Structure of Super Hopf Algebras

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March 17, 2008

Abstract

In these notes we give some elementary definitions and properties involving super Hopf algebras. For simplicity we only treat complex super Hopf algebras, but the extension to super Hopf algebras over other fields is straightforward. With linear we mean $\mathbb{C}$-linear.

1 Definitions

With $\mathrm{Exp}(a)$ we mean $(-1)^a$. The notion of a super algebra over $\mathbb{C}$ is assumed to be understood. If A, B are super algebras then also $A \otimes B$ with product given by $x \otimes y \cdot z \otimes w = \mathrm{Exp}(|y||z|)xz \otimes yw$. If A, B, C, D are superalgebras and $f : A \to C$ and $g : B \to D$ are linear maps, then $f \otimes g : A \otimes B \to C \otimes D$ is given by $f \otimes g(x \otimes y) = \mathrm{Exp}(|x||g|)f(x) \otimes g(y)$.

1.1 Super co-algebras

A super co-algebra (SCA) is a super vector space C over $\mathbb{C}$ together with a coproduct $\Delta : C \to C \otimes C$ and a co-unit $\epsilon : C \to \mathbb{C}$ such that Δ and ϵ are both linear and even and satisfy:

$$\mathrm{id} \otimes \epsilon \circ \Delta = \epsilon \otimes \mathrm{id} \circ \Delta = \mathrm{id}\,, \tag{1}$$

where we identify $\mathbb{C} \otimes C \cong C \otimes \mathbb{C} \cong C$. In Sweedler notation the last property means: $f = f'\epsilon(f'') = \epsilon(f')f''$, $\forall f \in C$ - we will always omit the summation signs in Sweedler notation -; for more on Sweedler notation see e.g. [1],[2].

We will always assume that a coalgebra is co-associative, which means: $\Delta \otimes \mathrm{id} \circ \Delta = \mathrm{id} \otimes \Delta \circ \Delta$.

One defines a co-ideal as a super sub vector space $I \subset C$ such that $\Delta(I) \subset I \otimes C + C \otimes I$ and $\epsilon(I) = 0$.

1.2 Super bi-algebras

A super bi-algebra (SBA) is an associative super algebra B (over $\mathbb{C}$) that is at the same time a co-associative SCA, such that the comultiplication and the co-unit are even super-algebra morphisms: $\Delta(xy) = \Delta(x)\Delta(y)$ and $\epsilon(xy) = \epsilon(x)\epsilon(y)$.

A bi-ideal in an SBA is a super sub vector space I that is an ideal of the super algebra and a co-ideal of the SCA.

1.3 Super Hopf algebra

A super Hopf algebra (SHA) is an SBA H together with an even linear map $S : H \to H$, called the antipode, such that for all $x \in H$ we have $x'S(x'') = S(x')x'' = \epsilon(x)$.

A Hopf-ideal is a bi-ideal that is stable under the action of S.

2 Constructions

2.1 Cosets of SCA's

If C is an SCA with an co-ideal I, then one defines $\pi : C \to C/I$ as a map of super vector spaces, which is well-defined since I is assumed to be a graded ideal. Then one can give C/I the structure of a SCA when one finds a way to define a comultiplication $\bar{\Delta} : C/I \to C/I \otimes C/I$. If $a \in C/I$ then there is an $x \in C$ such that $\pi(x) = a$, we then define

$$\bar{\Delta}(a) = \pi^{\otimes} \circ \Delta(x)\,, \tag{2}$$

where $\pi^{\otimes}$ is the canonical projection

$$\pi^{\otimes}\ : C/I \to \frac{C}{I} \otimes \frac{C}{I} \cong \frac{C \otimes C}{C \otimes I + I \otimes C}\ . \tag{3}$$

It is easy to check that this definition is independent of the choice of x. One defines for the same pair x, a the map $\bar{\epsilon} : C/I \to \mathbb{C}$ by

$$\bar{\epsilon}(a) = \epsilon(x)\,, \tag{4}$$

of which it is also easily verified that it is independent of the choice of x (hint: $\epsilon(I) = 0$).

2.2 Quotients of SHA's

If H is super Hopf algebra and I a Hopf ideal, then $\Delta I \subset I \otimes H + H \otimes I$, $\epsilon(I) = 0$ and $S(I) \subset (I)$. The quotient H/I is again a super Hopf algebra with the obvious maps (indicated with a bar): $\bar{\Delta}([x]) = \Delta x \mathrm{mod}(I \otimes H + H \otimes I)$, $\bar{\epsilon}([x]) = \epsilon(x)$ and $\bar{S}([x]) = [S(x)]$, where we denoted the equivalence classes of x by $[x]$. Later we will see that the antipode is unique and hence $\bar{S}$ is the only choice to make H/I into a super Hopf algebra.

2.3 Tensor products of SCA's and SBA's

The tensor product of two super algebras we already gave a super algebra structure. Now we will do this for SCA's.

Let C and D be two SCA's (we will write Δ and ϵ for the maps in both C and D), then we introduce the even linear operator $T^{(4)}_{23}$: $C \otimes C \otimes D \otimes D \to C \otimes D \otimes C \otimes D$ by:

$$T^{(4)}_{23} : x \otimes y \otimes z \otimes w \to \mathrm{Exp}(|y||z|) x \otimes z \otimes y \otimes w\,. \tag{5}$$

We then define the map $\Delta^{\otimes} : C \otimes D \to C \otimes D \otimes C \otimes D$ by

$$\Delta^{\otimes} = T^{(4)}_{23} \circ \Delta \otimes \Delta\,. \tag{6}$$

For $x, y \in C, D$ we thus have $\Delta^{\otimes}(x \otimes y) = \mathrm{Exp}(|x''||y'|) x' \otimes y' \otimes x'' \otimes y''$.

The map $\epsilon^{\otimes} : C \otimes D \to \mathbb{C}$ we define by $\epsilon^{\otimes}(x \otimes y) = \epsilon(x)\epsilon(y)$

The above definitions of $\Delta^{\otimes}$ and $\epsilon^{\otimes}$ we also want to extend to SBA's, but then we need to check that they are super-algebra morphisms. Therefore we calculate $\Delta^{\otimes}(a \otimes b \cdot c \otimes d)$:

$$\begin{aligned}
\Delta^{\otimes}(a \otimes b \cdot c \otimes d) &= \Delta^{\otimes}(\mathrm{Exp}(|b||c|)ac \otimes bd) \\
&= \mathrm{Exp}(|b||c|)T^{(4)}_{23} \circ \Delta \otimes \Delta(ac \otimes bd) \\
&= \mathrm{Exp}(|b||c|)T^{(4)}_{23}\Delta(ac) \otimes \Delta(bd) \\
&= \mathrm{Exp}(|b||c|)T^{(4)}_{23}\Delta a \Delta c \otimes \Delta b \Delta d \\
&= \mathrm{Exp}(|b||c|)T^{(4)}_{23}((a' \otimes a'' \cdot c' \otimes c'')\otimes \\
&\quad (b' \otimes b'' \cdot d' \otimes d'')) \\
&= T^{(4)}_{23}\mathrm{Exp}(|b||c| + |a''||c'| + |b''||d'|) \\
&\quad a'c' \otimes a''c'' \otimes b'd' \otimes b''d'' \\
&= L\, a'c' \otimes a''c'' \otimes b'd' \otimes b''d'' \\
\text{where } L &= \mathrm{Exp}(|b||c| + |a''||c'| + |b''||d'| + \\
&\quad (|a''| + |c''|)(|b'| + |d'|))\,.
\end{aligned} \tag{7}$$

On the other hand we have:

$$\begin{aligned}
\Delta^{\otimes}(a \otimes b)\Delta^{\otimes}(c \otimes d) &= \mathrm{Exp}(|a''||b'| + |c''||d'|)a' \otimes b' \otimes a'' \otimes b'' \cdot \\
&\quad c' \otimes d' \otimes c'' \otimes d'' \\
&= R\, a'c' \otimes b'd' \otimes a''c'' \otimes b''d'' \\
\text{where } R &= \mathrm{Exp}(|a''||b'| + |c''||d'| + |c'||b''| + |c'||a''| \\
&\quad + |c'||b'| + |d'||b''| + |d'||a''| + |c''||b''|)
\end{aligned} \tag{8}$$

After comparing the terms in exponents one concludes $R = L$ and hence $\Delta^{\otimes}$ is a super algebra morphism. That $\epsilon^{\otimes}$ is a super algebra morphism is trivial.

2.4 The algebra of linear maps

Let B_1 and B_2 be two SBA's. In both SBA's we denote multiplication by μ; $\mu : a \otimes b \mapsto ab$. We consider the $\mathbb{C}$-linear maps from B_1 to B_2 and we provide this super vector space with a product structure:

$$f * g = \mu \circ f \otimes g \circ \Delta\,,\ \ f * g(x) = \mathrm{Exp}(|g||x'|)f(x')g(x'') \tag{9}$$

We view the map $\epsilon : B_1 \to \mathbb{C}$ as a map to B_2 by considering the image within B_2. We have

$$f * \epsilon(x) = f(x')\epsilon(x'') = f(x'\epsilon(x'')) = f(x)\,, \tag{10}$$

and similarly $\epsilon * f = f$, hence ϵ is an identity element with respect to the product $*$. We denote by $\mathcal{A}(B_1, B_2)$ the algebra of linear maps from B_1 to B_2 with the product $*$ and identity element ϵ.

We claim that $\mathcal{A}(B_1, B_2)$ is an associative algebra. We will show this in two ways. The first approach:

$$
\begin{aligned}
(f*g)*h &= \mu\circ(f*g)\otimes h\circ\Delta\\
&= \mu\circ(\mu\circ f\otimes g\circ\Delta)\otimes h\circ\Delta\\
&= \mu\circ\mu\otimes \mathrm{id}\circ f\otimes g\otimes \mathrm{id}\circ\\
&\quad \Delta\otimes\mathrm{id}\circ\mathrm{id}\otimes h\circ\Delta\\
&= \mu\circ\mu\otimes \mathrm{id}\circ f\otimes g\otimes \mathrm{id}\circ\\
&\quad \mathrm{id}\otimes h\circ\Delta\otimes\mathrm{id}\circ\Delta\\
&= \mu\circ\mu\otimes\mathrm{id}\circ f\otimes g\otimes h\circ\mathrm{id}\otimes\Delta\circ\Delta\\
&= \mu\circ\mathrm{id}\otimes\mu\circ f\otimes g\otimes h\circ\mathrm{id}\otimes\Delta\circ\Delta\\
&= \mu\circ\mathrm{id}\otimes\mu\circ\mathrm{id}\otimes g\otimes h\circ\\
&\quad f\otimes\mathrm{id}\circ\mathrm{id}\otimes\Delta\circ\Delta\\
&= \mu\circ\mathrm{id}\otimes\mu\circ\mathrm{id}\otimes g\otimes h\circ\mathrm{id}\otimes\Delta\circ f\otimes\mathrm{id}\circ\Delta\\
&= \mu\circ\mathrm{id}\otimes(g*h)\circ f\otimes\mathrm{id}\circ\Delta\\
&= \mu\circ f\otimes(g*h)\circ\Delta\\
&= f*(g*h)\,.
\end{aligned}
\tag{11}
$$

For the second approach we first note that:

$$
|f*g(x)| = |f(x')|+|g(x'')| = |f|+|g|+|x'|+|x''| = |f|+|g|+|x|\,, \tag{12}
$$

from which it follows that $|f*g| = |f|+|g|$ and hence $\mathcal{A}(B_1, B_2)$ is even a super algebra. For associativity we observe:

$$
\begin{aligned}
f*(g*h)(x) &= \mathrm{Exp}((|g|+|h|)|x'|)f(x')(g*h)(x'')\\
&= \mathrm{Exp}((|g|+|h|)|x'|+|(x'')'||h|)\\
&\quad f(x')g((x'')')h((x'')'')\\
&= f\otimes g\otimes h\,(x'\otimes(x'')'\otimes(x'')'')\\
&= f\otimes g\otimes h\,((x')'\otimes(x')''\otimes x'')\\
&= \mathrm{Exp}(|(x')'|(|g|+|h|)+|(x')''||h|)\\
&\quad f((x')')g((x')'')h(x'')\\
&= \mathrm{Exp}(|(x')'||g|+|(x')'|(|h|+|g|)+|(x')''||h|)\\
&\quad (f*g)(x')h(x'')\\
&= \mathrm{Exp}(|x'||h|(f*g)(x')h(x''))\\
&= (f*g)*h(x)\,.
\end{aligned}
\tag{13}
$$

We conclude that $\mathcal{A}(B_1, B_2)$ is an associative superalgebra with identity. This implies that a left inverse of an element is at the same time a right inverse (hence one can omit the prefix left/right) and that inverses are unique.

From the above we conclude that an SBA H is an SHA if there is an inverse to the element $\mathrm{id}_H : h \mapsto h$;

$$
S*\mathrm{id}_H(x) = S(x')x'' = \epsilon(x)\,,\quad \mathrm{id}_H*S(x) = x'S(x'') = \epsilon(x)\,, \tag{14}
$$

which is consistent with the requirement that the antipode S be even.

3 Properties

Lemma 3.1. *The antipode is unique.*

Proof. This follows from the uniqueness of inverses. ☐

Theorem 3.2. *The antipode S of an SHA H satisfies: $S(xy) = Exp(|x||y|)S(y)S(x)$, $\forall x, y \in H$.*

Proof. We consider the algebra $\mathcal{A}(H \otimes H, H)$ and claim that $\rho : x \otimes y \mapsto S(xy)$ is a left inverse of $\mu : x \otimes y \mapsto xy$ and that $\nu : x \otimes y \mapsto \mathrm{Exp}(|x||y|)S(y)S(x)$ is a right inverse to μ. The claim then follows from the uniqueness of inverses and equality of left and right inverses.

We have

$$\begin{aligned} \rho * \mu(x \otimes y) &= \mathrm{Exp}(|x''||y'|)\rho(x' \otimes y')x''y'' \\ &= \mathrm{Exp}(|x''||y'|)S(x'y')x''y'' \\ &= \mathrm{Exp}(|x''||y'|)S \otimes \mathrm{id}(x'y' \otimes x''y'') \\ &= S \otimes \mathrm{id}\,(x' \otimes x'' \cdot y' \otimes y'') \\ &= S \otimes \mathrm{id}\,(\Delta x \Delta y) \\ &= S \otimes \mathrm{id}\,(\Delta(xy)) \\ &= S \otimes \mathrm{id}\,(xy)' \otimes (xy)'' \\ &= S((xy)')(xy)'' \\ &= \epsilon(xy)\,, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \mu * \nu(x \otimes y) &= \mu \circ \mu \otimes \nu \circ \Delta^{\otimes}(x \otimes y) \\ &= \mathrm{Exp}(|x''||y'|)\mu \circ \mu \otimes \nu(x' \otimes y' \otimes x'' \otimes y'') \\ &= \mathrm{Exp}(|x''||y'| + |x''||y''|)\mu(x'y' \otimes S(y'')S(x'')) \\ &= \mathrm{Exp}(|x''||y|)x'y'S(y'')S(x'') \\ &= \mathrm{Exp}(|x''||y|)x'\epsilon(y)S(x'') \\ &= x'S(x'')\epsilon(y) \\ &= \epsilon(x)\epsilon(y) \\ &= \epsilon(xy)\,, \end{aligned} \tag{16}$$

where we used that $|\epsilon(y)| = |y|$. This might look a bit strange since if y is odd ϵ maps it to zero, but one must keep in mind that in every equation where primes are involved the summation signs are implicit. Hence it is a fancy way of keeping up with the book keeping. ☐

Proposition 3.3. *Let H be an SHA. Then we have $S(1) = 1$.*

Proof. $\mathrm{id}_H * S(1) = 1 \cdot S(1) = \epsilon(1) = 1$. ☐

Proposition 3.4. *In an SHA we have $\epsilon \circ S = \epsilon$.*

Proof.

$$\begin{aligned} \epsilon(S(x)) &= \epsilon(S(x'\epsilon(x''))) \\ &= \epsilon(S(x')\epsilon(x'')) \\ &= \epsilon(S(x'))\epsilon(x'') \\ &= \epsilon(S(x')x'') \\ &= \epsilon(\epsilon(x)) = \epsilon(x)\,. \end{aligned} \tag{17}$$

☐

Proposition 3.5. *If H is a commutative SHA - meaning that it is commutative as a super algebra -, then $S^2 = id_H$.*

Proof. We show that in $\mathcal{A}(H,H)$ the map S^2 is also an inverse to S, implying it must be id_H.

$$\begin{aligned} S * S^2(x) &= S(x')S^2(x'') \\ &= S(\mathrm{Exp}(|x'||x''|)S(x'')x') \\ &= S(x'S(x'')) \\ &= S(\epsilon(x)) \\ &= \epsilon(x)S(1) = \epsilon(x)\,. \end{aligned} \tag{18}$$

□

Lemma 3.6. *Let x be an element of an SBA, then we have the identity:*

$$(x')' \otimes (x')'' \otimes (x'')' \otimes (x'')'' = x' \otimes ((x'')')' \otimes ((x'')')'' \otimes (x'')'' \tag{19}$$

Proof. The equality follows from:

$$\begin{aligned} \Delta \otimes \Delta \circ \Delta &= \Delta \otimes \mathrm{id} \otimes \mathrm{id} \circ \mathrm{id} \otimes \Delta \circ \Delta \\ &= \Delta \otimes \mathrm{id} \otimes \mathrm{id} \circ \Delta \otimes \mathrm{id} \circ \Delta \\ &= (\Delta \otimes \mathrm{id} \circ \Delta) \otimes \mathrm{id} \circ \Delta \\ &= (\mathrm{id} \otimes \Delta \circ \Delta) \otimes \mathrm{id} \circ \Delta \\ &= \mathrm{id} \otimes \Delta \otimes \mathrm{id} \circ \Delta \otimes \mathrm{id} \circ \Delta \\ &= \mathrm{id} \otimes \Delta \otimes \mathrm{id} \circ \mathrm{id} \otimes \Delta \circ \Delta\,, \end{aligned} \tag{20}$$

and applying the derived equality to x, we obtain the desired result. □

Theorem 3.7. *Let H be an SHA, then for all $x \in H$ we have:*

$$\Delta(S(x)) = \sum (-1)^{|x'||x''|} S(x'') \otimes S(x')\,. \tag{21}$$

Proof. We consider $\mathcal{A}(H, H \otimes H)$ and claim that $\rho = \Delta \circ S$ is a left inverse to Δ and that $\nu = S \otimes S \circ T \circ \Delta$ is a left inverse to Δ, where $T : x \otimes y \mapsto \mathrm{Exp}(|x||y|)y \otimes x$. The theorem then follows. Multiplication in $H \otimes H$ will also be denoted μ.

We have:

$$\begin{aligned} \rho * \Delta(x) &= \rho(x')\Delta(x'') \\ &= \Delta(S(x'))\Delta(x'') \\ &= \Delta(S(x')x'') \\ &= \Delta(\epsilon(x)) = \epsilon(x)\,. \end{aligned} \tag{22}$$

On the other hand we have

$$
\begin{aligned}
\Delta * \nu(x) &= \Delta(x')\nu(x'') \\
&= \mathrm{Exp}(|(x'')'||(x'')''|)\Delta(x')S((x'')'') \otimes S((x'')') \\
&= \mathrm{Exp}(|(x'')'||(x'')''|)\mu \circ x' \otimes x'' \otimes S((x'')'') \otimes S((x'')') \\
&= \mathrm{Exp}(|(x'')'||(x'')''|)\mu \circ \mathrm{id}_2 \otimes S \otimes S \\
&\quad \big((x')' \otimes (x')''\big) \otimes (x'')'' \otimes (x'')' \\
&= \mu \circ (\mathrm{id}_2 \otimes S \otimes S) \circ T^{(4)}_{34} \left((x')' \otimes (x')'' \otimes (x'')' \otimes (x'')''\right) \\
&= \mu \circ (\mathrm{id}_2 \otimes S \otimes S) \circ T^{(4)}_{34} \left(x' \otimes ((x'')')' \otimes ((x'')')'' \otimes (x'')''\right) \\
&= \mathrm{Exp}(|((x'')')''||(x'')''|)\mu \circ (\mathrm{id}_2 \otimes S \otimes S) \\
&\quad \big(x' \otimes ((x'')')' \otimes (x'')'' \otimes ((x'')')''\big) \\
&= \mathrm{Exp}(|((x'')')''||(x'')''|) \cdot \\
&\quad \mu \left(x' \otimes ((x'')')' \otimes S((x'')'') \otimes S(((x'')')'')\right) \\
&= \mathrm{Exp}(|(x'')''||(x'')'|)x'S((x'')'') \otimes ((x'')')'S(((x'')')'') \\
&= \mathrm{Exp}(|(x'')''||(x'')'|)x'S((x'')'') \otimes \epsilon((x'')'') \\
&= x'S(\epsilon((x'')')(x'')'') \\
&= x'S(x'') = \epsilon(x)\,,
\end{aligned}
\tag{23}
$$

where we used id_2 to denote the linear map in $H \otimes H$ sending $x \otimes y$ to $x \otimes y$ and the map $T^{(4)}_{34}$ is the exchange map $T^{(4)}_{34} : x \otimes y \otimes z \otimes w \mapsto \mathrm{Exp}(|z||w|)x \otimes y \otimes w \otimes z$. □

References

[1] Quantum Groups, Christian Kassel.

[2] Hopf algebras, M. Sweedler.