SUPERMATRICES

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ABSTRACT. This text is devoted to develop a consistent set of rules for operations in modules of commutative superalgebras. Most of the definitions can also be found in [1], but there are some small differences; our definition of supertransposition is different (and better). For other expositions on superalgebras see e.g. [2, 3, 4].

1. SUPERALGEBRAS

A vector space V is a super vector space (over a field k) if it admits a decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The elements of $V_{\bar{0}} \cup V_{\bar{1}}$ are called homogeneous elements and we have a map $V_{\bar{0}} \cup V_{\bar{1}} \to \mathbb{Z}_2$ defined by $|v| = i$ for $v \in V_i$, $i \in \mathbb{Z}_2$. If $v \in V_{\bar{0}}$ we call V even (of even parity) and if $v \in V_{\bar{1}}$ we call v odd (of odd parity). If we use parity assignments in formulas and definitions, it is understood that it is meant for homogeneous elements and extended by linearity to inhomogeneous elements.

The morphisms between super vector spaces are the vector space morphisms that preserve the parity assignment. If V and W are superspaces then the space of linear maps from V to W gets the structure of a super vector space when we take the even elements to be the linear maps that preserve the parity and the odd elements to be the linear maps that change the parity. The vector space $V \otimes W$ becomes a super vector space when we set $(V \otimes W)_i = \oplus_{k+l=i} V_k \otimes W_l$. For the direct product $V \oplus W$ we take the grading $(V \oplus W)_i = V_i \oplus W_i$.

A superalgebra Λ is a super vector space $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$ with a map $\Lambda \times \Lambda \to \Lambda$ called multiplication, written $(x, y) \mapsto xy$, that satisfies $|xy| = |x| + |y|$, $\forall x, y \in \Lambda$.

The tensor product of two superalgebras Λ_1 and Λ_2 is as a vector space the tensor product defined as above, with the following multiplication rule:

$$(a \otimes b)(c \otimes d) = (-)^{|b||c|} ac \otimes bd \quad a, c \in \Lambda_1\,,\ b, d \in \Lambda_2\,. \tag{1}$$

We call a superalgebra commutative if

$$ab = (-1)^{|a||b|} ba\,. \tag{2}$$

In the sequel Λ always denotes a commutative superalgebra with identity element 1.

We define $\mathcal{N}_\Lambda$ to be the ideal generated by the odd part $\Lambda_{\bar{1}}$. By definition, the elements of $\mathcal{N}_\Lambda$ are finite sums of elements of the form

an, with $a \in \Lambda$ and $n \in \Lambda_{\bar{1}}$. The body of Λ is the quotient algebra $\bar{\Lambda}$ defined by

$$\bar{\Lambda} = \Lambda/\mathcal{N}_\Lambda \,. \tag{3}$$

The body of an element a is the image under the projection $\Lambda \to \bar{\Lambda}$ and is denoted $\bar{a}$. It is easy to see that $\bar{\Lambda} \neq 0 \leftrightarrow \bar{1} \neq 0$.

A superalgebra is finitely generated if there exists a set of elements $a_1, \ldots, a_N$ of homogeneous elements such that the a_i generate the superalgebra as an algebra. A superalgebra is called affine if it is finitely generated and the body is an affine algebra. Note that the body of a finitely generated superalgebra is finitely generated, hence the body of a finitely generated superalgebra is affine if the body contains no nilpotent elements.

The polynomial algebra (over k) in n even variables X_i, $1 \leq in$, and m odd variables Θ_α, $1 \leq \alpha \leq m$, is defined to be the free algebra over k generated by the X_i and Θ_α subject to the relations $X_i X_j = X_j X_i$, $X_i \Theta_\alpha = \Theta_\alpha X_i$ and $\Theta_\alpha \Theta_\beta = \Theta_\beta \Theta_\alpha$ for all i, j, α, β.

2. Modules and matrices

Let M be an abelian group M that decomposes into an odd and even subgroup $M = M_{\bar{0}} \oplus M_{\bar{1}}$. We call M a left Λ-module if there is a map $\Lambda \times M \to M$, $(a, m) \mapsto am$ for $a \in \Lambda$ and $m \in M$, such that the parity of am is given by: $|am| = |a| + |m|$ and such that for any $a, b \in \Lambda$ and $m, n \in M$:

$$a(m+n) = am + an \,, \tag{4}$$

$$(a+b)m = am + bm \,, \tag{5}$$

$$(ab)m = a(bm) \,, \tag{6}$$

$$1m = m \,. \tag{7}$$

The definition of a right Λ-module is similar. Given a left Λ-module M, we can define a right action of Λ on M by $ma = (-1)^{|m||a|} am$ for $a \in \Lambda$ and $m \in M$. The right action of Λ is then compatible with the left action $a(mb) = (am)b$, for $a, b \in \Lambda$ and $m \in M$. Hence we call a left Λ-module with a right multiplication defined in this way a Λ-module.

If M and N are Λ-modules then a linear map from M to N is a map $F : M \to N$ such that $F(m_1 + m_2) = F(m_1) + F(m_2)$ and $F(ma) = (F(m))a$, for all $m, m_1, m_2 \in M$, $a \in \Lambda$. The set of linear maps from M to M is denoted by $\mathrm{End}(M)$. The set of linear invertible maps from M to M is denoted $\mathrm{GL}(M)$.

Among the modules a special place is taken by the free modules. Let I be an index set $I = I_{\bar{0}} \oplus I_{\bar{1}}$ such that $I_{\bar{0}} \cap I_{\bar{1}} = \emptyset$. A set of homogeneous elements $\{m_i | i \in I\}$ with parity assignment $|m_i| = \bar{0}$ for $m_i \in I_{\bar{0}}$ and $|m_i| = \bar{1}$ for $i \in I_{\bar{1}}$, is a basis for the module M if every element $x \in M$ can be uniquely expressed as

$$x = \sum_{i \in I} m_i \mu_i \,, \tag{8}$$

such that only a finite number of terms is nonzero. The module M is free if it possesses a basis. We only consider finite-dimensional free modules, for which the index set I is finite. The dimension of a free module is denoted $p|q$ where p is the number of elements in $I_{\bar{0}}$ and q is the number of elements in $I_{\bar{1}}$. The elements m_i are called basis elements or generators. For convenience we give the set I a parity assignment; $|i| = |m_i|$.

Note that the Λ-elements in the expression (8) have been written to the right of the generators m_i. Therefore the μ_i are called right-coordinates. Similarly there exist unique left-coordinates ν_i such that $x = \sum_i \nu_i m_i$ and $\nu_i = (-1)^{|\mu_i||m_i|}\mu_i$.

We call a set of generators standard if $m_1, \ldots, m_p$ are even and $m_{p+1}, \ldots, m_{p+q}$ are odd. To each element $x = \sum_i m_i \mu_i$ we assign a column vector and a row vector as follows;

(9)

$$x \mapsto |\mu_i\rangle = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{p+q} \end{pmatrix}, \quad x \mapsto \langle \nu_i| = (\nu_1, \cdots, \nu_{p+q}), \quad \nu_i = (-1)^{|\mu_i||m_i|}\mu_i \,.$$

If $F : M \to M$ is a linear map we assign to F a matrix (F_{ij}) by $F(m_i) = \sum m_j F_{ji}$. With this rule we obtain the usual matrix multiplication for the composition of linear maps. The parity of a matrix is its parity as a linear transformation.

When given a standard basis for a $p|q$-dimensional Λ-module M the matrices can be decomposed into block matrices as follows

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{10}$$

where A is a $p \times p$-matrix, B a $p \times q$-matrix, C a $q \times p$-matrix and D a $q \times q$ matrix. In the sequel a block-decomposition as in (10) is always according to a standard basis. If X is even the elements of A and D are even and those of C and B odd. When X is odd the elements of A and D are odd and those of C and B even. The set of $(p+q) \times (p+q)$-supermatrices with entries in Λ is denoted $\mathrm{Mat}(p, q|\Lambda)$. The subset of $\mathrm{Mat}(p, q|\Lambda)$ of invertible supermatrices is denoted $GL(p, q|\Lambda)$.

We denote the invertible elements of $\Lambda_{\bar{0}}$ by $\Lambda_{\bar{0}}^*$. We define the Berezinian $\mathrm{Ber} : \mathrm{GL}(p, q|\Lambda)_{\bar{0}} \to \Lambda_{\bar{0}}^*$ as

$$\mathrm{Ber} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{\det(A - BD^{-1}C)}{\det D} \,. \tag{11}$$

The Berezinian satisfies[3, 1]: $\mathrm{Ber}(XY) = \mathrm{Ber}(X)\mathrm{Ber}(Y)$.

Let $\{m_i\}$ and $\{m_i'\}$ be bases for M. Then there is a matrix (C_{ij}) such that $m_i = \sum m_k' C_{ki}$. The matrix (C_{ij}) is even and invertible. If we write $x = \sum m_i \mu_i = \sum m_i' \mu_i'$, then $\mu_k' = \sum_i C_{ki}\mu_i$ and the left-coordinates ν_i and ν_i' are related by

$$\nu_i' = \sum \nu_k C_{ki}^{ST}\,, \quad C_{ij}^{ST} = (-1)^{|j|(|i|+|j|)} C_{ji}\,. \tag{12}$$

The equation (12) defines the supertranspose of an even matrix. For any homogeneous matrix $X = (X_{ij})$ we define

$$X_{ij}^{ST} = (-1)^{(|i|+|j|)(|j|+|X|)} X_{ji}\,. \tag{13}$$

It then follows that $\langle \nu_i| = (|\mu_i\rangle)^{ST}$ and we have

$$(XY)^{ST} = (-1)^{|X||Y|} Y^{ST} X^{ST}\,. \tag{14}$$

The Berezinian is inert under composition with supertransposition: $\mathrm{Ber} X^{ST} = \mathrm{Ber} X$.

We note further that the action of Λ on $\mathrm{Hom}_\Lambda(M,N)$ is given by $(aF)_{ij} = (-1)^{|a||i|} aF_{ij}$, and $(aF)_{ij} = (-1)^{|a||j|} F_{ij} a$.

3. Grassmann envelope and Grassmann hull

Given a super vector space V of dimension $p|q$ over a field k, we can turn it into a super vector space over Λ by considering its Grassmann envelope V_Λ. The Grassmann envelope of V is defined as

$$V_\Lambda = \Lambda \otimes_k V\,. \tag{15}$$

The parity assignment is as follows:

$$V_{\Lambda,\bar 0} = \Lambda_{\bar 0} \otimes V_{\bar 0} \oplus \Lambda_{\bar 1} \otimes V_{\bar 1}\,, \quad V_{\Lambda,\bar 1} = \Lambda_{\bar 1} \otimes V_{\bar 0} \oplus \Lambda_{\bar 0} \otimes V_{\bar 1}\,. \tag{16}$$

Multiplication from the left by Λ-elements is defined by $a(b\otimes v) = (ab)\otimes v$ and multiplication from the right by $(b\otimes v)a = (-1)^{|m||a|} ba \otimes v$. With this structure the right and left multiplication are compatible. If V is an algebra the multiplication in V_Λ is as in (1); $(c\otimes v)(d\otimes w) = (-1)^{|d||v|} cd \otimes vw$.

The Grassmann hull of a vector space is the even part of the Grassmann envelope.

$$V[\Lambda] = V_{\Lambda,\bar 0} = \Lambda_{\bar 0} \otimes V_{\bar 0} \oplus \Lambda_{\bar 1} \otimes V_{\bar 1}\,. \tag{17}$$

Later we will see that the Grassmann hull of the Lie algebra $\mathfrak{osp}(m|2n)$ is isomorphic to the Lie algebra associated to the group $OSp(m|2n)$.

We now turn to the description of relating the endomorphisms of V_Λ, with $\dim V = p|q$, to the set $\mathrm{Mat}(p,q|\Lambda) \cong \mathrm{End}(V_\Lambda)$ of $(p+q)\times(p+q)$-matrices with entries in Λ.

If V has standard-basis elements e_i - $1 \le i \le p+q$ and such that $|e_i| = bar0$ for $1 \le i \le p$ and $|e_i| = \bar 1$ for $p+1 \le i \le p+q$ -, then V_Λ is a free module with standard-basis elements $\hat e_i = 1 \otimes e_i$. Matrix elements with respect to the basis $\{\hat e_i\}$ are given by $F(\hat e_i) = \sum_j \hat e_j F_{ji}$. In $\mathrm{End}(V)$ we define the basis elements $\mathcal{E}^{(kl)}$ by $\mathcal{E}^{(kl)}(e_i) = \delta_i^k e_l$.

We define the map $\tau : \mathrm{End}(V_\Lambda) \to \Lambda\hat{\otimes}\mathrm{End}(V)$ by

$$\tau : F = (F_{ij}) \mapsto \sum_{kl} (-1)^{|k|(|k|+|l|+|F|)} F_{kl} \otimes \mathcal{E}^{(lk)} \,. \tag{18}$$

With the definition (18) we can formulate the following easy result:

Theorem 3.1. *The map $\tau : End(V_\Lambda) \to \Lambda\hat{\otimes}End(V)$ is an isomorphism of superalgebras.*

Proof. It is clear that the map is surjective and injective. It only needs to be checked that $\tau(xy) = \tau(x)\tau(y)$, which is a direct calculation. □

Next we define the map $\sigma : \Lambda \otimes \mathrm{End}(V) \to \Lambda \otimes \mathrm{End}(V)$ by $\sigma c \otimes X \mapsto c \otimes X^{ST}$. It is to be understood that the matrices in $\mathrm{End}(V)$ have real entries and hence the off-diagonal parts are odd; therefore the choice of the minus sign is as that for an odd supermatrix. The map σ fulfills the relation $\sigma(xy) = (-1)^{|x||y|}\sigma(y)\sigma(x)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{End}(V_\Lambda) & \xrightarrow{\tau} & \Lambda\hat{\otimes}\mathrm{End}(V) \\ \Big\downarrow{\scriptstyle ST} & & \Big\downarrow{\scriptstyle \sigma} \\ \mathrm{End}(V_\Lambda) & \xrightarrow{\tau} & \Lambda\hat{\otimes}\mathrm{End}(V) \end{array} \quad .$$

We can therefore safely identify the maps ST and σ, and supertransposition in $\mathrm{End}(V)$ with supertransposition in $\mathrm{End}(V_\Lambda)$.

References

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