SRT Addendum: Accelerations

DBW

March 18, 2017

In this very short note you will find the following derivation: If an object X experiences some force F' and hence undergoes some acceleration a', then an observer O sees the following parallel and perpendicular accelerations $a_{||}$ and $a_\perp$ given by:

$$a'_\perp = \gamma^2 a_\perp \,, \qquad a'_{||} = \gamma^3 a_{||} \,,$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is the usual Lorentz factor.

We take K' to be an instantaneous rest frame for the object X, and the coordinates in K' are primed. K' is chosen to coincide at time $t' = 0$ with the object X. K is the rest frame of an observer, which we also take to coincide with K' at time $t = 0$ – the coordinates in K are unprimed. In K the object has a velocity v in the x-direction at time $t = 0$. We thus have the following transformation rules between K' and K:

$$x' = \gamma(x - \beta t) \,, \quad y' = y \,, \quad z' = z \,, \quad t' = \gamma(t - \beta x) \,.$$

We work in units such that $c = 1$, and thus replace v by β. Later we can restore units by putting $\beta = v/c$ and $t \mapsto ct$. Let us first consider the case in which the acceleration a' is in the x-direction. Then X follows a trajectory $x' = \frac{1}{2} a' t'^2$ in K'. Inserting the expressions for x' and t' one arrives at the following quadratic equation for x:

$$a'\gamma\beta^2 \cdot x^2 - 2(1 + a'\gamma\beta) \cdot x + a'\gamma t^2 + 2\beta t = 0 \,.$$

Calculating the discriminant one finds $D = 4(1 + 2a'\gamma\beta t(1 - \beta^2)) = 4(1 + 2\frac{a' t\beta}{\gamma})$. Therefore we find

$$x = \frac{1 + a'\gamma\beta t \pm \sqrt{1 + \frac{2a'\beta t}{\gamma}}}{a'\gamma\beta} \,.$$

In order to recover $x = 0$ at $t = 0$, we need the minus sign in the solution for x, so we only have one solution. Writing out to second order in t we find

$$x = \beta t + \frac{1}{2} \cdot \frac{a'}{\gamma^3} \cdot t^2 + O(t^3) \,,$$

in which we recognize the linear motion of K' $x = \beta t$ and the second order contains the acceleration a in the frame K: $a = \frac{a'}{\gamma^3}$. This proves the second of the identities to be proven.

Now consider the motion $y' = \frac{1}{2}b't'^2$. Putting in the expressions for y' and t' from the Lorentz transformations we find

$$y = \frac{b'}{2}\gamma^2(t - \beta x)^2 \,.$$

Now we put in the previous expression for x but, as we are only interested in second order contributions, we may safely put $x = \beta t + O(t^2)$, and thus find

$$y = \frac{b'}{2}\gamma^2 t^2(1 - \beta^2)^2 + O(t^3) \,.$$

Since $1 - \beta^2 = \gamma^{-2}$ we find

$$y = \frac{b'}{2}\gamma^{-2}t^2 + O(t^3) \,.$$

Reading of, we see $a_\perp = \frac{b'}{\gamma^2}$, but $b' = a'_\perp$ and thus both equations follow.

The use of these formulae is for example in extending the Larmor formula for the power radiated by a nonrelativistic accelerating charged particle to a relativistic particle. Larmor's formula states that the radiated power P radiated by a charged accelerating particle is given by a formula of the form $P = ka^3$. Since this formula only holds for $\beta << 1$ it can be extended by using this formula to obtain the power P in the instantaneous rest frame K' and using that $P' = P$, since the energies dW and dW' transform in the same way as the time intervalls dt and dt'. Then one uses $P = P' = ka'^3 = k(a'^2_\perp + a'^2_{||}) = k\gamma^4(a^2_\perp + \gamma^2 a^2_{||})$. Furthermore

$$a^2_\perp + \gamma^2 a^2_{||} = \gamma^2(a^2_\perp + a^2_{||} - (\gamma^2 - 1)a^2_\perp = \gamma^2(a^2 - \beta^2 a^2_\perp)$$

and $a^2_\perp = \frac{(\mathbf{a}\times\beta)^2}{\beta^2}$ resulting in

$$P = k\gamma^6(a^2 - (\mathbf{a} \times \beta)^2) \,.$$