SUPERSYMMETRY AND SUPERGEOMETRY

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Abstract. Dissertantenkolloquium at the university of Vienna on 29 November 2007. Some references are given, without attemtping to be complete or correct.

1. On the title and the motivation

The title of this talk consists of a mathematical part and of a physical part. Supergeometry is mathematical and originated from the works of Berezin, Leites, Kostant (see e.g. in [1, 2, 3, 4]) and many others from 1960-1970. Supersymmetry is a tool in theoretical physics developped by Salam, Strathdee, Wess, Zumino (see e.g. [5, 6, 7, 8, 9]) and many many others. Since their origin both worlds have contributed to each other and one can say in a certain sense that one without the other does not make sense.

I am from the physics world; I am just a simple theoretical physicist by education. The reason I am studying supergeometry now is that I am interested in supersymmetry from a mathematical point of view. Supersymmetry is an interesting symmetry that has many facets and surprises, by my humble opinion. Also from a pure mathematical point of view supersymmetry is interesting; choosing the right supersymmetrical model might give a grip on some fundamental mathematical problem. Among others, supersymmetry helped in giving insight into some index theorems and helped in proving other theorems [10] [11, 12]. In that area I am not an expert, but you might try to look at the work of Witten to see what I mean.

Nonetheless, since the tight interplay between physics and mathematics in the playground area of supersymmetry, I will start the talk with saying some words on supersymmetry. After the supersymmetry part, I will give a definition of a superalgebra, give examples and give an example of a supergroup. Then, if time permits, I will say some words on supervarieties. In this talk I will avoid the rich area of super differential geometry - see for example the book by Varadarajan[13], the notes of Nelson [14], Deligne and Morgan[15], the book of Tuynman[16],. . .

2. Supersymmetry

I have to start with an apology since time is to short to give all details. Hence I cannot give a more or rigorous and coherent construction of where supersymmetry comes from and what it means. Simply because time does not permit to treat all the physics necessary to understand all details, and I am afraid I must admit I don't know all details myself. Hence, the part on supersymmetry is done with a lot of handwaving and pictorial talking. I hope to convey a picture, not a framework.

It is a long-standing history in physics to try and combine different theories in one single theory, that is, to describe seemingly different phenomena by one theory. Let me shortly mention some of the famous and at the same time simple examples.

In the 18th century magnetism and electricity were known, more or less, but seen as different things [1]. After the work of among others Faraday and Maxwell, it was recognized that electricity and magnetism were best seen as two different sides of one object electromagnetism. Through this combined framework it was Maxwell who predicted the existence of light on theoretical grounds.

Another example is special relativity. To all of us it is known that the universe around as has in principle some fundamental symmetries. Translation in time, rotations around an axis or translating in space are supposed not to give different outcomes of the same experiment. But, in all these symmetries we treat space and time differently. It was Einstein who tried to combine both and investigated what happened when space and time coordinates were rotated in to each other with some symmetry. By doing this he brought a whole world of new physics to us. GPS and space-shuttles would not work without the framework of special relativity.

The two examples above show us to things: first, combining different theories into a single theory can be useful and marks progress in physics and second, a way to combine different theories is by using symmetries. At first glance it might not be clear, but the unification of electricity and magnetism can be seen as a same manifestation of the symmetry underlying special relativity, which is called Lorentz symmetry. In fact more is true, the Lorentz covariance of Maxwell's theory of electromagnetism pushed Einstein to invent special relativity.

So we have reached an intermediate conclusion; a guideline for (the useful hobby of) combining theories can be symmetry. That the two combined theories then appeared before as different has to be explained by some symmetry breaking process. The approached outlined I will now sketchily present for supersymmetry.

Nature has fermions and bosons. In the sixties of the 20th century the big particle accelerators found many elementary particles; it turned out that there are way more particles than the electrons, protons and neutrons. The particles fell into two classes, fermions and bosons. It is difficult to explain in simple language what makes a particle a fermion or a boson. But, every elementary particle is either a fermion or a boson, and the difference lies in the collective behavior; bosons are social particles and fermions are anti-social particles. Two fermions cannot be in the same physical state whereas for bosons this is no problem. As examples, electrons are fermions, and as you might remember from high school, electrons fill out shells in atoms. Electrons in particular do not fit into one shell and this is not only due to the fact that electrons repell each other. On the other hand, light particles, photons, are bosons; you can put as much light together as you want and this is precisely used in constructing lasers.

[1] The treatment in high schools (in the Netherlands at least) does mainly highlight the different behavior and not the similarity.

In order to describe the aforementioned difference, we look at it from a quantum mechanics point of view. In quantum mechanics, one works with vectors in a Hilbert space (just think of finite-dimensional vector spaces). A vector describes a state of a physical state. There is one special vector, the vacuum, the state with no particles present. To create particles one uses creation operators (think of matrices). Let us be simple-minded and think of just two kinds of particles, one of which is a fermion and the other kind is a boson. I denote with $Q_F(x)$ the operator that creates a fermion at place x. Similarly, $Q_B(x)$ creates a boson at place x. It turns out that these operators have to satisfy some relation like

$$\begin{aligned} Q_B(x)Q_B(y) - Q_B(y)Q_B(x) &= 0 \\ Q_B(x)Q_F(y) - Q_F(y)Q_B(x) &= 0 \\ Q_F(x)Q_F(y) + Q_F(y)Q_F(x) &= 0\,. \end{aligned} \tag{1}$$

Note that of $x = y$ we see that creating two fermions at the same time gives zero; a forbidden proces. We see a clear problem if we want to have symmetry that rotates the fermionic creation operators into bosonic creation operators. How to go from commuting operators to anticommuting operators? What the problem in this example seems to be is that complex numbers commute, so do we need anticommuting numbers??? This is where supersymmetry starts with: numbers that do not commute.

Remark 2.1. Of course, superalgebras are already used without supersymmetry; fermions have to be quantized using Grassmann algebras.

To finish the physics part of the talk, let me mention some areas where supersymmetry is used. The prime example is string theory, the new candidate for a grand unification theory, is inconsistent without supersymmetry. You might sometimes heard about string theory predicting 26 dimensions, or ten, or 11. String theory without fermions and supersymmetry gives 26 dimensions, with supersymmetry gives 10 dimensions and it is speculated that super string theory is a limit of another theory, M-theory, which lives in 11 theory. The standard model of elementary particles has been extended with supersymmetry; in this way it turns out to be more consistent ... Also in condensed matter physics, supersymmetry is found; in trying to describe vortices in superfluids some models with supersymmetry are proposed.

An interesing idea from physics is to enrich our four-dimensional world with some directions in which the coordinated do not commute. The approach using this idea is marked with the name 'superspace'-techniques,or 'superspace'-approach. Supersymmetry then becomes just a rotational symmetry in this enriched superspace. In order to make sense of superspace, we need some superalgebras ...

3. Superalgebras

Definition 3.1. *A superalgebra is: We have a vector space A over some field k (take $\mathbb{C}$) with a $\mathbb{Z}_2$-grading $A = A_0 \oplus A_1$. Elements of A_0 are called even, elements of A_1 are called odd, and a homogeneous element is an element that is either even or odd. We have a map $|\cdot|$ defined on homogeneous elements and given by $|a| = 1$ if a is odd and $|a| = 0$ if a is even. To make the*

$\mathbb{Z}_2$-graded vector space into a superalgebra we need to give a multiplication. We require this multiplication to be associative and to satisfy the following rule for homogeneous elements $|ab| = |a| + |b|$. The mentioned ingredients define a superalgebra.

Remark 3.2. Note that the direct sum means that all elements are uniquely decomposed in an even and an odd part. Therefore it suffices to determine and give properties on homogeneous elements and then extend by linearity to the rest.

Furthermore I require some additional properties to make life easy for me (us). I want the superalgebras to contain a 1. I want the superalgebras to be finitely-generated (if you don't understand this, skip the requirement). Last but not least, I want the superalgebras to be commutative, and in the superalgebra setting that means that

$$ab = (-1)^{|a||b|} ba \,, \tag{2}$$

on homogeneous elements and extended by linearity on inhomogeneous elements.

Definition 3.3. *An ideal in a superalgebra is a $\mathbb{Z}_2$-graded ideal $I = (I \cap A_0) \oplus (I \cap A_1)$ such that I is a superalgebra itself and $IA \subset I$.*

Every superalgebra A has a canonical ideal J_A given by the ideal generated by all the odd elements. That is, the ideal J_A consists of all the elements $x \in A$ such that there exists a finite number of odd elements ξ_i and an equal number of $x_i \in A$ such that

$$x = \sum x_i \xi_i \,. \tag{3}$$

It is not too hard to check that J_A is indeed an ideal. We have a corresponding canonical projection and quotient:

$$\pi : A \to A/J_A \equiv \bar{A} \,. \tag{4}$$

The algebra $\bar{A}$ is called the body of A and is an ordinary commutative algebra.

Example 3.4. Grassmann algebras. Consider the algebra over the complex numbers generated by generators ξ_i $1 \le i \le n$ with the relations

$$\xi_i \xi_j + \xi_j \xi_i = 0 \,. \tag{5}$$

This is called a Grassmann algebra and denoted by $\Lambda_{\mathbb{C}}[\xi_1, \dots, \xi_n]$. We cannot get arbitrarily high degree, at most n since multiplying $\xi_1 \cdots \xi_n$ by any element not in $\mathbb{C}1$ we get zero. Hence we have a 2^n-dimensional algebra over $\mathbb{C}$. The body is contains all polynomials of degree bigger than 1. Therefore the body is $\mathbb{C}$.

Example 3.5. Consider now the $\mathbb{C}$-algebra

$$A = \mathbb{C}[X] \otimes \Lambda_{\mathbb{C}}[\xi_1, \xi_2] \,, \tag{6}$$

which is an infinite-dimensional algebra, but the expansion in the ξ_i stops rather quickly. Let us consider the ideal I generated by $x^2 - \xi_1 \xi_2$ and define

$$B = A/I \,. \tag{7}$$

The superalgebra B is rather simple. It is in particular finite dimensional with basis

$$1\,, x\,, \xi_1\,, \xi_2\,, x\xi_1\,, x\xi_2\,, x\xi_1\xi_2\,. \tag{8}$$

Note that $x^3 = x\xi_1\xi_2$, and $x^4 = 0$.

4. Example: Supergroup $GL_{n|m}$

I would like to use matrices with entries in some superalgebra $\mathcal{A}$ (with the properties listed above). Actually, one has to develop a theory for that first since one has to know what a module is. And it turns out that one has to be rather precise in the definitions, since multiplying from the left and the right is not the same. Unfortunately, I don't have time for that. For those interested, I will put something on my website, or just ask me, or read in the lecture notes of Leites[3], or the book of Berezin[1]. Since when you work with matrices, you can hide the differences, I will just proceed as would I have already told you the nasty details; it will look like natural, I hope ...

Consider the set of 'matrices' of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\,, \tag{9}$$

where A is an $n \times n$ matrix with even elements in some superalgebra $\mathcal{A}$, B is an $n \times m$ matrix with odd elements, C is an $m \times n$ matrix with odd elements and D is an $m \times m$ matrix with even elements. Matrices of this kind I will call even $n|m \times n|m$ matrices. One can add such matrices as one is used to and also ordinary matrix multiplication works in the usual sense and one checks easily that the product of two even matrices is again even.

Definition 4.1. *The group $GL_{n|m}(\mathcal{A})$ is the group of invertible $n|m \times n|m$ matrices.*

Let us show a criterion to see when an even $n|m \times n|m$ matrix is invertible.

Lemma 4.2. *An $n|m \times n|m$ matrix X of the form 9 is invertible if and only if $\bar{X}$ is, where $\bar{X}$ is the matrix with the entries $(\bar{X})_{ij} = (\bar{X}_{ij})$, that is, it is the body of the matrix X.*

Proof. If X is invertible, we have that there is a matrix Y (same size, also even ...) with $XY = YX = 1$, where the last symbol 1 now means unit matrix. Than $\overline{XY} = \bar{X}\bar{Y} = 1$. On the other hand, if $\bar{X}\bar{Y} = 1$, there is a matrix Y with $XY = 1 - N$ where N only contains elements in $J_{\mathcal{A}}$. But all elements in $J_{\mathcal{A}}$ square to zero, thus $N^{(n+m)^2}$ has entries such that at least one of the entries of N appears twice; thus N is nilpotent. Thus $(1-N)^{-1} = \sum_{k=0}^{M} N^k$ for some finite M and $XY(1-N)^{-1} = 1$. □

Corollary 4.3. *It follows that a matrix of the form 9 is invertible if and only if $\bar{A}$ and $\bar{D}$ are, thus if and only if A and D are invertible.*

Remark 4.4. Define the map

$$Ber : GL_{n|m}(\mathcal{A}) \to (\mathcal{A}_0)^*, , Ber \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\det(A - BD^{-1}C)}{\det D}\,, \tag{10}$$

where $(\mathcal{A}_0)^*$ means the invertible elements of $\mathcal{A}_0$. The map Ber satisfies $Ber(XY) = Ber(X)Ber(Y)$[1]. The map Ber is called the Berezinian. The Berezinian is a representation of the group $GL_{n|m}(\mathcal{A})$, but it is not tensorial!

5. Varieties

This is the last part of the talk, here I need to see how much time I have. Then I will decide what I will treat.

In ordinary algebraic geometry one defines an algebraic variety more or less as the zero locus of a set of polynomial equations. For superalgebras the approach of classical algebraic geometry clearly is less sensible. But, to every algebraic variety V there is a ring of functions $k[V]$ and the category of algebraic varieties is contra-equivalent to the category of affine algebras (affine = reduced and finitely generated). Using algebras is something we can do with superalgebras.

The points on an algebraic variety with "coordinates" in a certain field k correspond to morphisms from $k[V]$ to k. Varying k not only over fields but also over algebras we get the functor of points $\mathrm{Hom}(k[V], \cdot)$. Knowing the functor is enough to recover all information of V. This approach we take for supergeometry.

Definition 5.1. *An affine superalgebra is one that is finitely generated and the body contains no nilpotents.*

Definition 5.2. *An affine algebraic supervariety is a representable functor from the category of commutative superalgebras with 1 to sets such that the representing superalgebra is affine.*

Definition 5.3. *An algebraic supervariety is a representable functor from the category of commutative superalgebras with 1 to sets.*

We obtain a category in a certain way. The morphisms are the natural maps between the representing superalgebras. A nontrivial example of an affine algebraic supervariety:

Example 5.4. The functor $\mathcal{A} \to GL_{n|m}(\mathcal{A})$ is an affine algebraic supervariety and the representing superalgebra is a Hopf superalgebra:

$$k[GL_{n|m}] = \frac{k[A, D, \lambda, \mu | B, C]}{(\lambda \det A - 1, \mu \det D - 1)} \,. \tag{11}$$

see e.g. [18, 19].

We get the so-called underlying varieties by considering

$$V^u = \mathrm{Hom}(k[V], \mathbb{C}) \,, \tag{12}$$

for some affine algebraic supervariety V with representing superalgebra $k[V]$.

Example 5.5. Let us consider an algebraic variety with underlying variety the sphere. We take the polynomial ring in even variables X_1, X_2 and X_3, and odd variables ξ_1, ξ_2 and ξ_3 modulo some ideal;

$$\begin{aligned} k[V] &= k[X_1, X_2, X_3, \xi_1, \xi_2, \xi_3] \,, \\ I &= (X_1^2 + X_2^2 + X_3^2 - 1, X_1\xi_1 + X_2\xi_2 + X_3\xi_3) \,. \end{aligned} \tag{13}$$

To each open set U on the underlying sphere one can associate a certain ring of functions:

$$A(U) = \left\{ \frac{p}{q} |Big| p \in k[V]\,, q \in k[V]_0, \bar{q}(u) \neq 0 \forall u \in U \right\} . \tag{14}$$

Thus we get a ringed space.

More generally, for a superalgebra A we can study SpecA_0 and then consider the sheaf of superalgebras determined by A as in the example.

Birational supergeometry. Let now A be a superalgebra, such that J_A is prime. Then $\bar{A}$ is an integral domain. Then we can form the following superalgebra of fractions:

$$\mathrm{Frac}(A) \equiv \left\{ \frac{p}{q} \middle| p \in A, q \in A_0, \bar{q} \neq 0 \right\} / \sim \tag{15}$$

where the equivalence relation is the usual one

$$\frac{p}{q} \sim \frac{a}{b} \leftrightarrow \exists s \in A_0, \bar{s} \neq 0\,,\; s(pb - aq) = 0\,. \tag{16}$$

We call an affine algebraic supervariety irreducible if $\bar{A}$ is an integral domain. The underlying variety is then irreducible. We can define a subcategory of irreducible affine algebraic supervarieties with morphisms the dominant maps, that is, the maps between the superalgebras is injective.

End of talk

References

[1] F. A. Berezin, A. A. Kirillov and D. Leites, Introduction to Superanalysis, *Dordrecht, Netherlands: Reidel (1987) 424 P. (Mathematical Physics and Applied Mathematics, 9).*

[2] F. A. Berezin, Dokl. Akad, Nauk SSSR 137 (1961), 311.

[3] D.A. Leites, Introduction to the theory of supermanifolds, Russian Math. Surveys, v. 35, n.1, 1980, 3-53.

[4] B. Kostant, Graded Manifolds, Graded Lie Theory, And Prequantization, *In *Bonn 1975, Proceedings, Differential Geometrical Methods In Mathematical Physics*, Berlin 1977, 177-306*

[5] J. Wess and B. Zumino, Supergauge Transformations in Four-Dimensions, Nucl. Phys. B **70** (1974) 39.

[6] J. Wess and B. Zumino, Superspace Formulation Of Supergravity, Phys. Lett. B **66** (1977) 361.

[7] J. L. Martin, Proc. Roy. Soc. Lond. A251 (1959) 536; 543.

[8] A. Salam and J. A. Strathdee, Supersymmetry And Superfields, Fortsch. Phys. **26** (1978) 57.

[9] R. Delbourgo, A. Salam and J. A. Strathdee, Supersymmetric V-A Gauges And Fermion Number, Phys. Lett. B **51** (1974) 475.

[10] B. Zumino, Supersymmetry And The Index Theorem,

[11] L. Alvarez-Gaume, Supersymmetry And The Atiyah-Singer Index Theorem, Commun. Math. Phys. **90** (1983) 161.

[12] L. Alvarez-Gaume, Supersymmetry And Index Theory, *In *Bonn 1984, Proceedings, Supersymmetry*, 1-44*

[13] V. S. Varadarajan, Supersymmetry for mathematicians: An introduction, *New York, USA: Courant Inst. Math. Sci. (2004) 300 p.*

[14] P. C. Nelson, Lectures on supermanifolds and strings,

[15] P. Deligne *et al.*, Notes on Supersymmetry. In: Quantum fields and strings: A course for mathematicians. Vol. 1& 2, *Providence, USA: AMS (1999) 1-1501*

[16] G. M. Tuynman, Supermanifolds and Supergroups, Basic Theory, Mathematics and Its Applications, Kluwer Academic Publishers.

[17] Yu. I. Manin, Gauge field theory and complex geometry. Berlin, Germany: Springer (1988) 295 P. (Grundlehren der mathematischen Wissenschaften, 289)

[18] R. Fioresi and M. A. Lledo, On algebraic supergroups, coadjoint orbits and their deformations, Commun. Math. Phys. **245** (2004) 177, `math.RA 0301362`.

[19] R. Fioresi, On algebraic supergroups and quantum deformations, Journal of Algebra and Its Applications, vol.2 (2003), no.4, 403-423; `math.QA/0111113`.