Elementary triangle geometry

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Abstract

In this short note we discuss some fundamental properties of triangles up to the construction of the Euler line.

Contents

Lemma 1. *See figure 1. If P is a point outside a circle C with center Q, h and h' the two lines emanating from P and tangent to C and let S and S' be the points where h, respectively h', intersect C. The triangle $\triangle PQS$ and $\triangle PQS'$ are isomorphic (i.e., there is a distance-preserving transformation such that the points P, Q and S are mapped to P, Q and S' respectively).*

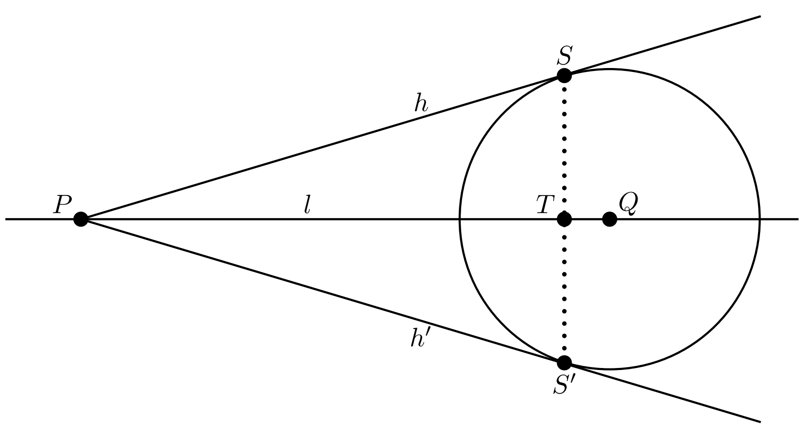


Figure 1: Two tangents to a circle emanating from a point outside the circle.

Proof. The proof is more or less obvious from figure 1; if we reflect the whole figure in the axis through P and Q, we end up with the same figure. Hence by symmetry considerations we conclude that the segment SS' intersects the line through P and Q at half of SS' and at an angle of ninety degrees. Triangles $\triangle PQS$ and $\triangle PQS'$ are mapped to each other under the reflection in the axis through P and Q and are hence isomorphic. □

Angle bisectors

Definition 2. *Let a triangle $\triangle ABC$ with vertices A, B and C be given. We call the line through A that divides the angel $\angle BAC$ into two equal parts the angle bisector through A.*

Clearly, a triangle has three angle bisectors. The next proposition shows, that the three angle bisectors meet in one point:

Proposition 3. *Let a triangle $\triangle ABC$ with vertices A, B and C be given. The angle bisectors intersect at a single point inside the triangle.*

Proof. It is clear that the intersection of two angle bisectors lies inside the triangle and hence we only need to prove that the angle bisectors intersect in a single point. We will show that the point of intersection is the inscribed circle of $\triangle ABC$. The inscribed circle is the maximal circle that fits inside the triangle.

Consider two lines l_1 and l_2 in the plane intersecting at a point P and consider a circle, of which the two lines are tangents. By symmetry arguments, as in the proof of lemma 1, the line through P and the center of the circle divide the angle α in two equal parts.

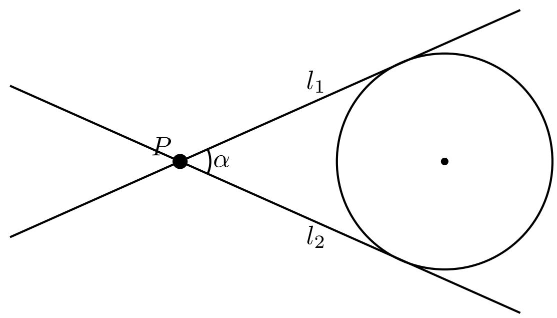


Figure 2: A circle tangent to two lines.

Thus, in the family of circles that have the sides AB and AC as tangents, all centers lie on the angle bisector through A. However, in the family of circles that have the sides AB and AC as tangents, there can be only one that has exactly one intersection point with the line through B and C. This circle has all the sides of the triangle as tangents to the circle and hence the line through the center of this circle and B bisects the angle $\angle ABC$ and the line through the center of this circle and C bisects the angle $\angle ACB$. This proves the proposition. □

Bisectors

Definition 4. *Let $\triangle ABC$ be a triangle with vertices A, B and C. A bisector is a line perpendicular to a side and passing to its midpoint. The bisector through AB is called the bisector of AB.*

It follows that a triangle has three bisectors. With the aid of lemma 1 we can deduce that all three bisectors are concurrent, by which we mean, that they intersect in a single point:

Proposition 5. *In a triangle $\triangle ABC$ the three bisectors intersect in a single point, i.e., they are concurrent.*

Proof. We first draw two bisectors in a triangle and show that the point of intersection is the center of the unique circle through all vertices of the triangle. Then the third bisector must also go through this point, as the center of the circle through all vertices of a triangle is unique.

Consider the figure below: the midpoints of $\triangle ABC$ are denotes A', B' and C', such that A' lies on the side opposite to A, B' lies on the side opposite to B and C' lies on the side opposite to C:

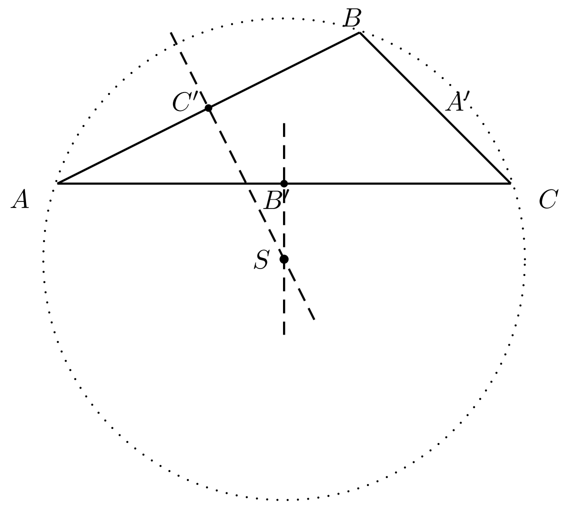


Figure 3: The intersection point of the medians through A and B is called S. The bisectors of AB and AC are dashed, the circle through the vertices is dotted.

We call the point of intersection of the drawn medians S. Next we consider the circle through the vertices of the triangle. It requires of course proof, that there is a unique circle through these vertices, but this is a small challenge[1]. The two tangents at the circle in the points A and C intersect in a point outside of the circle and by lemma 1, the line connecting this intersection point with the center of the circle intersects AC perpendicular in B'. This line is therefore the bisector of AC. Similarly the line through the midpoint of the circle and the intersection point of the tangents at the circle at A and B is the bisector of AB. Hence the bisectors intersect at the midpoint of the unique circle through the vertices. □

Medians

Definition 6. *Let $\triangle ABC$ be a triangle with vertices A, B and C. A median is a line through a vertex and the midpoint of the opposite side.*

In a triangle $\triangle ABC$ there are thus three medians; through A and the middle of BC, through B and the middle of AC and through C and the middle of AB.

Proposition 7. *In a triangle $\triangle ABC$ the three medians intersect in a single point, i.e., the medians are concurrent.*

Proof. We use a useful trick in triangle geometry; we enlarge the triangle by gluing three copies on the sides of the original triangle, as shown in the picture below:

We could equally as well enlarge all sides of the original triangle by a factor of two, and then connect the midpoints of the enlarged triangle. The resulting triangle is easily seen to be isomorphic to the original triangle (it is rotated by 180 degrees). We call this process a duplication of a triangle.

In the duplicated triangle the side AC is parallel to $A'C'$, BC is parallel to $B'C'$ and AB is parallel to $A'B'$. The median of the triangle $\triangle ABC$ through B intersects $A'C'$ at its midpoint. Hence the medians of $\triangle ABC$ coincide with the medians of the smaller triangle $\triangle A'B'C'$. Let us consider the medians through A and B; these are the line through A and A' and the line through B and B'.

[1] One first considers the family of circles through two vertices, and then one notes, that in this family there is a unique one also going through the third vertex.

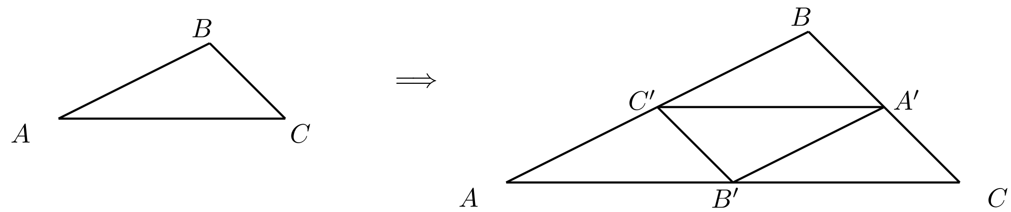


Figure 4: Left the original triangle and right the duplicated triangle.

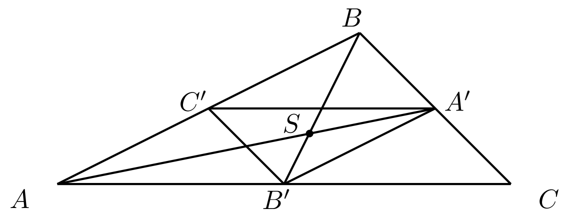


Figure 5: The intersection of the medians through A and B.

Let us call the intersection point S. Since the line through A and A' is a median as from the middle smaller as from the bigger triangle, which is larger by a factor of two, we have $|AS| = 2|A'S|$, where $|XY|$ denotes the distance between arbitrary points X and Y. If T were the intersection point from the median through C and C' and the median through A and A', the same reasoning would lead to $|AT| = 2|A'T|$, which implies $T{=}T'$. □

Altitudes

Definition 8. *Let $\triangle ABC$ be a triangle with vertices A, B and C. An altitude is a line through a vertex and perpendicular to the opposite side.*

In a triangle $\triangle ABC$ there are thus three altitudes; through A and perpendicular to BC, through B and perpendicular to AC and through C and perpendicular to AB.

Proposition 9. *In a triangle $\triangle ABC$ the three altitudes are concurrent.*

Proof. We start with an arbitrary triangle $\triangle ABC$ and duplicate it. Then we draw the bisectors, of which we know that they are concurrent; see figure 6. But the bisectors of triangle $\triangle ABC$ are the altitudes of triangle $\triangle A'B'C'$, which is congruent to triangle $\triangle ABC$. This proves the proposition. □

Eulers' line

The main point of this section is to prove the following statement:

Theorem 10. *In any triangle the intersection points of the medians, altitudes and bisectors lie on a line, i.e., they are colinear.*

The line on which these points lie is called the Euler line, after Leonard Euler, who seemed to have described this line first - in fact, he proved that the midpoint of the nine-point-circle also lies on this line, but we won't delve into that. Before we come to the proof of this statement, we shortly review some elementary transformations of the plane.

A rotation is characterized by two properties; the center of the rotation - the point around which we rotate - and the angle about which we rotate. It is fairly easy to see, that if we rotate around some point over an angle of 180 degrees, then any line is carried to a line that is parallel to the

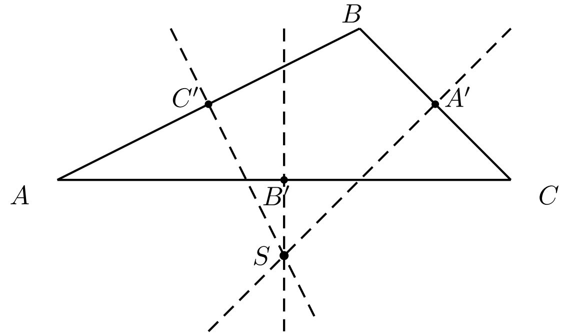
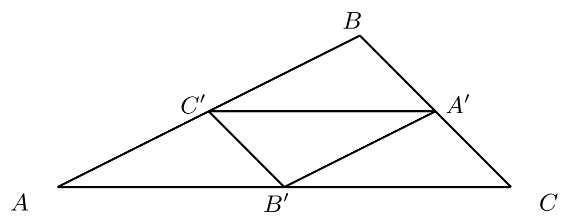


Figure 6: Constructing the altitudes from a given triangle by duplication and drawing the bisectors of the duplicated triangle.

original line. We remark that a rotation of 180 degrees around a point p is the same transformation as a point reflection across the point p.

A central scaling around a point p of a factor λ is a map that sends any point q to the point that lies on the line pq a distance $\lambda|pq|$ away from p and a distance $(\lambda - 1)|pq|$ away from q. In words; you just send any point λ times further away from p in the radial direction away from p. In vector notation this would mean: $q \mapsto p + (q - p)\lambda$. A central scaling around p of a factor 0 sends all points to p and a central scaling around p of any factor sends p to p and a central scaling around p of a factor 1 is the identity map. It is not difficult to see that a central scaling around any point of any nonzero positive factor sends lines to lines and if two lines are parallel, then the images are also parallel.

of thm. 10. Consider again a consellation of a duplicated triangle:



Let P be the intersection point of the medians of the larger triangle; then P is also the intersection point of the medians of the smaller triangle. The main observation is the following: we obtain the larger $\triangle ABC$ triangle from the smaller triangle $\triangle A'B'C'$ by first rotating around P over an angle of 180 degrees - or equivalently, by first performing a reflection across P - followed by a central scaling around P of a factor of two.

Let P' be the intersection point of the altitudes of the smaller triangle $\triangle A'B'C'$, then P' is the intersection point of the bisectors of the larger triangle $\triangle ABC$. Let then P'' be the intersection point of the altitudes of the larger triangle $\triangle ABC$. We denote the line through the points P and P' by l and the line through the points P and P'' by l'. We thus have to prove that l and l' are the same line. But in fact, since they share one point it suffices to show that l and l' are parallel.

Now the reader might have guessed our strategy already: we can obtain the line l' as the image of the transformation that is the successive application of a point reflection across P and a central scaling around P of a factor two. This follows, since the image of P under this transformation is P and the image of P' of this transformation is P'' - which again follows from the fact that $\triangle ABC$ is the image of $\triangle A'B'C'$ under this transformation. By the above observations we know that l is parallel to l'' and hence the theorem is proved. □