1 The affine superscheme

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a superring. The commutative ring $A/(A_{\bar{1}})$ is denoted $\bar{A}$ and is called the body of A. We know there is an inclusion preserving one-to-one correspondence between the prime ideals in A and the prime ideals in $\bar{A}$. Now consider $A_{\bar{0}}$ as a commutative ring. It has an ideal $A_{\bar{1}}^2$, which is contained in all prime ideals. Furthermore, we have $\bar{A} \cong A_{\bar{0}}/A_{\bar{1}}^2$ so that we conclude that there is an inclusion preserving one-to-one correspondence between the prime ideals of $A_{\bar{0}}$ and A. We use this fact to associate a topological space to each superring. This topological space we equip with a sheaf of superrings. The result is called an affine superscheme and a general superscheme has to look locally like an affine superscheme. The presentation below is very similar to the usual expositions of the construction of the spectrum of a commutative ring. Therefore the discussion will be rather short and some proofs are omitted. All omitted proofs for the commutative case can be found in the textbooks [1, 2] and can be copied almost literally for the super case.

1.1 The topological space

For any commutative ring R, we denote $\mathrm{Spec}(R)$ the topological space of prime ideals. The topology is defined by prescribing the closed sets. The closed sets of $\mathrm{Spec}(R)$ are the sets

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supset \mathfrak{a}\} \ , \tag{1}$$

where $\mathfrak{a}$ is any ideal in R. If $\mathfrak{a}$ is generated by a single element $f \in R$, we also write $V(f)$ for $V(\mathfrak{a})$. One checks easily that $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$ and $V(\sum_i \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$, so that the closed sets $V(\mathfrak{a})$ indeed define a topology. For any $f \in R$ we define the principal open set $D(f)$ by

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\} \ . \tag{2}$$

Equivalently, we can define $D(f)$ as the complement of $V(f)$. If J is any ideal in R that is contained in the nilradical of R, then the projection $\pi : R \to R/J$ induces a continuous map

$$\pi' : \mathrm{Spec}(R/J) \to \mathrm{Spec}(R) \,, \quad \pi'(\mathfrak{p}) = \pi^{-1}(\mathfrak{p}). \tag{3}$$

As J is contained in the nilradical, the map π' is a bijection and one checks easily that its inverse, which sends $\mathfrak{q} \in \mathrm{Spec}(R)$ to $\mathfrak{q} \mod J \in \mathrm{Spec}(R/J)$, is continuous. Hence π' is in fact a homeomorphism. Thus for any superring A there is a homeomorphism of topological spaces:

$$\mathrm{Spec}(\bar{A}) \cong \mathrm{Spec}(A_{\bar{0}}) \,, \tag{4}$$

induced by the projection $A_{\bar{0}} \to \bar{A} \cong A_{\bar{0}}/A_{\bar{1}}^2$. To any superring A we now associate the topological space $\mathrm{Spec}(A_{\bar{0}})$ and call it the spectrum of A. For convenience, we mostly work with the description of the spectrum of A as the topological space of prime ideals of $A_{\bar{0}}$, although describing the spectrum as the prime ideals of $\bar{A}$ is equivalent. The following lemma gives some properties of the principal open sets on $\mathrm{Spec}(A_{\bar{0}})$:

Lemma 1.1. *Let A be a superring, then we have:*

(i) If $F = \{f_i \mid i \in I\}$ is a set of elements of $A_{\bar{0}}$, then $\bigcup_{i \in I} D(f_i) = \mathrm{Spec}(A_{\bar{0}})$ if and only if the ideal generated by the f_i is A. Thus, the principal open sets $D(f_i)$ cover $\mathrm{Spec}(A_{\bar{0}})$ if and only if there are finitely many $f_{i_1}, \ldots, f_{i_k}$ in F and finitely many $a_1, \ldots, a_k \in A$ such that $1 = \sum_{j=1}^k a_j f_{i_j}$.

(ii) $D(\mathfrak{a}) \subset D(\mathfrak{b})$ if and only if $V(\mathfrak{b}) \subset V(\mathfrak{a})$ if and only if $\sqrt{\mathfrak{a}} \subset \sqrt{\mathfrak{b}}$. In particular, $D(f) \subset D(g)$ if and only if $f \in \sqrt{g}$.

(iii) $D(f) \cap D(g) = D(fg)$.

(iv) *If* $D(f) = D(g)$, *then* $A_f \cong A_g$.

(v) *The principal open sets are a basis of the topology.*

Proof. (i): The functions f_i generate $A_{\bar{0}}$ in $A_{\bar{0}}$ if and only if they generate A in A. Hence we are back in the commutative case. (ii): The first equivalence is trivial. For the second; $V(\mathfrak{b}) \subset V(\mathfrak{a})$ if and only if any prime ideal that contains $\mathfrak{b}$ also contains $\mathfrak{a}$. But $\sqrt{\mathfrak{a}}$ is the intersection of the prime ideals that contain $\mathfrak{a}$. (iii): A prime ideal $\mathfrak{p}$ does not contain f and g if and only if it does not contain fg. (iv): By (ii) there are $a, b \in A$ and integers m, n such that $f^n = ag$ and $g^m = bf$. We have a natural map $A_f \to A_g$ that maps c/f^s to cb^s/g^{ms} and a natural map $A_g \to A_f$ that maps d/g^t to da^t/f^{nt}. Since $f^{mn} = a^m bf$ and $ab^n g = g^{mn}$, these maps are isomorphisms. (v): Any open set U is the complement of a set $V(\mathfrak{a})$ for some $\mathbb{Z}_2$-graded ideal in A. This complement is not empty if there exists an even $a \in \mathfrak{a}$ that is not contained in some prime ideal $\mathfrak{p}$. Then $D(a) \subset U$. $\square$

1.2 The sheaf

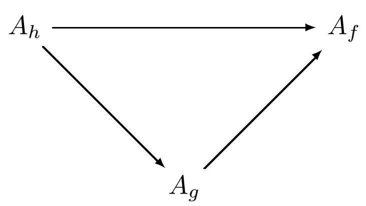
We now equip the topological space $\mathrm{Spec}(A_{\bar{0}})$ with a sheaf $\mathcal{O}$ of superrings, such that the stalks are local rings. In this way, $\mathrm{Spec}(A_{\bar{0}})$ becomes a locally superringed space. First we describe some properties that we want the sheaf to have. Then we define the sheaf and state as a proposition that the sheaf indeed has the desired properties.

The stalk at $\mathfrak{p}$ of the sheaf $\mathcal{O}$ we want to be $A_{\mathfrak{p}}$. This is in analogy with the case for commutative rings. Also, if we have a map of superrings $\psi : A \to B$, then for any prime ideal $\mathfrak{q}$ of B, the preimage $\psi^{-1}(\mathfrak{p})$ is a prime ideal of A and we have an induced map $\psi_{\mathfrak{p}} : A_{\psi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$.

On the principal open sets $D(f)$ we want that $\mathcal{O}(D(f)) = A_f$. We thus only allow even f for the principal open sets. If $D(f) \subset D(g)$ then $f^n = ag$ for some $a \in A$ and thus there is a natural map $A_g \mapsto A_f$ given by

$$\frac{b}{g^m} \mapsto \frac{a^m b}{f^{mn}} . \qquad (5)$$

The map $A_g \mapsto A_f$ is well-defined and does not depend on the choice of the exponent n and the element a. If we have inclusions $D(f) \subset D(g) \subset D(h)$, then one easily checks that the following diagram commutes:



(6)

Furthermore, if $D(f) = D(g)$, then by lemma 1.1 we have $A_f \cong A_g$. The assignment $D(f) \mapsto A_f$ therefore defines a presheaf on the principal open sets. The following lemma shows that the assignment in fact gives a sheaf on the principal open sets:

Lemma 1.2. *Suppose* $D(f) = D(f_1) \cup \ldots \cup D(f_k)$.

(i) If $s \in A_f$ *goes to zero in all maps* $A_f \to A_{f_i}$, *then* $s = 0$ *in* A_f.

(ii) If a set of elements $s_i \in A_{f_i}$ is given, such that s_i and s_j have the same image in $A_{f_i f_j}$ for all $1 \leq i, j \leq k$, then there is an element $s \in A_f$ that has image s_i in A_{f_i} for all i.

Proof. As $D(f) \cong \mathrm{Spec}(A_f)$ it is sufficient to prove the lemma for $f = 1$. For (i): s goes to zero in A_{f_i} if and only if there is an integer N such that $f_i^N s = 0$ for all i. But as the $D(f_i)$ form a cover, there is a relation $1 = f_1 a_1 + \dots f_k a_k$. But then $s = 1s = (f_1 a_1 + \dots f_k a_k)^{Nk} s = 0$. For (ii): We can write $s_i = a_i / f_i^n$ for some n and some $a_i \in A$. That the images of s_i and s_j agree in $A_{f_i f_j}$ means that $a_i f_j^n / (f_i f_j)^n = a_j f_i^n / (f_i f_j)^n$. But then there is an integer m such that

$$(f_i f_j)^m \Big(a_i f_j^n - a_j f_i^n \Big) = 0 \,, \qquad \forall i, j \,. \tag{7}$$

We define $b_i = a_i f_i^m$ and write $s_i = b_i / f_i^m$. Then eqn.(7) reads $b_i f_j^{n+m} = b_j f_i^{n+m}$. From $\mathrm{Spec}(A_{\bar{0}}) = \cup D(f_i) = \cup D(f_i^{n+m})$ we infer that there is a relation $1 = \sum_i c_i f_i^{n+m}$ for some even c_i. Define $s = \sum_i c_i b_i$, then as $s f_i^{n+m} = b_i$ we see that the image of s in A_{f_i} is s_i. □

With these preliminaries we can give the general definition of the sheaf on $\mathrm{Spec}(A_{\bar{0}})$. To check that it is really a sheaf is then further an exercise in dealing with sheaves and can be found in the textbooks [2] and [1]. On an arbitrary open set $U \subset \mathrm{Spec}(A_{\bar{0}})$ we define $\mathcal{O}(U)$ as follows: $\mathcal{O}(U)$ is the superring of all functions $s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$, $s : \mathfrak{p} \mapsto s_{\mathfrak{p}}$, such that $s_{\mathfrak{p}} \in A_{\mathfrak{p}}$ and for all $\mathfrak{p}$ in U there exists an open neighborhood V of $\mathfrak{p}$ and elements $a \in A$ and $f \in A_{\bar{0}}$, with $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in V$, such that $s_{\mathfrak{q}}$ equals the image of a/f in $A_{\mathfrak{q}}$ for all $\mathfrak{q} \in V$. As the principal sets form a basis of the topology, we may in fact always assume that $V = D(f)$. We call the sheaf $\mathcal{O}$ associated to A the structure sheaf of $\mathrm{Spec}(A)$. We now state that the given definition of the structure sheaf $\mathcal{O}$ has the required properties:

Proposition 1.3. *Let A be a superring and let $\mathcal{O}$ be the sheaf of superrings on* $\mathrm{Spec}(A_{\bar{0}})$ *defined above. Then*

(i) On the principal open sets we have $\mathcal{O}(D(f)) = A_f$.

(ii) The stalk at $\mathfrak{p}$ is $A_{\mathfrak{p}}$.

Proof. We only indicate the proof and refer for the details to [2, 1]. (i) is proved in 1.2(ii). For (ii): the principal open sets are a basis for the topology and hence in calculating the inductive limit $\varinjlim_{\mathfrak{p} \in U} \mathcal{O}(U)$ we can take the limit over the principal open sets. If $f \in \sqrt{g}$ we have a map $A_g \to A_f$, which we use to construct the inverse system ($A_g \to A_f : f, g \notin \mathfrak{p}$, and $f \in \sqrt{g}$); the limit of this inverse system is then the stalk at $\mathfrak{p}$. But the map

$$\varphi : \varinjlim_{f \notin \mathfrak{p}} A_f \to A_{\mathfrak{p}} \,, \qquad \frac{a}{f} \mapsto \frac{a}{f} \in A_{\mathfrak{p}} \,, \tag{8}$$

is an isomorphism of superrings. □

From proposition 1.3 we see that the stalks are local rings, and the global sections are the elements of A.

If $\mathfrak{p}$ is a prime ideal in A and s is a section that is defined in some open neighborhood U of $\mathfrak{p}$, then we write $s_{\mathfrak{p}}$ for the image of s in $A_{\mathfrak{p}}$. The 'function value' of s in $\mathfrak{p}$ is defined as the image of $s_{\mathfrak{p}}$ in $A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$, and is denoted $s(\mathfrak{p})$. All odd sections thus have zero value at any point, but as sections they are not zero.

2 The general superscheme

To define the category of superschemes is then done in a similar way as the category of schemes is constructed. We define a locally superringed space to be a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of superrings on X such that the stalks $\mathcal{O}_{X,x}$ are local superrings for all $x \in X$. A morphism of locally superrings spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (φ, ψ) with $\varphi : X \to Y$ a continuous map and ψ a set of morphisms $\psi_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ for all open sets $U \subset Y$ satisfying the following two conditions: (i) The ψ_U are compatible with restrictions, that is, for all inclusions $V \subset U \subset Y$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{\psi_U} & \mathcal{O}_X(\varphi^{-1}(U)) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(V) & \xrightarrow{\psi_V} & \mathcal{O}_X(\varphi^{-1}(V)) \end{array} \tag{9}$$

In diagram 9 the vertical arrows are the restriction maps. (ii): Because of the compatibility of the morphisms ψ_U, there is an induced map on the stalks $\psi_x : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$, which we require to be a local morphism. That is, if $\mathfrak{m}_{\varphi(x)}$ is the maximal ideal of $\mathcal{O}_{Y,\varphi(x)}$ and $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$ then $\psi_x(\mathfrak{m}_{\varphi(x)}) \subset \mathfrak{m}_x$, or equivalently, $\psi_x^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{\varphi(x)}$.

Definition 2.1. *Let X be a topological space with a sheaf $\mathcal{O}_X$ of local superrings. We say that $(X, \mathcal{O}_X)$ is a superscheme if for each point $x \in X$ there is an open neighborhood U of x such that the topological space U equipped with the restriction of the sheaf $\mathcal{O}_X$ to U is isomorphic as a locally superringed space to an affine superscheme* $\mathrm{Spec}(A_{\bar{0}})$ *with its sheaf of superrings. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two superschemes then a morphism of superschemes is morphism $f = (\varphi, \psi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally superringed spaces.*

It follows from the definition that for a superring A, the topological space $X = \mathrm{Spec}(A_{\bar{0}})$ equipped with the sheaf of superrings $\mathcal{O} : D(f) \mapsto A_f$ defines a superscheme. We denote Ssch the category of superschemes.

Proposition 2.2. *Let $(X, \mathcal{O}_X)$ be a superscheme and let A be a superring. Denote $(Y, \mathcal{O}_Y)$ the superscheme associated to the spectrum of A. Then the natural map*

$$\Gamma : \mathrm{Hom}_{\mathrm{Ssch}}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \to \mathrm{Hom}_{\mathrm{sRng}}(A, \mathcal{O}_X(X)) \tag{10}$$

that sends a morphism of superschemes $f = (\varphi, \psi)$ to the morphism of superrings $\psi_Y : A \to \mathcal{O}_X(X)$, is a bijection.

Proof. First we show that Γ is injective. Let $f = (\varphi, \psi)$ be a morphism and let $\Gamma(f) : A \mapsto \mathcal{O}_X(X)$ be the map on global sections. For any $x \in X$ the image $\varphi(x)$ is recovered by

$$\begin{aligned} \varphi(x) &= \left\{ a \in A \mid a_{\varphi(x)} \in \mathfrak{m}_{\varphi(x)} \right\} \\ &= \left\{ a \in A \mid \psi_x(a_{\varphi(x)}) \in \mathfrak{m}_x \right\} \\ &= \left\{ a \in A \mid (\psi_X(a))_x \in \mathfrak{m}_x \right\} . \end{aligned} \tag{11}$$

Indeed, the first equality follows from the fact that for any prime ideal $\mathfrak{p} \subset A$ the kernel of the map $A \to A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ is precisely $\mathfrak{p}$. The second equality follows from the fact that

$\psi_{\mathfrak{p}}$ is a local morphism. The third equality follows since ψ is compatible with restrictions to any open subset in X, so that the following diagram commutes

$$\begin{array}{ccc} A = \mathcal{O}_Y(Y) & \xrightarrow{\psi_X} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ A_{\varphi(x)} = \mathcal{O}_{Y,f(x)} & \xrightarrow[\psi_x]{} & \mathcal{O}_{X,x} \end{array} \tag{12}$$

Thus the continuous map $\varphi : X \to Y$ is uniquely determined by $\psi_X = \Gamma(f)$. To show that the maps ψ_U for open sets $U \subset Y$ are uniquely determined by ψ_X as well, we only need to check this on a basis of the topology, which is given by the principal open sets $D(g)$, for an even element g of A. Consider the following diagram:

$$\begin{array}{ccc} A = \mathcal{O}_Y(Y) & \xrightarrow{\psi_X} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ A_g = \mathcal{O}_{Y,D(g)} & \overset{}{\underset{\psi_{D(g)}}{\dashrightarrow}} & \mathcal{O}_{X,\varphi^{-1}(D(g))} \end{array} \tag{13}$$

The map $\psi_{D(g)}$ is the unique map $A_g \to \mathcal{O}_X(\varphi^{-1}(D(g)))$ that makes the diagram commutative. Indeed, the image of the elements in the multiplicative set generated by g under the composition $A \to \mathcal{O}_X(X) \to \mathcal{O}_X(\varphi^{-1}(D(g)))$ is contained in the set of invertible elements of $\mathcal{O}_X(\varphi^{-1}(D(g)))$. But then there is one unique map $A_g \mapsto \mathcal{O}_X(\varphi^{-1}(D(g)))$ such that the diagram commutes by the universal property of the localized superring A_g. Thus the map Γ is injective.

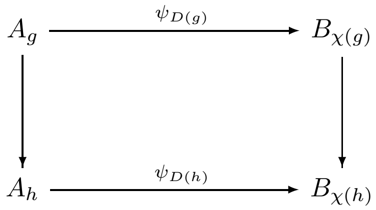
To prove surjectivity of Γ, we first assume $X = \mathrm{Spec}(B)$ for some superring B and that a morphism of superrings $\chi : A \to B$ is given. As alluded before, this induces a map $\chi_{\mathfrak{p}} : A_{\chi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p} \subset B$. By the above, we know that there is only one possible continuous map $\varphi : X \to Y$, which is easily seen to be $\varphi(\mathfrak{p}) = \chi^{-1}(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \subset B$. The map φ is then continuous since for any principal open subset $D(g) \subset Y$ we have

$$\begin{aligned} \varphi^{-1}(D(g)) &= \big\{\mathfrak{p} \subset B\,, \text{prime ideal} \mid g \notin \chi^{-1}(\mathfrak{p})\big\} \\ &= \{\mathfrak{p} \subset B\,, \text{prime ideal} \mid \chi(g) \notin \mathfrak{p}\} \\ &= D(\chi(g))\,. \end{aligned} \tag{14}$$

As the $D(g)$ are a basis, φ is continuous. Since $\varphi^{-1}(D(g)) = D(\chi(g))$ we immediately see that on the principal open subsets we can define $\psi_{D(g)}$ by the map

$$\psi_{D(g)} = \chi_g : A_g \to B_{\chi(g)}\,, \qquad \frac{a}{g^r} \mapsto \frac{\chi(a)}{\chi(g)^r}\,. \tag{15}$$

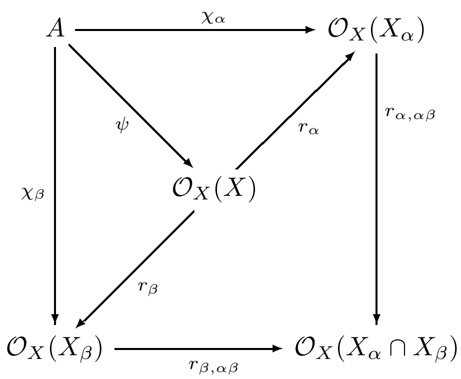
The map $\psi_{D(g)}$ is then compatible with restrictions: Let $D(h) \subset D(g)$, then $h^n = vg$ for some $v \in A_{\bar{0}}$ and $\chi(h)^n = \chi(v)\chi(g)$. Thus we have maps $A_g \to A_h$, sending a/g^r to av^r/h^{nr} and $B_{\chi(g)} \to B_{\chi(h)}$ sending $b/\chi(g)^r$ to $b\chi(v)^r/\chi(h)^{nr}$. Then the following diagram commutes:



(16)

For any prime ideal $\mathfrak{p}$ in B, we then consider the compositions $A_g \to B_{\chi(g)} \to B_{\mathfrak{p}}$, where g runs over all the even elements of A with $\chi(g) \notin \mathfrak{p}$. As the diagram 16 commutes, there is a unique map $A_{\chi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$, which is the induced map on the stalks $\psi_{\mathfrak{p}}$. We see that $\psi_{\mathfrak{p}}$ maps a/g to $\chi(a)/\chi(g)$ for any $g \notin \psi^{-1}(\mathfrak{p})$. Thus ψ coincides with the natural map $\chi_{\mathfrak{p}} : A_{\chi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ and we see that $\psi_{\mathfrak{p}}$ is local. The maps ψ_U for open $U \subset Y$ thus combine with φ to give a morphism $f = (\varphi, \psi)$ of superschemes and on the global sections we see that $\psi_X = \chi$.

Assuming surjectivity of Γ for affine superschemes, let $(X, \mathcal{O}_X)$ be any superscheme. We can cover X by affine superscheme $(X_\alpha, \mathcal{O}_{X_\alpha})$, where $\mathcal{O}_{X_\alpha}$ is the restriction of $\mathcal{O}_X$ to X_α. For any $\chi : A \to \mathcal{O}_X(X)$ define χ_α as the composition of χ with the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_{X_\alpha}(X_\alpha) = \mathcal{O}_X(X_\alpha)$. Then there are morphisms $f_\alpha = (\varphi_\alpha, \psi_\alpha)$ with $\psi_\alpha(X_\alpha) = \chi_\alpha$. The following diagram commutes due to the definition of χ_α:



(17)

where r_α, r_β, $r_{\alpha,\alpha\beta}$ and $r_{\beta,\alpha\beta}$ are the restriction morphisms. Hence, we have two morphisms of superschemes f_α, f_β from $(X_\alpha \cap X_\beta, \mathcal{O}_{X_\alpha \cap X_\beta})$ to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ that give the same morphism of superrings $A \to \mathcal{O}_X(X_\alpha \cap X_\beta)$. But the map Γ is injective and therefore on the intersections $X_\alpha \cap X_\beta$ the morphisms f_α agree. We can thus glue the f_α together to form a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. □

The following two corollaries are immediate:

Corollary 2.3. *The object* $\mathrm{Spec}(\mathbb{Z})$ *is terminal in the category of superschemes.*

Corollary 2.4. *The category of affine superschemes is contravariant equivalent to the category of superrings.*

An important example of an affine superscheme is the space $\mathbb{A}_k^{n|m}$, which is defined as $\mathrm{Spec}(A_{n|m})$ where $A_{n|m} = k[x_1, \ldots, x_n | \vartheta_1, \ldots, \vartheta_m]$.

3 Projective superschemes

We say a superring A is a $\mathbb{Z}$-graded superring if A is a direct sum $A = \bigoplus_{i\geq 0} A_0$ of abelian groups such that $(A_i)_{\bar{0}} = A_{\bar{0}} \cap A_i$, $(A_i)_{\bar{1}} = A_{\bar{1}} \cap A_i$ and the multiplication map satisfies $A_i A_j \subset A_{i+j}$. We say an element a of A is homogeneous if it lies in some $A_{i,\bar{0}}$ or $A_{i,\bar{1}}$. We call an element $\mathbb{Z}$-homogeneous if it lies in some A_i. We write $\deg(a)$ for the $\mathbb{Z}$-degree of a $\mathbb{Z}$-homogeneous element of A. An ideal $\mathfrak{a} \subset$ is called homogeneous if for any element $a \in \mathfrak{a}$ also all its homogeneous components $a_{i,\bar{0}} \in A_{i,\bar{0}}$ and $a_{i,\bar{1}} \in A_{i,\bar{1}}$ lie in $\mathfrak{a}$. Intersection, sum and product of homogeneous ideals are again homogeneous and an ideal is homogeneous if and only if it can be generated by homogeneous elements. A homogeneous ideal $\mathfrak{p}$ is prime if and only if for any homogeneous $a, a' \in A$ that are not in $\mathfrak{p}$ also $aa' \notin \mathfrak{p}$. We denote A_+ the homogeneous ideal given by $A_+ = \bigoplus_{i\geq 1} A_i$.

We want to associate a topological space to a $\mathbb{Z}$-graded superring. Following for example [2] we define $\mathrm{Proj}(A)$ to be the set of homogeneous prime ideals in A that do not contain A_+. We give $\mathrm{Proj}(A)$ the topology defined by the closed sets $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(A) \mid \mathfrak{p} \supset \mathfrak{a}\}$, where $\mathfrak{a}$ is any homogeneous ideal in A. As in the affine case, one checks easily that this indeed defines a topology. For any even homogeneous $f \in A_+$ we define the principal open subset $D_+(f)$ to be the subset of $\mathrm{Proj}(A)$ of the homogeneous prime ideals $\mathfrak{p}$ that do not contain f. As in the affine case, the set $D_+(f)$ is the complement of $V(f)$ and the set of all $D_+(f)$, where f runs over all homogeneous even elements of A_+, forms a basis of the topology. Indeed, if $V(\mathfrak{a})$ is a closed subset, we choose $a \in \mathfrak{a}_{\bar{0}} \cap A_+$, then $D_+(a)$ lies in the complement of $V(\mathfrak{a})$.

Let $\{f_i \in A_+ \cap A_{\bar{0}} \mid i \in I\}$ be a set of even $\mathbb{Z}$-homogeneous elements in A_+ such that the f_i generate A_+. Then we have $\mathrm{Proj}(A) = \bigcup_{i\in I} D_+(f_i)$, since if $\mathfrak{p}$ is a homogeneous prime ideal not in the union, it contains all the f_i and thus A_+, which is impossible. The converse need not hold. A counterexample is provided by the commutative $\mathbb{Z}$-graded ring $A = k[x,y]/(y^2)$, where $A_0 = k$ and the $\mathbb{Z}$-degrees of x and y are both 1. Then $\mathrm{Proj}(A)$ contains only the element $\mathfrak{p} = (y)$: Suppose a homogeneous prime ideal $\mathfrak{p}$ contains $f = x^m + \lambda x^{m-1} y$ for some $\lambda \in k$. Since $y \in \mathfrak{p}$ it follows that $x^m \in \mathfrak{p}$. But then we need that $x \in \mathfrak{p}$ and it follows that $A_+ = (x,y) \subset \mathfrak{p}$, which is impossible by the definition of $\mathrm{Proj}(A)$. But then $D_+(x)$ is an open subset of $\mathrm{Proj}(A)$ that covers $\mathrm{Proj}(A)$. However, x does not generate A_+. The following lemma does give a sufficient and necessary condition for a set of even $\mathbb{Z}$-homogeneous elements to give rise to a covering of $\mathrm{Proj}(A)$:

Lemma 3.1. *Let $\{f_i \mid i \in I\}$ be a set of even $\mathbb{Z}$-homogeneous elements in A_+. Then $\bigcup_{i\in I} D_+(f_i) = \mathrm{Proj}(A)$ if and only if the radical ideal of the ideal $(f_i : i \in I)$ contains A_+.*

Proof. Suppose that the radical ideal of $(f_i : i \in I)$ contains A_+. Then any homogeneous prime ideal $\mathfrak{p}$ not contained in the union $\bigcup_{i\in I} D_+(f_i)$ must contain $(f_i : i \in I)$ and therefore also the radical of $(f_i : i \in I)$. But then $\mathfrak{p} \supset A_+$, and thus $\mathfrak{p}$ does not correspond to a point in $\mathrm{Proj}(A)$. For the converse, suppose that the radical ideal of $(f_i : i \in I)$ does not contain A_+. Then there is a homogeneous a such that no power of a lies in $(f_i : i \in I)$. Consider the set Ω of homogeneous ideals that contain $(f_i : i \in I)$ but do not contain any power of a. Then Ω is not empty as $(f_i : i \in I) \in \Omega$ and Ω can be partially ordered by inclusion. If $\{I_\alpha\}$ is some totally ordered subset of Ω, then $\bigcup_\alpha I_\alpha$ is homogeneous, contains $(f_i : i \in I)$ and does not contain any power of a. By Zorn, there is a maximal element $\mathfrak{m} \in \Omega$. Suppose, that there are $x, y \notin \mathfrak{m}$ that are homogeneous and that $xy \in \mathfrak{m}$. Then the homogeneous ideals $(\mathfrak{m}, x)$ and $(\mathfrak{m}, y)$ both properly contain $\mathfrak{m}$ and thus contain some power of a. There are thus $r, s, r', s' \in A$ and $m, m' \in \mathfrak{m}$ such that $a^k = (rx + sm)$ and $a^l = (r'y + s'm')$ for some positive integers k, l. It follows then that $a^{k+r} = (rx + sm)(r'y + s'm') \in \mathfrak{m}$, which is a contradiction; hence no such x and

y exist. Therefore $\mathfrak{m}$ is a prime ideal. Since $a \notin \mathfrak{m}$, $\mathfrak{m}$ does not contain A_+. But then $\mathfrak{m}$ corresponds to a point in $\mathrm{Proj}(A)$ not contained in the union $\bigcup_{i\in I} D_+(f_i)$. Hence the $D_+(f_i)$ do not cover $\mathrm{Proj}(A)$. □

If S is a multiplicative set inside A that only contains even $\mathbb{Z}$-homogeneous elements, then the localization $S^{-1}A$ has a natural $\mathbb{Z}$-grading and is again a superring: For homogeneous a we define the $\mathbb{Z}$-grading of a/s to be the $\mathbb{Z}$-degree of a minus the $\mathbb{Z}$-degree of s and we call a/s even (odd) if a is even (odd). The subsuperring of all elements of $\mathbb{Z}$-degree zero we denote $(S^{-1}A)_0$. This is again a superring and all elements are of the form a/s for some $\mathbb{Z}$-homogeneous $a \in A$ with $\mathbb{Z}$-degree equals to the $\mathbb{Z}$-degree of s. In the case where $S \subset A_0$, one easily sees that $(S^{-1}A)_0$ is naturally isomorphic to the superring obtained by localizing A_0 with respect to S. If S is the multiplicative set generated by an even $\mathbb{Z}$-homogeneous element f, then we write $A_{(f)}$ for $(S^{-1}A)_0$.

Lemma 3.2. *Let S be a multiplicative set of A that only contains even $\mathbb{Z}$-homogeneous elements. Let T' be a multiplicative set inside $(S^{-1}A)_0$ that only contains even elements. Consider the set T of even elements $t \in A$ such that $t/s \in T'$ for some $s \in S$. Then T is a multiplicative set inside A that only contains even $\mathbb{Z}$-homogeneous elements and*

$$(T')^{-1}(S^{-1}A)_0 \cong ((TS)^{-1}A)_0 \,. \tag{18}$$

Proof. We denote the elements of $(T')^{-1}(S^{-1}A)_0$ by $(a/s, t/z)$, where $a/s \in (S^{-1}A)_0$ and $t/z \in T'$. All elements of $(T')^{-1}(S^{-1}A)_0$ are of the form $(a/s, t/z)$, with $\deg(a) = \deg(s)$, $\deg(z) = \deg(z)$ and $t \in T$.

We define a map $\varphi : (T')^{-1}(S^{-1}A)_0 \to ((TS)^{-1}A)_0$ by

$$\varphi : (a/s, t/z) \mapsto \frac{az}{st} \,. \tag{19}$$

It is easily checked that φ is well-defined and preserves the $\mathbb{Z}_2$-grading. Furthermore, we see that $\varphi(x) + \varphi(y) = \varphi(x + y)$ and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in (T')^{-1}(S^{-1}A)_0$. Therefore φ is a morphism of superrings. Suppose $\varphi((a/s, t/z)) = 0$, then there are $s' \in S$ and $t' \in T$ so that $s't'az = 0$ in A. But $(a/s, t/z)$ is zero in $(T')^{-1}(S^{-1}A)_0$ if and only if there are s'' and t'' in S and T respectively, such that $s''t''a = 0$. It follows that $\mathrm{Ker}(\varphi) = 0$. To prove surjectivity, let $\frac{a}{st} \in ((ST)^{-1}(A))_0$ be given with $s \in S$ and $t \in T$. It follows that $\deg(a) = \deg(s) + \deg(t)$. As $t \in T$, there is an s' with $\deg(s') = \deg(t)$ such that $s'/t \in T'$. The element $(a/ss', t/s')$ of $(T')^{-1}(S^{-1}A)_0$ is thus well-defined and

$$\varphi : (a/ss', t/s') \mapsto \frac{as'}{tss'} \sim \frac{a}{st} \,. \tag{20}$$

□

Lemma 3.3. *Let f be an even $\mathbb{Z}$-homogeneous element of nonzero $\mathbb{Z}$-degree. Then the map $\varphi : D_+(f) \to \mathrm{Spec}(A_{(f)})$ given by $\varphi(\mathfrak{p}) = (\mathfrak{p}A_f)_0$ is a homeomorphism.*

Proof. We define an inverse map as follows (also see [3, 4]): suppose $\mathfrak{q}$ is a prime ideal in $A_{(f)}$, define $\psi(\mathfrak{q})$ as the set of all elements that are sums of $\mathbb{Z}$-homogeneous elements y for which there exist integers m, n with $y^m/f^n \in \mathfrak{q}$. Then $\psi(\mathfrak{q})$ is by construction homogeneous, does not contain f and contains all odd elements. We thus need to show that $\psi(\mathfrak{q})$ is an ideal and is prime. Suppose $x, y \in \psi(\mathfrak{q})$ and x, y are homogeneous. If x, y do not have the same $\mathbb{Z}$-degree, then by definition $x + y$ lies in $\psi(\mathfrak{q})$. If x and y have the same $\mathbb{Z}$-degree and the integers a, b, c, d are such that x^a/f^b and y^c/f^d lie in $\mathfrak{q}$, then it follows that $ad = bc$. Thus $x^{ac}/f^{bc} \in \mathfrak{q}$ and $y^{ac}/f^{bc} \in \mathfrak{q}$, from which we deduce $(x+y)^{2ac}/f^{2bc} \in \mathfrak{q}$ so that $x+y \in \psi(\mathfrak{q})$. If $a \in A$ is homogeneous and x is a homogeneous element in $\psi(\mathfrak{q})$ with $x^m/f^n \in \mathfrak{q}$, then $(rx)^{m\deg(f)}/f^{n\deg(r)} \in \mathfrak{q}$, from which we conclude that $rx \in \psi(\mathfrak{q})$. Now suppose $x, y \in A$ are homogeneous such that $xy \in \psi(\mathfrak{q})$, so that there are positive integers m, n with $z = x^n y^n / f^m \in \mathfrak{q}$. Then $n\deg(x) + n\deg(y) = m\deg(f)$ and raising z to the power $\deg(f)$ gives

$$\frac{x^{n\deg(f)}}{f^{n\deg(x)}} \frac{y^{n\deg(f)}}{f^{n\deg(y)}} \in \mathfrak{q}\,, \tag{21}$$

from which we conclude that $\psi(\mathfrak{q})$ is a prime ideal. It is furthermore straightforward to check that $\psi(\varphi(\mathfrak{p})) = \mathfrak{p}$ for any homogeneous prime ideal $\mathfrak{p}$ in $D_+(f)$ and that $\varphi(\psi(\mathfrak{q})) = \mathfrak{q}$ for any prime ideal $\mathfrak{q}$ in $A_{(f)}$.

One verifies easily that $\mathfrak{p} \in D_+(f)$ contains the homogeneous even element $g \in A$ if and only if $\varphi(\mathfrak{p})$ contains $h = g^{\deg(f)}/f^{\deg(g)} \in A_{(f)}$. Thus $\varphi(V(g) \cap D_+(f)) \subset V(h)$ and $\varphi^{-1}(V(h)) \subset V(g) \cap D_+(f)$. But then we must have equalities and thus φ is continuous and sends open sets to open sets. Since we already proved φ is a bijection, φ is a homeomorphism. □

Corollary 3.4. *Let f be an even $\mathbb{Z}$-homogeneous element of nonzero $\mathbb{Z}$-degree. There is a one-to-one correspondence between the homogeneous prime ideals of A_f and the prime ideals of $A_{(f)}$.*

Proof. There is a one-to-one correspondence between the prime ideals in A not containing f and the prime ideals in A_f. It is easily seen that this correspondence preserves the $\mathbb{Z}$-grading. (For a direct proof see [3].) □

For any prime ideal $\mathfrak{p}$ not containing A_+, consider the multiplicative set S of all even $\mathbb{Z}$-homogeneous elements of A that are not in $\mathfrak{p}$. Then the localization $S^{-1}A$ is a $\mathbb{Z}$-graded superring and contains a subsuperring of elements of $\mathbb{Z}$-degree zero. We define $A_{(\mathfrak{p})}$ to be this subsuperring of elements of $\mathbb{Z}$-degree zero. It is easily seen that this again defines a superring. We then obtain a sheaf $\mathcal{O}$ on $\mathrm{Proj}(A)$ as follows: for any open set $U \subset \mathrm{Proj}(A)$ we define $\mathcal{O}(U)$ to be the superring of all functions $s : U \to A_{(\mathfrak{p})}$, such that the image $s_\mathfrak{p}$ of $\mathfrak{p} \in U$ under s lies in $A_{(\mathfrak{p})}$ and such that for all $\mathfrak{p} \in U$ there is an open neighborhood $V \subset U$ containing $\mathfrak{p}$ for which there are $\mathbb{Z}$-homogeneous $a \in A$ and $f \in A_{\bar{0}}$ of the same $\mathbb{Z}$-degree and $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in V$, such that $s_\mathfrak{q}$ coincides with the image of a/f in $A_{(\mathfrak{q})}$ for all $\mathfrak{q} \in V$. We call $\mathcal{O}$ the structure sheaf of $\mathrm{Proj}(A)$. We use the notation $(\mathcal{O}, \mathrm{Proj}(A))$ for the topological space $\mathrm{Proj}(A)$ equipped with the sheaf of superrings $\mathcal{O}$ that we just defined.

Lemma 3.5. *Let $\mathfrak{p} \in \mathrm{Proj}(A)$, then the stalk of the sheaf $\mathcal{O}$ at $\mathfrak{p}$ is $A_{(\mathfrak{p})}$.*

Proof. The stalk can be constructed as follows: Consider the set of all pairs (s, U), where U is an open neighborhood of $\mathfrak{p}$ and $s \in \mathcal{O}(U)$. We define the pairs (s, U) and (t, V) to be equivalent if there is an open subset W contained in U and V such that the restrictions of s and t to W are equivalent as elements in $\mathcal{O}(W)$. The equivalence classes of the pairs (s, U) can then be equipped with the structure of a superring and the superring obtained in this way is the stalk $\mathcal{O}_\mathfrak{p}$.

By the very definition of the sheaf $\mathcal{O}$ on $\mathrm{Spec}(A)$ we can assume that for any element $(s, U) \in \mathcal{O}_\mathfrak{p}$ the open set U is so large that s is given by a/g for some $g \notin \mathfrak{p}$ and $a \in A$ with equal degree. Then there is an obvious map $\mathcal{O}_\mathfrak{p} \to A_{(\mathfrak{p})}$ that sends $(a/g, U)$ to the element a/g in $A_{(\mathfrak{p})}$. This implies that the map is well-defined. Clearly, it is surjective.

To check injectivity, we assume (s, U) is mapped to zero. We may then assume $s = a/g$ for some $g \notin \mathfrak{p}$. By assumption there is an even element $h \in \mathfrak{p}$ such that $ha = 0$. But then the section s vanishes on $U \cap D_+(h)$ and hence (s, U) is zero in $\mathcal{O}_\mathfrak{p}$. □

Proposition 3.6. *Let f be an even homogeneous element of A_+. Then the restriction of the structure sheaf $\mathcal{O}$ of A to $D_+(f)$ makes $D(f)$ into an affine superscheme, which is isomorphic to* $\mathrm{Spec}(A_{(f)})$.

Proof. By lemma 3.3 the spaces $D_+(f)$ and $\mathrm{Spec}(A_{(f)})$ are homeomorphic with the homeomorphism $\varphi : D_+(f) \to \mathrm{Spec}(A_{(f)})$ defined by $\varphi(\mathfrak{p}) = (\mathfrak{p}A_f)_0$. From the proof of the same lemma we know that $\varphi(D_+(f) \cap D_+(g)) = D(g^{\deg(f)}/f^{\deg(g)})$.

Call $(X, \mathcal{O}_X)$ the superringed space $D_+(f)$ with the sheaf $\mathcal{O}_{\mathrm{Proj}(A)} |_{D_+(f)}$ and $(Y, \mathcal{O}_Y)$ the superscheme of $\mathrm{Spec}(A_{(f)})$. Call S the multiplicative set of even $\mathbb{Z}$-homogeneous in A_f not contained in $\mathfrak{p}A_f$ for some homogeneous prime ideal $\mathfrak{p}$ in $D_+(f)$. By lemma 3.2 we have $\mathcal{O}_{Y,\varphi(\mathfrak{p})} = (S^{-1}A_f)_0$, which is again isomorphic to $A_{(\mathfrak{p})}$ since $f \in S$. Therefore $\mathcal{O}_{X,\mathfrak{p}} \cong \mathcal{O}_{Y,\varphi(\mathfrak{p})}$ as superrings. But since the stalks are local rings, the isomorphism must be a local morphism. Explicitly, the morphism is given by $\chi_\mathfrak{p} : \mathcal{O}_{Y,\varphi(\mathfrak{p})} \to \mathcal{O}_{X,\mathfrak{p}}$, $\chi_\mathfrak{p} : (a/f^r, g/f^t) \mapsto af^t/gf^r$.

A principal open set $D(g/f^t) \subset Y$ is the spectrum of $A_{(f,g)}$ and thus we have an induced map $\chi_{D(g/f^t)} : \mathcal{O}_Y(D(g/f^t)) \to \mathcal{O}_X(D_+(g) \cap D_+(f))$ given by $(a/f^r, g/f^t) \mapsto af^t/gf^r$. But then there is only one way to extend this to a map of sheaves. Suppose U is an open set of Y and s is a section over U. Then we define $\chi_U(s)$ to be the map that sends $\mathfrak{p} \in \varphi^{-1}(U)$ to $\chi_\mathfrak{p}(s_{\varphi(\mathfrak{p})})$. Then $\chi_U(s)$ is indeed a section of $\mathcal{O}_X(U)$ since if on $V \subset U$ the section s is given by $s_\mathfrak{q} = (a/f^r, g/f^t) \in \mathcal{O}_{Y,\mathfrak{q}}$ then we have $\chi_U(s) : \mathfrak{p} \mapsto af^t/gf^r \in \mathcal{O}_{X,\mathfrak{p}}$ for all $\mathfrak{p} \in \varphi^{-1}(U)$. It is then easily seen that the maps χ_U are compatible with restrictions and that the induced map on the stalks is precisely the local morphism $\chi_\mathfrak{p}$. As $\chi_\mathfrak{p}$ is an isomorphism, we have an isomorphism of sheaves $\mathcal{O}_X \cong \mathcal{O}_Y$ and the proposition is proved. □

Corollary 3.7. *For any $\mathbb{Z}$-graded superring,* $\mathrm{Proj}(A)$*, together with its structure sheaf $\mathcal{O}$ of superrings, is a superscheme.*

We define the projective superspace $\mathbb{P}_k^{n|m}$ to be $\mathrm{Proj}(A_{n+1|m})$, where $A_{n+1|m} = k[x_0, \ldots, x_n \mid \vartheta_1, \ldots, \vartheta_m]$. The elements x_i and ϑ_α we give $\mathbb{Z}$-degree 1. The elements $x_0, \ldots, x_n$ define an ideal whose radical is A_+, and thus the subsets $D_+(x_i)$ provide a cover. One easily sees that $D_+(x_i) \cong \mathrm{Spec}(A_{n|m}) = \mathbb{A}_k^{n|m}$.

References

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