A proof of Van der Waerden's theorem

D.B. Westra

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In this text we try to give a proof of a version of a theorem dating back to 1927 due to Van der Waerden. Let us state the theorem first:

Theorem 1. Let the set of natural integers be colored using r colors. Then for any k, we can find a number $W(r, k)$, such that if $N > W(r, k)$ we can find an arithmetic progression of length k in the set $\{1, 2, 3, \ldots, N\}$.

The theorem thus states if the natural numbers are divided into r classes, one can find a an arithmetic progression of length k in one class. Independent of how the division is made, one can give a number $W(r, k)$, dependent on r and k, such that one is sure to find the arithmetic progression among the first N natural numbers if N is larger than $W(r, k)$. The proof shown below does not give a value for $W(r, k)$; it only shows a very bad upper bound for $W(r, k)$ and thus proves existence.

The proof is surely not a grant new proof, neither is it shorter, nor uses new concepts, nor provides new insights; we just found it more natural, or easier to understand. We definitely think it is completely equivalent to an already existing proof - see for example the works of Graham, Rothschild and Khinchin for more complete accounts and references. We wrote it down to provide a kind of notes that can be read by everyone; it also is a way to archive our knowledge.

Let us define some of the objects:

A coloring of $\mathbb{N}$ is any map $c : \mathbb{N} \to C$, where the set C is equivalent to the set $\{1, 2, \ldots, r\}$. We will call C the set of colors. If two numbers x and y have equal color, we write $c(x) = c(y)$. If A and B are two subsets of $\mathbb{N}$, we will write $c(A) = c(B)$ only if $B = A + d$ for some integer d, so that B is a translate of A, $B \cap A = \emptyset$ and $c(a) = c(a + d)$ for all $a \in A$.

We use the short-hand k-progression for arithmetic progression of length k, that is, a set $\{a_1, a_2, a_3, \ldots, a_k\}$ such that $a_{i+1} - a_i = d$ for some positive integer d; $a_j = a_1 + (j - 1)d$. We call a monochromatic k-progression any such set whose members all have equal color. To simplify reading, we use the word integer and number for natural numbers.

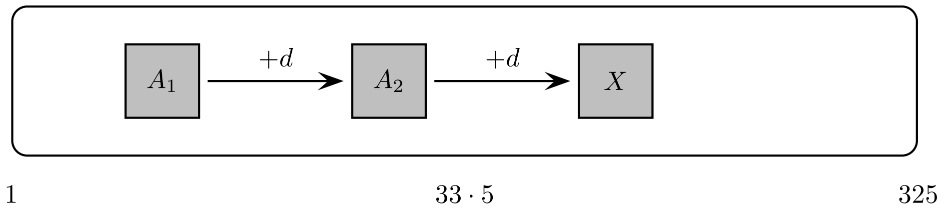
To find a monochromatic 1-progression is trivial. If $\mathbb{N}$ is colored with r colors, then any set of $r + 1$ elements will contain a monochromatic 2-progression. This is just an easy application of the pigeon-hole principle, a principle that will play an important role in the sequel.

We have not tried to make this text as compact as possible. Due to our almost nonexistent knowledge in this branch of mathematics, we have spelled out more details than necessary. Our hope is that others find in this text an easy to read account of a well-known fact from a branch algebra called Ramsey theory.

1 Finding a monochromatic 3-progression: two colors

Let us try with a rather easy exercise: We assume that $\mathbb{N}$ is colored with two colors and we want to find a monochromatic 3-progression. In fact, we show that a monochromatic 3-progression can always be found among the first 325 numbers. This result is rather poor; the number 325 is much too high; already among the first 9 natural numbers a monochromatic 3-progression can be found. However, the method provides a way to get a general proof. Although the number 325 might first seem rather arbitrary, we will shortly see the logic behind this choice.

We divide the set $\{1, 2, 3, \dots, 325\}$ into 65 blocks[1] of 5 consecutive integers. A block of 5 integers can be colored in $2^5 = 32$ ways. Therefore we can find at least two identically colored blocks of five integers among the first 33 blocks. Let us call the first block A_1, the second A_2 can be written as $A_1 + d$ for some integer d bounded by $32 \cdot 5$, since A_1 and A_2 are among the first 33 blocks of five. Therefore the block $X = A_2 + d = A_1 + 2d$ is a subset of $\{1, 2, 3, \dots, 325\}$.



If the color of one element $x \in X$ equals the color of $x - d \in A_2$ then we have a monochromatic 3-progression. But this need not be the case. Let us first consider the structure in A_1.

Since A_1 has five elements, among the first three at least two must be equally colored. Thus we can find three elements a_1, $a_2 = a_1 + e$, and $a_3 = a_1 + 2e$ in A_1 such that a_1 and a_2 have the same color. If the color a_3 is equal to this color, we are done, so assume the contrary.

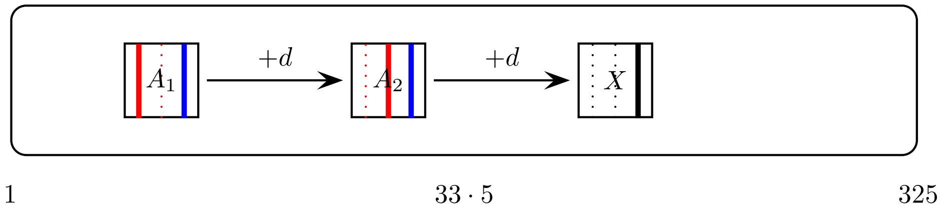
We know that $a_1 + d$ is an element of A_2 and since A_1 and A_2 are colored the same way, the colors of a_1 and $a_1 + d$ are identical. Similarly, the colors of a_2 and $a_2 + d = a_1 + d + e$ are identical, hence $c(a_1) = c(a_1 + d + e)$. Also $c(a_3) = c(a_3 + d)$. Now consider the following 3-progressions:

$$N_1 = \{a_1, a_1 + d + e, a_2 + 2(d + e)\}, \quad N_2 = \{a_3, a_3 + d, a_3 + 2d\}.$$

Since $a_3 = a_1 + 2e$ the final elements of N_1 and N_2 are equal; they focus on the same element so to say. Since there are only two colors, the color of $a_1 + 2d + 2e$ must equal one

[1] A block of integers we understand in the sequel as a set of consecutive integers.

of the two $c(a_1)$ and $c(a_3)$. Furthermore, $a_1 + 2(d+e)$ lies in the set X. Hence we are sure to find a monochromatic 3-progression among the first 325 integers.



The solid lines represent the progressions N_1 and N_2. Both end in the same black solid line. Since there are only two colors, the solid black line needs to be either red or blue, completing one of the two of N_1 and N_2 to a monochromatic 3-progression.

The above proof used that a block of five is necessary to get a 3-progression whose first two members are of equal color. Also, a block of five can be colored in 32 two ways, thus $2 \cdot 32 + 1 = 65$ of such blocks are needed to get a system of the type AAX, where the three sets form a 3-progression and the first two blocks are colored identically.

2 Finding a monochromatic 3-progression: three colors

We can mimic the above strategy to find a monochromatic 3-progression if $\mathbb{N}$ is colored using 3 colors. Since there are now 3 colors, we need to find 3 3-progressions with their first two elements of the same color that end in the same point. We thus need one step more compared to the strategy above.

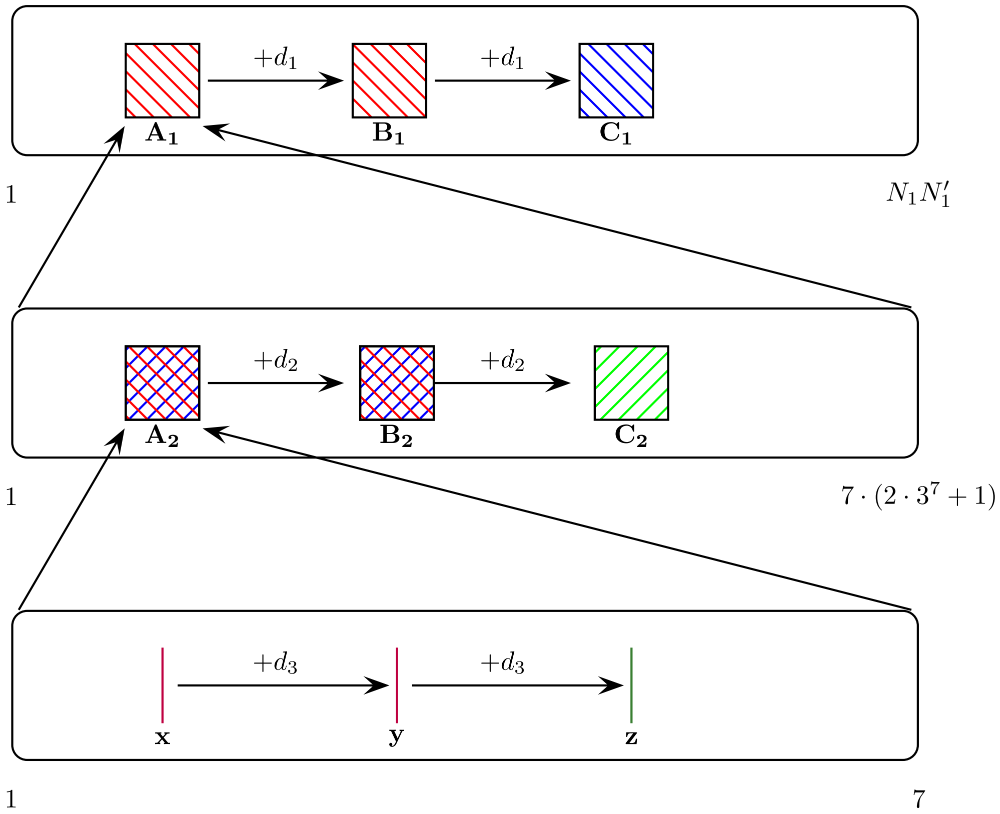
In order to find two numbers with the same color, we need to consider at least 4 elements. To complete these two to a 3-progression at most 7 numbers are required. Blocks of 7 can be colored in 3^7 ways. To get a 3-progression of such blocks where the first two blocks are identically colored, takes thus at most $2 \cdot 3^7 + 1$ such blocks, with a total of $N_1 = 7 \cdot (2 \cdot 3^7 + 1)$. A super-block of N_1 integers can be colored in 3^{N_1} ways. To find a 3-progression of such super-blocks with the first two super-blocks colored identically, we thus consider $N_1' = 2 \cdot 3^{N_1} + 1$ such superblocks. We define $N = N_1 N_1'$.

Divide the set $\{1, 2, 3, \ldots, N\}$ with $N = N_1 N_1'$ into N_1' superblocks of $N_1 = 7 \cdot (2 \cdot 3^7 + 1)$ consecutive integers. Since $N_1' = 2 \cdot 3^{N_1} + 1$ we can find three superblocks A_1, $B_1 = A_1 + d_1$, $C_1 = A_1 + 2d$ such that A_1 and B_1 are colored in the same way. Thus for each $a \in A_1$ we have $c(a) = c(a + d_1)$.

Consider the set A_1. It has size $N_1 = 7 \cdot (2 \cdot 3^7 + 1)$. We can divide it into $2 \cdot 3^7 + 1$ blocks with 7 consecutive numbers each. To B_1 and C_1 this division can be applied as well. Since we have $2 \cdot 3^7 + 1$ of these blocks of seven, we can find a 3-progression of these blocks A_2, $B_2 = A_2 + d_2$ and $C_2 = A_1 + 2d_2$, such that A_2 and B_2 are colored identically. But then the blocks A_2, $A_2 + d_1$, $A_2 + d_2$ and $A_2 + d_2 + d_1$ are all colored identically; adding d_2 to an element of A_2 preserves the color, and adding d_1 to an element of A_1 also preserves color, but $A_2 \subset A_1$.

Now we consider the block A_2. It consists of seven elements. Hence we can find a 3-

progression x, $y = x + d_3$ and $z = x + 2d_3$ inside A_2 such that $c(x) = c(y)$. Now we can work our way back up to A_1, B_1 and C_1.



If we have $c(x) = c(z)$, then x, y and z define a monochromatic 3-progression and we are done. So assume the contrary, then we consider the following 3-progressions

$$\begin{aligned} N_1 &= \{x, x + d_2 + d_3, x + 2(d_2 + d_3)\} \subset A_1 \\ N_2 &= \{x + 2d_3, x + d_2 + 2d_3, x + 2d_2 + 2d_3\} \subset A_1 \,. \end{aligned}$$

We already know that $c(x) = c(x+d_3)$. Furthermore, for all $u \in A_2$, we have $c(u+d_2) = c(u)$. Hence $c(x) = c(x + d_2 + d_3)$ and $c(x + 2d_3) = c(x + d_2 + 2d_3)$. Thus the first two elements of N_1 are colored identically and also N_2 has its first two elements colored identically. The common final element may be colored in the same color as x or $x + 2d_3$, in which case we are done. So we assume the contrary; $c(x)$, $c(x + 2d_3)$ and $c(x + 2d_2 + 2d_3)$ are of different color, which already exhausts all colors.

Now we consider the 3-progressions

$$\begin{aligned} M_1 &= \{x, x + d_1 + d_2 + d_3, x + 2(d_1 + d_2 + d_3)\} \\ M_2 &= \{x + 2d_3, x + d_1 + d_2 + 2d_3, x + 2d_1 + 2d_2 + 2d_3\} \\ M_3 &= \{x + 2d_2 + 2d_3, x + d_1 + 2d_2 + 2d_3, x + 2d_1 + 2d_2 + 2d_3\} \,. \end{aligned}$$

We have already seen that x and $x + d_2 + d_3$ are of equal color. But since both are elements of A_1, adding d_1 preserves color. Thus M_1 is a 3-progression whose first two elements have the same color. By the same reasoning $c(x+2d_3) = c(x+d_2+2d_3) = c(x+d_1+d_2+2d_3)$ and thus M_2 is a 3-progression with its first two elements equally colored as well. The same conclusion holds for M_3. But then we have three 3-progressions with a common final element such that the first two elements of any of them are equally colored but all three progressions differ in the colors of their first two elements. Since their are only three colors, the common final element must complete one of the three to a monochromatic 3-progression.

3 Finding a monochromatic 3-progression: four colors

By now the picture might have become clear, so let us consider the case of 3-progressions in $\mathbb{N}$ colored with 4 colors as an exercise to get the notation for the general case.

We first put $N_3 = 9$, $N_2 = N_3 N_3'$ with $N_3' = 2 \cdot 4^{N_3} + 1 = 2 \cdot 4^9 + 1$, $N_1 = N_2 N_2'$ with $N_2' = 2 \cdot 4^{N_2} + 1$, $N_0 = N_1 N_1'$ with $N_1' = 2 \cdot 4^{N_1} + 1$. We thus have sequence of numbers (N_3, N_2, N_1, N_0) defined by a recurrence relation: $N_r = N_{r+1}(2 \cdot 4^{N_{r+1}} + 1)$ and $N_3 = 9$. In any case, we have $N_0 = N_1 N_1'$ with

$$N_1 = 9 \cdot (2 \cdot 4^9 + 1) \cdot \left(2 \cdot 4^{9 \cdot (2 \cdot 4^9 + 1)} + 1\right), \quad N_1' = 2 \cdot 4^{N_1} + 1\,.$$

We divide the set $\{1, 2, \ldots, N_0\}$ into N_1' blocks of N_1 consecutive integers. We then find a 3-progression of sets A_1, $A_1 + d_1$, $A_1 + 2d_1$, where A_1 and $A_1 + d_1$ are colored identically. Thus for any $x \in A_1$ we have $c(x) = c(x + d_1)$.

Any block of size $N_1 = N_2 N_2'$ can be divided into N_2' blocks of N_2 consecutive numbers. Since $N_2' = 2 \cdot 4^{N_2} + 1$ we find another 3-progression of blocks A_2, $A_2 + d_2$, $A_2 + 2d_2$, all contained in A_1 such that A_2 and $A_2 + d_2$ are colored identically.

Any block of size $N_2 = 9 \cdot (2 \cdot 4^9 + 1)$ can be divided into $2 \cdot 4^9 + 1$ blocks of 9 consecutive integers. We can thus find inside A_2 a 3-progression of blocks of size 9 whose first two elements are equally colored: A_3, $A_3 + d_3$ and $A_3 + 2d_3$ and $c(A_3) = c(A_3 + d_3)$.

Inside A_3, which contains 9 elements, we find a 3-progression

$$\alpha_0 = \{x, x + d_4, x + 2d_4\} \subset A_3\,,$$

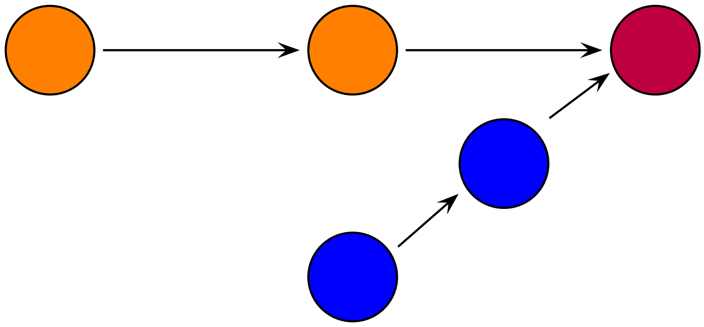
with $c(x) = c(x + d_4)$. We will call such a 3-progression a 2-strand, since its first two elements are of the same color, its final element might have a different color. If the final element has the same color as the other two, the 2-strand is a 3-progression. If α_0 is a monochromatic 3-progression we are done, so we assume the contrary.

Since $c(y) = c(y + d_3)$ for any $y \in A_3$, we have $c(x) = c(x + d_3) = c(x + d_3 + d_4)$. Choosing $y = x + 2d_4$ we also have $c(x + 2d_4) = c(x + d_3 + 2d_4)$. Therefore we find two 2-strands

$$\begin{aligned} \alpha_1 &= \{x, x + d_3 + d_4, x + 2(d_3 + d_4)\} \subset A_2 \\ \beta_1 &= \{x + 2d_4, x + d_3 + 2d_4, x + 2d_3 + 2d_4\} \subset A_2 \end{aligned}$$

with a common final element. Such a system of strands we will call a net. Since these strands differ in the colors of their first two elements, we call such a system a heterochromatic net.

What image we have in mind is given pictorially below; with more and more strands it becomes like a complicated net.



If the color of the final element $x+2d_3+2d_4$ equals $c(x)$ or $c(x+2d_4)$ we have a monochromatic 3-progression and we are done. So assume the contrary.

Consider the 3-progressions

$$\begin{aligned}
\alpha_2 &= \{x, x+d_2+d_3+d_4, x+2(d_2+d_3+d_4)\} \subset A_1 \\
\beta_2 &= \{x+2d_4, x+d_2+d_3+2d_4, x+2d_2+2d_3+2d_4\} \subset A_1 \\
\gamma_2 &= \{x+2d_3+2d_4, x+d_2+2d_3+2d_4, x+2d_2+2d_3+2d_4\} \subset A_1
\end{aligned}$$

with a common final element. Since $c(y) = c(y+d_2)$ for all $y \in A_2$ and since $c(x) = c(x+d_3+d_4)$ and $c(x+2d_4) = c(x+d_3+2d_4)$ – from the above 2-strands α_1 and β_1 – we see that these 3-progressions are 2-strands. Since the initial elements have different colors we have a heterochromatic net of 3-strands. If the common final element has a color equal to one of the initial elements, we have a monochromatic 3-progression and we are done, so we asumme the contrary.

Consider the 3-progressions

$$\begin{aligned}
\alpha_3 &= \{x, x+d_1+d_2+d_3+d_4, x+2(d_1+d_2+d_3+d_4)\} \\
\beta_3 &= \{x+2d_4, x+d_1+d_2+d_3+2d_4, x+2d_1+2d_2+2d_3+2d_4\} \\
\gamma_3 &= \{x+2d_3+2d_4, x+d_1+d_2+2d_3+2d_4, x+2d_1+2d_2+2d_3+2d_4\} \\
\delta_4 &= \{x+2d_2+2d_3+2d_4, x+d_1+2d_2+2d_3+2d_4, x+2d_1+2d_2+2d_3+2d_4\}
\end{aligned}$$

with a common final element. Comparing with the strands α_2, β_2 and γ_2 and using $c(y) = c(y+d_1)$ for all $y \in A_1$ we see that these 3-progressions define a heterochromatic net of strands. Since there are only 4 colors available, the common final element must complete one of the strands to a monochromatic 3-progression.

4 $j-k$-Nets, k-strands and all that

Let us try to formalize the concepts we have seen so far:

Definition 1. A **k-strand** is a $(k+1)$-progression whose first k elements define a monochromatic k-progression. The last element of a k-strand is called its **final element**.

If the color of the final element of a k-strand equals the color of the first element, it defines a monochromatic $(k+1)$-progression.

Definition 2. We call a $\mathbf{j-k}$**-net** a set of j different k-strands $A_1, \ldots, A_j$ such that $A_a \cap A_b$ equals the final element of A_a and A_b, if $a \neq b$.

The k-strands in a $j-k$-net only have their final elements in common. Therefore a $j-k$-net has $jk+1$ elements.

Definition 3. We call a $j-k$-net heterochromatic if the colors of the first elements of the k-strands are all different.

If $\mathbb{N}$ is colored using r colors, a heterochromatic $r-k$-net must contain a monochromatic $(k+1)$-progression; the color of the common final element must equal the color of the first elements of one of the k-strands as there are only r colors.

We now define some claims, which we want to prove to be true. These claims are to be seen as Boolean variables. Each of these claims states that a certain finite number N exists, such that among the N integers, one can find a desired structure.

Definition 4. We call $\Lambda(r,k)$ the claim: If $\mathbb{N}$ is colored using r colors, there exists a number $W(r,k) < \infty$ such that among the first $W(r,k)$ integers we can find a monochromatic k-progression.

Definition 5. We call Λ_k the claim: $\Lambda(r,k)$ is true for all r.

In the examples above, we have used a strategy of the form "either we already have a k-progression and we are done, or we proceed and find a net with another strand with a different color". To formalize this, we need:

Definition 6. We call $\Sigma(r,j,k)$ the claim: If $\mathbb{N}$ is colored using r colors, there exists a number $S(r,j,k) < \infty$ such that among the first $S(r,j,k)$ integers we can find a monochromatic $(k+1)$-progression or a heterochromatic $j-k$-net of k-strands.

We have already seen that $\Lambda(r,2)$ is true. We can take $W(r,2) = r+1$. We have also seen that $\Lambda(2,3) = \Lambda(3,3) = \Lambda(4,3)$ is true. Let us now collect some results about these statements and their relations.

Lemma 1. $\Sigma(r,1,k) \iff \Lambda(r,k)$.

A $1-k$-net is a k-strand. A k-strand starts with a monochromatic k-progression and is followed by final element in any color. Thus if one can find a k-strand, one surely can find a monochromatic k-progression and $S(r,1,k) \geq W(r,k)$. Conversely, any monochromatic k-progression can be completed to a k-strand. We can give a bound for $S(r,1,k)$ as follows: If we have a monochromatic k-progression among the integers 1 upto $W(r,k)$, then the difference between any two elements is at most $(W(r,k)-1)/(k-1)$. Even more, if this number is not an integer, we should round down to the next nearest integer, which has a fancy name, the floor value denoted by brackets $\lfloor \ \rfloor$. But then $S(r,1,k) \leq W(r,k) + \lfloor (W(r,k)-1)/(k-1) \rfloor$. For $k=2$ we indeed find $S(r,1,2) \leq r+1+\lfloor (r+1)/1 \rfloor = 2r+1$.

Lemma 2. If Λ_k is true, then $\Sigma(r, j, k) \Longrightarrow \Sigma(r, j+1, k)$.

Assume that $\Sigma(r, j, k)$ and Λ_k are both true. Denote $S(r, j, k)$ by N. Any block of N consecutive integers can be colored in $r' = r^N$ ways. Since $\Lambda(r', k)$ is true, also $\Sigma(r', 1, k)$ is true. Thus there is a number M such that if we divide the integers 1 upto NM into blocks of size N, we can find a k-strand of the blocks of size N. That is, we have blocks A, $A + d$, $\ldots$, $A + kd$ such that the first k blocks are colored identically. Since A has size N, we can find either a monochromatic $(k + 1)$-progression or a heterochromatic $j - k$-net. If we find a monochromatic $(k + 1)$-progression, we are done. So assume the contrary, then we have j k-strands $\alpha^{(1)}$, $\ldots$, $\alpha^{(j)}$ with a common final element but whose first members all have a different color. Since we do not have a $(k + 1)$-progression, the final element has a color different from all the first elements.

Let us consider any k-strand of the net

$$\alpha^{(l)} = \{a, a + \delta, a + 2\delta, \ldots, a + k\delta\} \subset A\,, \quad 1 \le l \le j\,,$$

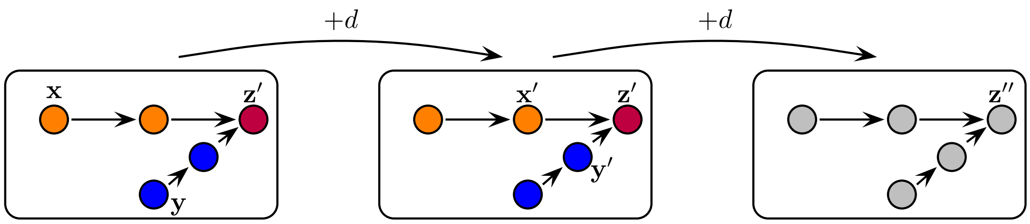
and let us write z for the common final element; $z = a + k\delta$. We modify $\alpha^{(l)}$ to the following $(k + 1)$-progression

$$\beta^{(l)} = \{a, a + (\delta + d), a + 2(\delta + d), \ldots, a + (k - 1)(\delta + d), a + k(\delta + d)\}\,.$$

Since adding d, $2d$, upto $(k-1)d$ to an element of A preserves the color, we see that $\beta^{(l)}$ is a k-strand with final element $a + k(\delta + d) = z + kd$. We can modify all $\alpha^{(l)}$ in this way, which all will have $z + kd$ as their final element and in addition consider the new $(k+1)$-progression

$$\beta^{(j+1)} = \{z, z + d, z + 2d, \ldots, z + kd\}\,.$$

The color of z is different from all the initial elements of any k-strand $\beta^{(1)}$, $\ldots$, $\beta^{(j)}$. Thus we have a heterochromatic $(j + 1) - k$-net. In fact $S(r, j + 1, k) \le NM$.



In the picture we have tried to present the idea of how to extend $\Sigma(r, j, k)$ to $\Sigma(r, j + 1, k)$ for the case $j = 2$ and $k = 2$. The new sequence are denoted $\{x, x', z''\}$, $\{y, y', z''\}$ and $\{z, z', z''\}$. Since the third block might be completely different colored, we have used lightgray to display an unknown coloring.

Lemma 3. If Λ_k is true, then so is Λ_{k+1}.

If Λ_k is true, then for any r also $\Sigma(r, 1, k)$ is true. By induction then also $\Sigma(r, r, k)$ is true. So either we have a monochromatic $(k + 1)$-progression or we can find a heterochromatic $r - k$-net. But since there are only r colors available, the common final element must complete one of the k-strands to a monochromatic $(k + 1)$-progression.

Proof of Van der Waerden's theorem: We already know that Λ_2 is true and $W(r,2) = r+1$. This serves as the induction start, and the induction step is then provided by the last lemma 3. $\square$

5 Final details

Let us first comment on a proof of $\Lambda(r,3)$: We already saw in sections 1, 2 and 3 how to construct a 2-strand among the first $2r+1$ integers. Then we found we can construct a heterochromatic $2-2$-net or a monochromatic 3-progression among the first $N_2 = (2r+1)\cdot(2\cdot r^{2r+1}+1)$ integers. If $r=2$ this will suffice, if $r>2$, we can go on and be sure to be able to find a $3-2$-net or a monochromatic 3-progression among the first $N_3 = N_2\cdot(2\cdot r^{N_2}+1)$ integers. If we then find a monochromatic k-progression we are done. Proceeding inductively, suppose we have a number N_j such that either we find a monochromatic 3-progression or a heterochromatic $j-3$-net. Assuming we don't have a monochromatic 3-progression, we divide the first $N_{j+1} = N_j(2\cdot r^{N_j}+1)$ integers into $2\cdot r^{N_j}+1$ blocks of size N_j. Among the first $r^{N_2}+1$ of these blocks, two of them must be colored identically, and thus we find a 3-strand of blocks of size N_j: A, $A+d$ and $A+2d$. Within A we find either a monochromatic 3-progression or a heterochromatic $j-3$-net. Assuming we have no monochromatic 3-progression, we find j 3-strands

$$\{a_1, a_1+\delta_1, z\}, \dots, \{a_j, a_j+\delta_j, z\}$$

inside A with common final element $z = a_l + 2\delta_l$ for all $1 \le l \le j$ and such that z has a color different from all $c(a_l)$ for $1 \le l \le j$. Since $A+d$ is colored identically as A, we consider the 3-progressions

$$\{a_1, a_1+\delta_1+d, z+2d\}, \dots, \{a_j, a_j+\delta_j+d, z+2d\}$$

and

$$\{z, z+d, z+2d\},$$

and conclude that they are 3-strands with common final element $z+2d$. Since they all differ in the color of their first element, we have found a heterochromatic $(j+1)-3$-net.

Hence we can go on up to $r-3$-nets, which must contain a monochromatic 3-progression as there are only r colors available. Hence Λ_3 is true.

Thus we can also use $\Lambda(r,3)$ as a start for the induction step. But indeed, having understood this proof amounts to the construction of the proof of the general case.

There exists the following version of Van der Waerden's theorem:

Theorem 2. Let $\mathbb{N}$ be colored using r colors. Then we can find a color, in which monochromatic k-progressions exists for any k.

It is clear that this version implies theorem 1. To see the converse, we first remark that a monochromatic k-progression contains several monochromatic k'-progressions for $k' < k$; indeed, we can just truncate a k-progression down to a k'-progression. Now, for each k we can find a color, so that we have a monochromatic k-progression in this color. We can view this as a map $w : \mathbb{N} \to \{1, 2, \ldots, r\}$. However, since there are only r colors, for at least one color $\alpha \in \{1, 2, , \ldots, r\}$ we must have infinitely k with $w(k) = \alpha$. But then we can find a monochromatic k-progression in color α for any k – if necessary we truncate a monochromatic K-progression of color α for $K > k$.

The theorem of Van der Waerden can be extended to a result called Szemerédi's theorem (1975), which generalizes the (finite) colors to subsets with a certain property, which goes under the name of positive upper density. This result was first generalized and then further used by Green and Tao in 2004 to prove that among the set of prime numbers, arbitrarily long arithmetic progressions can be found.