(Non)uniqueness of minimizers in least gradient problem

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Abstract

The least gradient problem, which is related to conductivity imaging in medical scans, see [5], and to free material design models, see [4], is a variational problem of the form

$$\min\{\int_\Omega |Du|, \quad u \in BV(\Omega), \quad Tu = f\}, \tag{1}$$

where $f \in L^1(\Omega)$ and T denotes the trace operator. It is well known that for continuous boundary data the solution exists and is continuous up to the boundary. Here, we focus on the two-dimensional case and extend the class of functions for which the solution exists to $BV(\partial\Omega)$ and discuss how the set of minimizers looks like. However for discontinuous f uniqueness of solutions may fail; it turns out that the structure of superlevel sets of all minimizers is very similar and we may characterize all of them.

Introduction

There are various ways to introduce problem (1) in the literature. The most general way, considered in [6] and requiring little regularity of the boundary and boundary data, would be to use the direct method for the functional

$$F(u) = \begin{cases} \int_\Omega |Du| & \text{if } u \in BV(\Omega) \text{ and } Tu = f; \\ +\infty & \text{otherwise.} \end{cases}$$

However, this functional is not lower semicontinuous with respect to L^1 convergence and even existence of solutions for arbitrary $f \in L^1(\partial\Omega)$ fails (see [7]). The counterexample was a characteristic function of a certain fat Cantor set; note that it does not lie in $BV(\partial\Omega)$. We can instead consider the relaxation $\overline{F}$ of the original functional, but we have to replace the Dirichlet boundary condition by a weaker one.

Another way, studied in [8], [5], and [4] ,is to impose stronger conditions on Ω and consider the Dirichlet boundary condition in the sense of traces. We follow this approach and assume $\Omega \subset \mathbb{R}^2$ to be an open, bounded, strictly convex set with Lipschitz (or C^1) boundary. The boundary datum f will belong to $L^1(\partial\Omega)$ or $BV(\partial\Omega)$. These assumptions are motivated mainly by the existence result for continuous boundary data in [8].

Existence in $\Omega \subset \mathbb{R}^2$

Since [8] it is well known that for an open, bounded, stricly convex set Ω with Lipschitz boundary and $f \in C(\partial\Omega)$ there exists a unique solution of the least gradient problem and that it is continuous up to the boundary. This result can be extended (see [3]):

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, strictly convex set with C^1 boundary. Then for every $f \in BV(\partial\Omega)$ there exists a solution of least gradient problem for f.*

Note that this condition is not necessary and if we denote by X the space of traces of least gradient functions, then

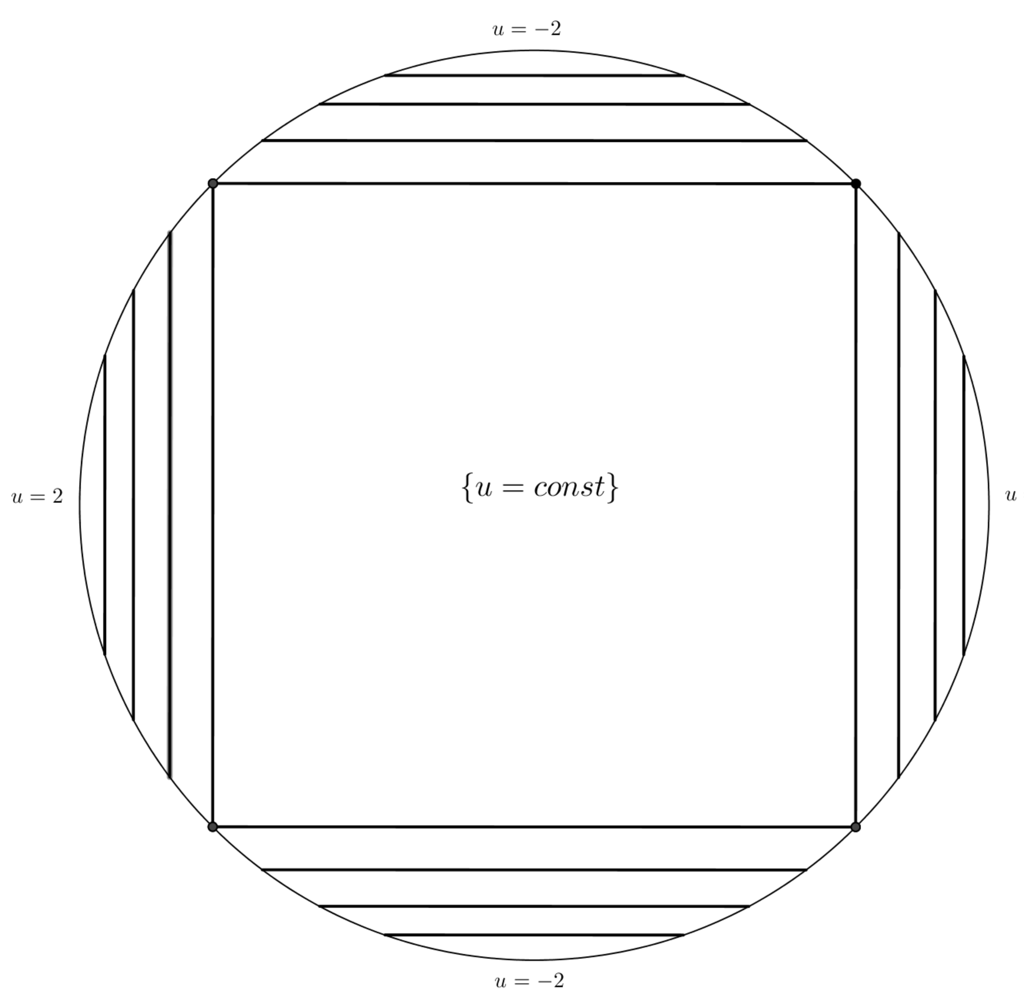
$$C(\partial\Omega) \cup BV(\partial\Omega) \subsetneq X \subsetneq L^1(\partial\Omega).$$

Outline of proof:

- Fix a sequence of smooth functions $f_n \to f$ in the strict topology. Consider $u_n \in BV(\Omega)$, the unique solutions to the least gradient problem for $f_n \in C^\infty(\partial\Omega)$.
- We prove that $u_n \to u$, a least gradient function, in $L^1(\Omega)$; two-dimensional geometry implies that $\chi_{\{u_n \geq t\}} \to \chi_{\{u \geq t\}}$ strictly for a.e. t. By the co-area formula $u_n \to u$ strictly.
- By continuity of the trace operator in the strict topology we have $Tu = f$.

Uniqueness of solutions

For continuous boundary data the solutions are unique. There are also examples with uniqueness of solutions constructed via decomposition of $\partial\Omega$ into two arcs, on which the boundary data are monotone. The first example for nonuniqueness is the Brothers example:



Consider $\Omega = B(0,1)$ and the boundary datum $f(x,y) = x^2 - y^2$. The unique solution u has boundaries of level sets as shown on the picture and constant value zero on the square $S = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^2$. Now we add jumps to the boundary datum: consider

$$\widetilde{f}(x,y) = \begin{cases} x^2 - y^2 + 1 & \text{if } |x| > \frac{1}{\sqrt{2}} \\ x^2 - y^2 - 1 & |x| < \frac{1}{\sqrt{2}}. \end{cases}$$

The solutions u_t all have the same structure of boundaries level sets (save for translation by ± 1), but may differ on S: there is a one-parametric family of solutions characterized by the constant value $t \in [-1,1]$ on S.

Uniqueness: structure of solutions

It turns out that all the examples of nonuniqueness of solutions in $\mathbb{R}^2$ are similar to the Brothers example and involve level sets of positive measure.

Theorem 2. *Let u, v be functions of least gradient in $\Omega \subset \mathbb{R}^2$ such that $Tu = Tv = h$. Then $u = v$ on $\Omega \backslash (C \cup N)$, where both u and v are locally constant on C and N has Hausdorff dimension at most 1.*

Note that this result involves no restrictions on the regularity of the boundary or the boundary datum. The set N is the sum of J_u and J_v, the jump sets of u and v respectively, and (for technical reasons) of B_u and B_v, boundaries of level sets of positive measure of u and v, i.e.

$$B_u = \bigcup_{\{t:\, l^2(\{u=t\})>0\}} (\partial\{u \geq t\} \cup \partial\{u \leq t\}).$$

The set N has Hausdorff dimension at most 1. Moreover, every connected component of C is a polygon (possibly with countably many sides) satisfying Green's formula, i.e. the sum of lengths of even-numbered sides equals the sum of lengths of odd-numbered sides.

Remarks about the proof:

- The proof is geometric in nature and is based on an observation from [2] that the boundaries of superlevel sets of least gradient functions are minimal surfaces. We rely heavily on the maximum principle for minimal graphs and on local properties of minimal surfaces in the neighbourhood of $\partial\Omega$;
- The most important step is to prove that $\partial\{u \geq t\}$ and $\partial\{v \geq s\}$ cannot intersect in $\Omega \backslash N$ for any $t \neq s$. This is precisely why u and v may differ on C; there are no boundaries of superlevel sets inside C.

Finding all the solutions

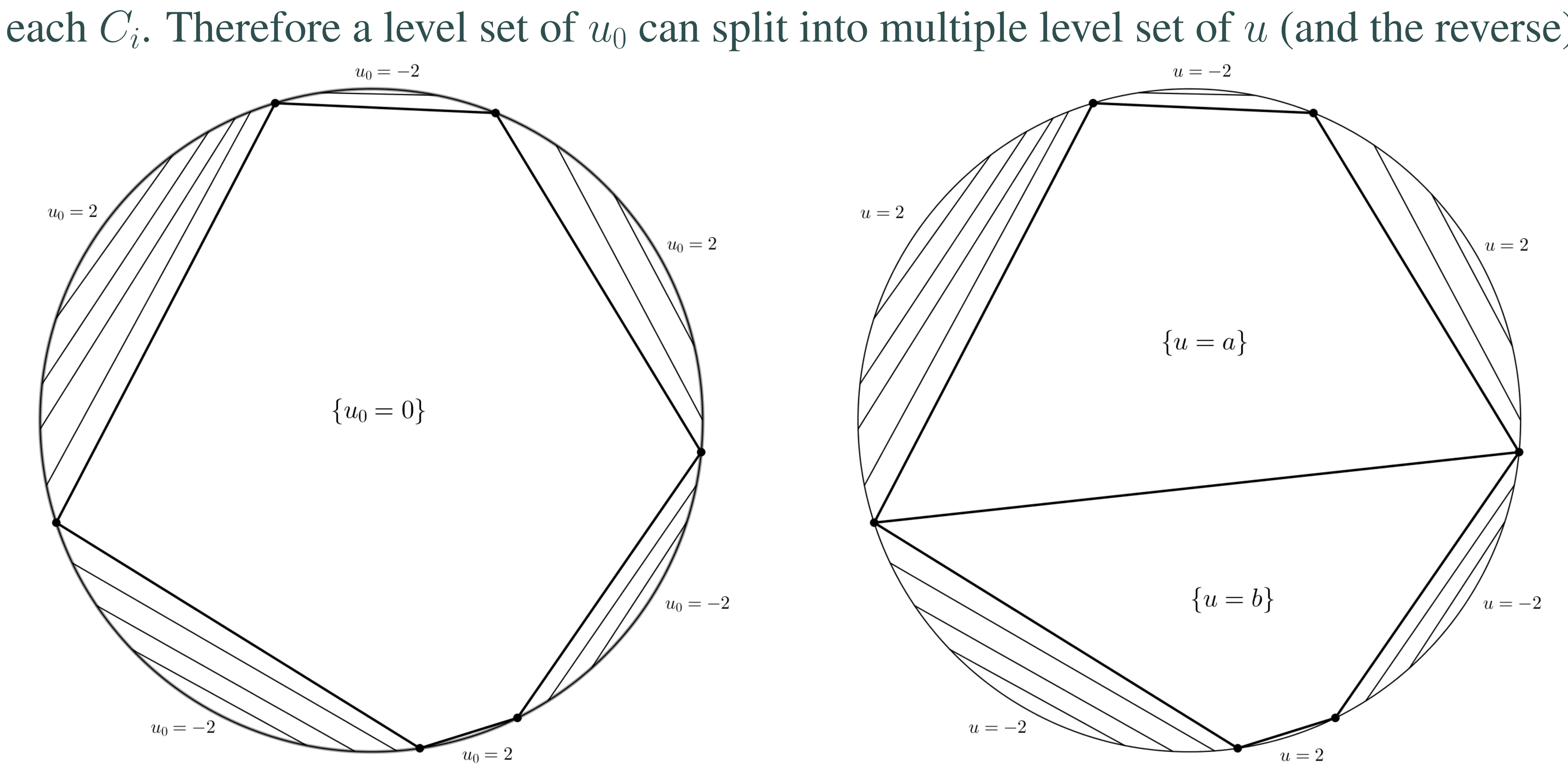
The proof of Theorem 1 gives us an algorithm which produces at least one solution u_0. By Theorem 2 any other solution may differ from u_0 only on a set C where u_0 is locally constant (we may assume it is connected). What happens next depends on the geometry of C:

Proposition 3. *If C is a finite polygon with sides Γ_i that has no subpolygons which obey Green's formula, then the class of solutions to least gradient problem with boundary data h contains precisely the functions u such that $u = u_0$ in $\Omega \backslash C$ with constant value t on C satisfying the inequality*

$$\max_k T^{\Gamma_{2k-1}} u_0 \leq t \leq \min_k T^{\Gamma_{2k}} u_0.$$

Here we assumed (without loss of generality) that $\Gamma_1 \subset \partial\{u_0 \geq t_0\}$, where t_0 is the value of u_0 on C. $T^{\Gamma_k} u_0$ denotes the trace of u_0 on Γ_k from $\Omega \backslash C$ (it turns out to be constant).

If C has subpolygons which obey Green's formula, situation becomes a bit harder. While in the Brothers example all the solutions u are constant on C, in general we have to split C into C_i, minimal subpolygons which satisfy Green's formula, and u is only constant on each C_i. Therefore a level set of u_0 can split into multiple level set of u (and the reverse):



Even in this case an analogue of Proposition 3 holds: we can describe the set of all solutions to the least gradient problem in terms of any single solution u_0.

Selection criterion

Motivated by the strain-gradient plasticity model considered in [1], we consider approximate minimalization problems:

$$G_\varepsilon(u) = \overline{F}(u) + \sqrt{\varepsilon}\|u\|_1 = \begin{cases} \int_\Omega \sqrt{\varepsilon}|u| + \int_\Omega |Du| + \int_{\partial\Omega} |Tu - f| & \text{if } u \in BV(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

The passage $\varepsilon \to 0$ models an increasing effective range of interaction. We may prove that $\Gamma - \lim_{\varepsilon \to 0} G_\varepsilon = \overline{F}$ and we have

Theorem 4. *Let $v_n \in \arg\min G_{\varepsilon_n}$ and suppose that $v_n \to v$ in $L^1(\Omega)$ as $\varepsilon_n \to 0$. Then $v \in \arg\min \overline{F}$. and it is the element with the smallest L^1 norm among minimizers of $\overline{F}$.*

- Theorem 2 implies that the minimizer of $\overline{F}$ with the smallest $1-$norm is unique. Thus the functional G_ε produces a selection criterion for minimizers of $\overline{F}$.
- For $f \geq 0$ Theorem 4 is stronger, as every sequence $v_n \in \arg\min G_{\varepsilon_n}$ is increasing, therefore convergent to a least gradient function.

References

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