/g$ for some polynomials f and g, we see that f and g are equal as polynomials on a set that contains an open set in the Euclidean topology. Hence $f = g$ everywhere.

Concluding, whenever O and O' are two frames and $s^2 = s'^2$ for all events for which one side is nonnegative, then equality holds for all events. Thus two observers in inertial frames are related by a linear transformation (linear as to have constant velocity, for which we have seen that time dilations and length contractions are linear) such that s^2 takes equal values. All linear transformations that preserve s^2 are Lorentz transformations; they preserve the bilinear form determined by the diagonal matrix with eigenvalues $(-1, +1, +1, +1)$.

Invariance of light-cone. For this section we neglect two spatial dimension and consider a two-dimensional spacetime. The reason is that two inertial frames differ in the velocity with respect to each other and this relative velocity gives a preferred spatial dimension; we expect that the coordinate transformation between two inertial frames does not affect the dimensions perpendicular to the relative velocity.

The constancy of the velocity of light gives $s^2 = t^2 - x^2 = 0$ in each frame. If we try to classify all coordinate transformations $\varphi : (t, x) \mapsto (t', x')$ such that $t'^2 - x'^2 = 0$, we get too much coordinate transformations; the transformation $t' = x$ and $x' = t$ is not a physical transformation, as it interchanges $s^2 > 0$ with $s^2 < 0$, so that events that can be connected by a traveling particle are interchanged by events that cannot be connected by a traveling particle. But also scaling transformations like $(t', x') = (\lambda t, \lambda x)$ for some $\lambda \neq 0$ are contained in the set of linear transformations preserving the light-cone $s^2 = 0$.

Factoring out the scaling transformations and requiring that we only consider invertible transformations, we may suppose that $\det(\varphi) = 1$. The reason why we not chose to take the branch $\det(\varphi) = -1$, is that these transformations cannot be continuously connected to the identity - i.e., there is no way to smoothly deform a transformation φ with determinant minus one so as to become the identity transformation, which has unit determinant. Hence, these negative determinant transformations are not physical; diminishing the relative velocity should end with the identity transformation.

Let us write the general coordinate transformation as

$$\varphi : \begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} ,$$

and we require now that $t'^2 - x'^2 = 0$, whenever $x = \pm t$ and in addition $ad - bc = 1$. We then get three equations

$$a^2 + b^2 = c^2 + d^2 \,, \quad ab = cd \,, \quad ad - bc = 1 \,.$$

If $a = 0$, we get $bc = -1$ and thus we need $d = 0$, so we obtain the unphysical transformation $x' = t$ and $t' = x$, which we disregard and hence we may assume $a \neq 0$. Solving $b = \frac{cd}{a}$ from the second equation and inserting this in the third, we get $d(a^2 - c^2) = a$, which we can solve for d. Using these relations, the first equation becomes

$$a^2 + \frac{c^2}{(a^2 - c^2)^2} = c^2 + \frac{a^2}{(a^2 - c^2)^2} \,,$$

which can be simplified and we get $a^2 - c^2 = 1$ - we disregard solutions $a = \pm c$, since these enforce $a = 0$. We then find $a = d$ and $b = c$. As $a^2 = 1 + c^2 \geq 1$, we have two options $a \geq 1$ or $a \leq -1$ and since we want the solution to be a continuous deformation of the identity transformation, we consider only those solutions that have $a \geq 1$. This being such, we can now introduce a parameter β and write $a = \cosh\beta$ and $c = \sinh\beta$. Hence the admissible transformations are precisely the transformation

$$\varphi : \begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} .$$

We now consider the measurement of a given length L' (of some rod, say, stretching between $x' = -L'$ and $x' = 0$) residing in the moving frame, where coordinates are indicated by a prime. We let the first end pass the origin of the fixed (unprimed) frame and then wait, till the other end of the rod passes the origin and measure the time interval. We get two events, given by

$$\text{measuring one end} : x' = x = t = t' = 0$$

and

$$\text{measuring the other end} : x = 0 \,, \quad x' = -L' \,, \quad t = L/v \,, \quad t' = L'/v \,.$$

Hence we have $L'/v = \cosh\beta L/v$ and $-L' = \sinh\beta L/v$. Comparing with the known formulas for length contraction we see that

$$\cosh\beta = \frac{1}{\sqrt{1-v^2}}\,, \qquad \sinh\beta = \frac{-v}{\sqrt{1-v^2}}\,.$$

Given v, these solutions can always be solved for β and conversely, given any β, there exists a unique v, such that the above relations between v and β are fulfilled. Hence we have found all coordinate transformation and shown that they have the usual parametrizations

$$x' = \frac{x - vt}{\sqrt{1-v^2}}\,, \quad t' = \frac{t - vx}{\sqrt{1-v^2}}\,.$$

Using dimensional analysis, one can recover the factors of x as

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}\,, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\,.$$

Searching observers. There exist two kinds of pairs events: spacelike separated events and timelike separated events. Let us fix an observer S who uses coordinates (t, x) and let us take event P to take place at $t = 0$ and $x = 0$ in some frame and let Q be an event taking place at $t = T$ and $x = X$, with $T, X > 0$. If $X^2 > c^2T^2$ then light is not fast enough to connect P and Q. Hence nobody can travel at a velocity such that both P and Q take place at the origin of the frame of the observer; no observer can connect these events. In this case we call the separation of the events spacelike.

If $X^2 < c^2T^2$ we call the separation of the events timelike and in this case it is possible that an observer connects the two events; in the frame of the observer that passes at $t = 0$ the point $x = 0$ and travels at a speed of $v = X/T < c$ in the direction of $x = X$, both P and Q take place at the origin.

Let us return to the spacelike case; we cannot find an observer, which sees that the two events take place at the same place, but can we find an observer, who sees both events taking place at the same time? Well, yes we can! Below we show how.

Let us imagine the following experiment. The event P coincides with the emision of a ray of light in the direction of $x = X$. The event Q coincides with the emission of a ray of light in the direction of $x = 0$. Also, we imagine an observer to travel along the line connecting $x = 0$ and $x = X$, such that at $t = 0$ the observer passes $x = 0$.

We thus consider the following four events: (I) A ray of light is emitted at $(t, x) = (0, 0)$. (II) A ray of light is emitted at $(x, t) = (X, T)$. (III) At $(t, x) = (X/c, X)$ the ray of light from (I) arrives at the point $x = X$. (IV) The light ray of (II) hits the travelling observer.

The travelling observer sees the two events at the same time if and only if events (III) and (IV) coincide for S. Although at first this seems trivial, it is tricky, since we require simultaneity in the frame of S! The real test comes from the Lorentz transformations.

If the ray of light from (I) hits $x = X$ it has travelled a distance $v \cdot X/c$ more than the ray of light from (II). The time it took to do so was T, and since as the speed of light equals c, we get

$$c = \frac{v \cdot X/c}{T} \quad \Longrightarrow \quad v = \frac{c^2 T}{x} \,.$$

Let us test this result with Lorentz transformations. As vX/c^2 now equals T, we see that indeed $T' = \gamma(T - \frac{vX}{c^2}) = \gamma(T - T) = 0$. Of course, we could have also gotten this result from the Lorentz transformations directly.

Another derivation of this velocity goes as follows. Clearly, the sought for velocity depends on the fraction X/T. The obvious candidate $v = X/T$ does not work, as this exceeds c. The other combinations of c and X/T we can make that have dimensions of velocity and not exceed c are of the form $c^{n+1}\frac{T^n}{X^n}$ for integers $n > 0$. For simplicity reasons, the best guess is $n = 1$.

Twins in a periodic universe[2] . We have seen that the twin paradox is resolved by observing that one observer has to make the return and thus, this observer is not always in an inertial frame. But what if topology is such that no observer needs to make the turn to return to the twin observer? For simplicity we assume that spacetime is two-dimensional and has the topology of a circle. We will consider the circe as the real line with an equivalence relation $\sim$ which identifies points periodically.

We would like to impose an equivalence condition like $(t, x) \sim (t, x + L)$, but this

[2]This section is based on *The twin paradox in compact spaces*, Phys. Rev. A 63, 044104 (2001).

already makes clear what will enforce the asymmetry in this case; in order to get the periodicity into the spacetime at hand, we use a frame to impose the equivalence relations. It is clear that there is one unique frame for which the identification of points has no time-components. In more dimensions there will be a larger class of preferred frames, all related by simple rotations - without Lorentz boosts!

So let us fix a frame S in which the equivalence relations read $(t, x) \sim (t, x + L)$ where L is a positive prescribed length. If S' is another frame, moving with positive velocity v with respect to S, the Lorentz transformations are

$$x' = \gamma(x - vt)\,, \qquad t' = \gamma(t - vx)\,.$$

The equivalence relations then give $(\gamma t - \gamma vx, \gamma x - \gamma vt) \sim (\gamma t - \gamma vx - \gamma vL, \gamma x + \gamma L - \gamma vt)$, and hence we have

$$\text{in} \;\; S' \qquad (t', x') \sim (t' - \gamma vL, x' + \gamma L)\,.$$

The fact that in S' the equivalence relation has time components makes it impossible that S' has synchronized clocks; it is impossible to place a clock at each point of the compact space such that all clocks run with equal time. Supposed that in frame S' one puts clocks at points in space (thus along the circle) starting at $x' = 0$, then walking in the positive direction put a second clock synchronized with the first clock and so on. The observer in S' will notice that the last clock is out off pace with the clock at $x' = 0$ and has a mismatch of γvL.

The best he can do to remedy this is to use two overlapping patches to cover spacetime and to glue the coordinates together on the cross-section. This is similar to the construction of the Möbius strip. Indeed, consider the two patches $U_0 = \{x' \mid -\gamma L/2 < x' < \gamma L/2\}$ and $U_1 = \{x' \mid 0 < x' < \gamma L\}$. Then, at the overlap $0 < x' < \gamma L/2$ we can use the identity function $(t', x') \mapsto (t', x')$ but on $\gamma L/2 < x' < \gamma L$ we have to use the function $(t', x') \mapsto (t' - \gamma vL, x')$ to glue the patches together. On each patch we can have synchronized clocks.

This being discussed, let us now show how the twin paradox resolves in this case. First, the observer in S will see the traveling person S' move with velocity v in the positive direction and returning from the other side after a time L/v. However, as both have been moving in inertial frames and S' moved with a velocity with respect to S, the observer in S will see that the clock in S' has only shifted by an amount of $L/\gamma v$; a typical case of time dilation - moving clocks run slower.

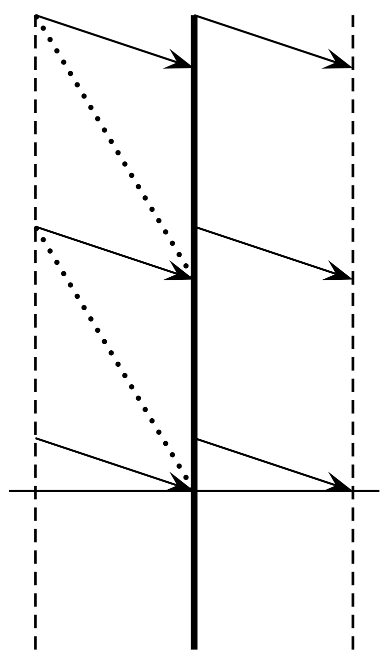
Secondly, the observer in S' sees a larger compact dimension; he will measure a distance γL. This again is typical for Lorentz contraction; lengths contract in going from the moving frame to the resting frame. Thus, S' sees S travel at a velocity $-v$ to the point $-\gamma L$, indeed in the opposite direction from that in which S sees that S' is traveling. But the point $(t', -\gamma L) \sim (t' - \gamma v L, -\gamma L + \gamma L) = (t' - \gamma v L, 0)$ and thus S' sees S reappear at the origin with an additional time shift of $-\gamma v L$. Therefore S' will agree that he meets S again at the time

$$t' = \gamma L/v - \gamma v L = \frac{\gamma L}{v}(1 - v^2) = vL/\gamma\,.$$

Thus S' will also agree that his clock shows a time vL/γ; both agree that S' is younger.

From the above discussion we see that in all cases the twin that is the oldest at re-meeting again is the one whose frame is characterized by the following: in his frame the equivalence relation that renders the compactness has no time components and in this frame the compact dimension has the smallest size.

The above discussion can be clarified using spacetime diagrams. In S' the identifications are done as is indicated by the arrows. The observer S moves from $(t' = 0, x' = 0)$ in the direction of $-\gamma L$ along the dotted line - the dotted line is the worldline of S as seen in S'. The fat vertical line is the worldline of the observer of S'. Every time the worldlines of S and S' cross they meet at the origin of S'.



Epilogue. In all cases we see that naive thinking is the source of all paradoxes.

Taking into account what is really simultaneous and which time intervals are really dilated, or which spatial intervals are really contracted, one saves himself from running into erroneous conclusions. However, the statement of Kant, that geometry *is* a manifestation of the mind, must be taken as less absolute. Apparently we need a different way of thinking in order to describe the world that is accessible through experiment. Hence, spacetime is not only a product of the mind; or we can change our minds and fit in new experiences.