Existence and regularity of minimizers in the anisotropic least gradient problem

Wojciech Górny
University of Warsaw
w.gorny@mimuw.edu.pl

Abstract
The anisotropic least gradient problem is a variational problem of the form

$$\min\{\int_\Omega |Du|_\phi, \quad u \in BV(\Omega), \quad Tu = f\}, \qquad \text{(ALGP)}$$

where ϕ is a fixed norm, $f \in L^1(\Omega)$ and T denotes the trace operator. When ϕ is the Euclidean norm, it is well known that for continuous boundary data the solution exists and is continuous up to the boundary. Here, we focus on the two-dimensional case and see if the result holds in the anisotropic setting. Our main focus in on two issues: regularity of a unique minimizer for a strictly convex norm and existence of low-regularity minimizers for a non-strictly convex norm.

Introduction

In the classical least gradient problem, when ϕ is the Euclidean norm, existence, regularity, and uniqueness of minimizers depends on the geometry of the set $\Omega \subset \mathbb{R}^2$. Here the situation is slightly more complicated, as we have additionally the interplay between the shapes of Ω and the unit ball in the anisotropic norm $B_\phi(0,1)$; our goal is to explore this relationship. We divide our reasoning into two stages:

- The unit ball $B_\phi(0,1)$ is strictly convex. Then, regardless of the regularity of ϕ, we are able to prove existence and uniqueness of minimizers for strictly convex Ω and obtain regularity estimates in terms of the modulus of continuity of the boundary data.
- The unit ball $B_\phi(0,1)$ has flat facets. Then, under stronger assumptions on Ω, we use the regularity estimates from the strictly convex case to prove existence of a single minimizer with the same regularity. However, we lose uniqueness of minimizers and the additional minimizers may have regularity no better than $BV(\Omega) \cap L^\infty(\Omega)$.

Existence and uniqueness for strictly convex $B_\phi(0,1)$

Firstly, we need to understand how the geometry of Ω influences the regularity of least gradient functions. For existence of minimizers, the crucial idea is the *barrier condition*:

Definition. ([4, Definition 3]) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Suppose that ϕ is an elliptic metric integrand. We say that Ω satisfies the barrier condition if for every $x_0 \in \partial\Omega$ and sufficiently small $\varepsilon > 0$, if V minimizes $P_\phi(\cdot\,; \mathbb{R}^N)$ in

$$\{W \subset \Omega : W \backslash B(x_0,\varepsilon) = \Omega \backslash B(x_0,\varepsilon)\}$$

then

$$\partial V \cap \partial\Omega \cap B(x_0,\varepsilon) = \emptyset.$$

This condition guarantees existence of minimizers to the anisotropic least gradient problem for continuous boundary data; see [4]. Now, we prove that the barrier condition is equivalent to strict convexity of Ω; we obtain

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded strictly convex set. Suppose that ϕ is a norm on $\mathbb{R}^2$ and $B_\phi(0,1)$ is strictly convex. Let $f \in C(\partial\Omega)$. Then then there exists a unique solution to Problem (ALGP).*

The uniqueness proof is based on a topological argument and does not require any regularity of Ω; this is a departure from the setting of [4].

Regularity for uniformly convex Ω

We briefly recall the main regularity result from [7] concerning the isotropic case. Assume that $\partial\Omega \in C^2$ and that Ω is uniformly convex, i.e. the mean curvature of $\partial\Omega$ is positive. Then, if the boundary data f is of class $C^{0,\alpha}(\partial\Omega)$, where $\alpha \in (0,1]$, then the (unique) minimizer of the least gradient problem u is in the class $C^{0,\alpha/2}(\Omega)$. As the (two-dimensional) examples provided by the authors show, this result is optimal. Here, we allow Ω to have only Lipschitz boundary and use the following definition of uniform convexity (which agrees with the classical definition for C^2 sets):

Definition. We say that the set an open bounded convex set Ω is uniformly convex, if the following condition is satisfied: let $P = \{y \geq ax^2\}$, where $a > 0$. Let $x_0 \in \partial\Omega$ and let l be a supporting line at x_0. Then there exists P', an isometric image of P, tangent to l at x_0 such that $\overline{\Omega} \subset P'$ and $\partial P' \cap \overline{\Omega} = \{x_0\}$.

Proposition 2. *Suppose that $\Omega \subset \mathbb{R}^2$ is uniformly convex and $B_\phi(0,1)$ is strictly convex. Let $f \in C(\partial\Omega)$ and take ω to be its modulus of continuity. Let u be the solution of Problem (ALGP) with boundary data f. Then $u \in C(\overline{\Omega})$ and it is continuous with modulus of continuity*

$$\overline{\omega}(|x-y|) = \omega(c(\Omega)|x-y|^{1/2}).$$

Let us stress the fact that in the Proposition above the constant $c(\Omega)$ depends only on Ω and not on the metric integrand ϕ (and can be explicitly calculated). If the boundary is C^2, then it depends only on the diameter and the lower bound c on curvature of Ω.

Remarks about the proof:

- It uses an anisotropic analogue of a classical result from [1] on the regularity of superlevel sets of least gradient functions;
- It uses heavily the geometry of Ω (it boils down to calculation of distances for some possible configurations of Ω and the parabolas from the Definition above), but does not use any regularity of $\partial\Omega$ or ϕ altogether;
- The method used adapts quite well to other situations, for instance if only parts of $\partial\Omega$ satisfy the uniform convexity condition or if the set Ω is only convex and the boundary data f satisfy some additional geometrical assumptions (for existence of minimizers in that case, see [6]).

Existence of minimizers for non-strictly convex $B_\phi(0,1)$

The existence of minimizers is based on an analogue of Miranda's theorem, which states that a limit of a sequence of least gradient functions is a least gradient function. In the next result we allow the norm to depend additionally on the location $x \in \Omega$. Define the following functional (which is a relaxation of the total variation functional with respect to Dirichlet boundary data, see [5]):

$$F_\phi(v) = \int_\Omega |Dv|_\phi + \int_{\partial\Omega} \phi(x,\nu^\Omega)|Tv - f|d\mathcal{H}^{N-1}.$$

Theorem 3. *Let ϕ and ϕ_n be metric integrands such that $\phi_n \to \phi$ in $C(\overline{\Omega} \times \partial B(0,1))$. Then the sequence of functionals F_{ϕ_n} $\Gamma-$converges (with respect to the L^1 convergence) to the functional F_ϕ.*

The assumption that $\phi_n \to \phi$ in $C(\overline{\Omega} \times \partial B(0,1))$ is quite natural in this context: as metric integrands are $1-$homogenous in the second variable, it is sufficient to check convergence only on the unit sphere. Furthermore, an easy example shows that we may not relax the assumption concerning uniform convergence in Ω.

Using Theorem 1 and Theorem 3, we see that we only have to take care of the boundary condition; we keep the boundary datum fixed and approximate arbitrary norm ϕ by (strictly convex) norms $\phi_n = \phi + \frac{1}{n}l_2$, where l_2 is the isotropic norm. Let u_n be a minimizer for the norm ϕ_n. $\Gamma-$convergence of the functionals above guarantees convergence (on a subsequence) of u_n to a function $u \in BV(\Omega)$, which is a minimizer of the functional F_ϕ. We need to see that $Tu = f$. There are two ways of doing that:

- If Ω is uniformly convex, then we may use the regularity estimates from Proposition 2 (which do not depend on ϕ!) and a maximum principle to use the Arzela-Ascoli theorem and obtain a minimizer with the same regularity estimate;
- If Ω is not uniformly convex, then we may blow up the boundary of Ω to prove that $Tu = f$ directly from the definition of trace. However, in this case we do not obtain any regularity estimates better than $u \in C(\overline{\Omega})$.

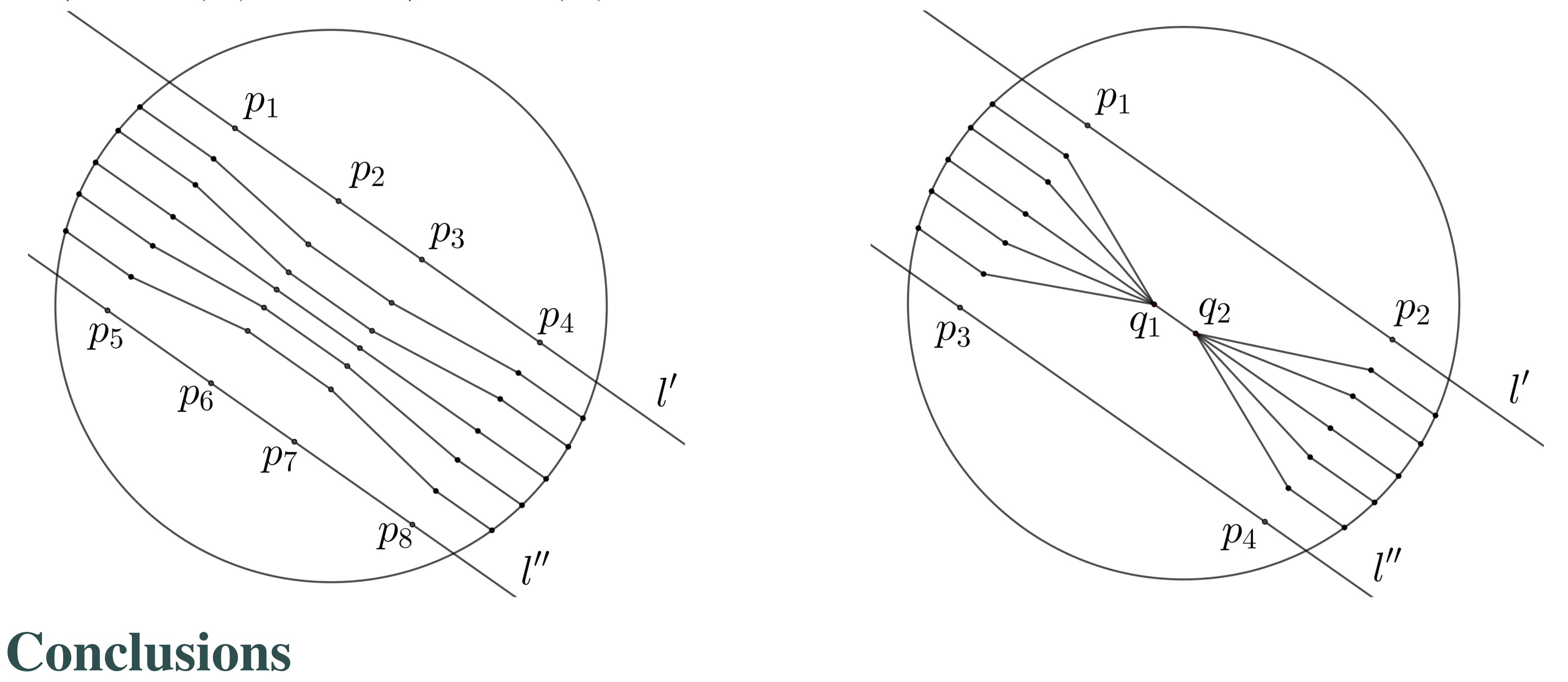
Minimizers with low regularity

As line segments are not the only connected $\phi-$minimal surfaces, we lose uniqueness of minimizers to the anisotropic least gradient problem. Firstly, we have to find a large class of $\phi-$minimal surfaces (we do not intend to find all of them, as they may form various types of singularities, see for instance [2]). The next Proposition gives such a class.

Proposition 4. *Let $\alpha\nu_0 \in$ int I, where I is a flat part of the boundary of $B_\phi(0,1)$. Define N to be an angular neighbourhood of ν_0 such that a positive multiple of every vector in N lies in I. Let F be a set such that its boundary is a piecewise C^1 curve from p_1 to p_2 such that the normal vector to ∂F at each point lies in N. Then, using an approximation in the $\phi-$strict topology by sets whose boundaries are polygonal chains, we see F that is a $\phi-$minimal set.*

We use this result to prove that for each ϕ such that $B_\phi(0,1)$ has flat facets there exist smooth boundary data such that uniqueness of minimizers and the regularity of the "additional" minimizers fails.

Proposition 5. *Suppose that Ω is convex and $B_\phi(0,1)$ is not strictly convex. Then there exists a boundary datum, for which there exist solutions u_1, u_2 to Problem (ALGP) such that $u_1 \notin W^{1,1}(\Omega)$ and $u_2 \notin SBV(\Omega)$.*



Conclusions

In this final Section we state some implications of the obtained regularity estimates for the validity of several known results from the isotropic case also in the anisotropic case.

Theorem 6. *Let Ω be an open bounded strictly convex set with C^1 boundary. Then for $f \in BV(\partial\Omega)$ there exists a solution to Problem (ALGP).*

Theorem 7. *Let Ω be an open bounded convex set. Suppose that $B_\phi(0,1)$ is strictly convex. Let u be a minimizer of Problem (ALGP). Then $u_j = u_c + u_j$, where u_c is continuous, u_j has only jump-type derivative and this decomposition is unique up to an additive constant.*

Funding

The work of the author has been partially supported by the National Science Centre, Poland, research project no. 2017/27/N/ST1/02418.

References

[1] E. Bombieri, E. de Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.
[2] W. Górny, *Planar least gradient problem: existence, regularity and anisotropic case*, Calc. Var. Partial Differential Equations **57** (2018), no. 4, 57-98.
[3] ———, *Least gradient problem with respect to a non-strictly convex norm*, arXiv:1806.01921 (2018).
[4] R.L. Jerrard, A. Moradifam, and A.I. Nachman, *Existence and uniqueness of minimizers of general least gradient problems*, J. Reine Angew. Math. **734** (2018), 71–97.
[5] J.M. Mazon, *The Euler-Lagrange equation for the anisotropic least gradient problem*, Nonlinear Anal. Real World Appl. **31** (2016), 452–472.
[6] P. Rybka and A. Sabra, *The planar least gradient problem in convex domains*, arXiv:1712.07150 (2017).
[7] P. Sternberg, G. Williams, and W.P. Ziemer, *Existence, uniqueness, and regularity for functions of least gradient*, J. Reine Angew. Math. **430** (1992), 35–60.