Least gradient problem on unbounded domains

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Abstract

The least gradient problem, which is related to the planar optimal transport problem, see [4], and to conductivity imaging in medical scans, see [5], is a variational problem of the form

$$\min\left\{ \int_\Omega |Du|, \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \right\}. \tag{LGP}$$

When Ω is an open bounded strictly convex set, it is well known that for continuous boundary data there exists a unique solution and it is continuous up to the boundary. Here, we apply a projection-based method to study existence of solutions for discontinuous boundary data and introduce the least gradient problem on unbounded domains. Moreover, we discuss the anisotropic case and minimise $\int_\Omega \phi(x, Du)$, where $\phi(x, Du)$ is convex, bounded, uniformly elliptic and $1-$homogenous in the second variable.

Introduction

The least gradient problem is typically considered for open bounded strictly convex sets $\Omega \subset \mathbb{R}^N$. Existence, uniqueness and continuity of minimisers for continuous boundary data is given by the Sternberg-Williams-Ziemer construction, see [8]; the idea is to reverse the fact that superlevel sets of a least gradient function are minimal (see [1]). For almost all $t \in \mathbb{R}$, the authors construct a uniquely determined minimal set E_t. Then, they show that $E_t \subset\subset E_s$ for $t > s$, so the sets E_t can be given a meaning as the superlevel sets of a function u, i.e. $E_t = \{u \geq t\}$, and that u constructed this way solves Problem (LGP).

Here, we present two extensions of the above framework, both developed in [3]. Firstly, we prove existence of minimisers to (LGP) for boundary data with a small discontinuity set. Then, we provide an analysis of the least gradient problem on unbounded domains. For clarity, we focus on the isotropic case and refer to the last Section for the anisotropic case.

Existence of minimisers for discontinuous boundary data

In the discontinuous case, we are motivated by the two following results:

- Let $\Omega \subset \mathbb{R}^2$ be an open bounded strictly convex set. If $f \in BV(\partial\Omega)$, then there exists a solution to (LGP) with boundary data f; see [2].
- Let $\Omega = B(0,1) \subset \mathbb{R}^2$. Then there exists a fat Cantor set C such that there is no solution of (LGP) with boundary data $f = \chi_C \in L^\infty(\partial\Omega)$; see [7]. Note that f is discontinuous on a set of positive measure.

The Theorem below is motivated by the above results. It generalises the first one and is optimal in view of the second one. We provide a sketch of the proof, as this type of reasoning is recurring in all the results presented here.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded strictly convex set. Suppose that $f \in L^1(\partial\Omega)$ is a function such that $\mathcal{H}^{N-1}$-almost all points of $\partial\Omega$ are continuity points of f. Then there exists a minimiser $u \in BV(\Omega)$ to* (LGP) *with boundary data f.*

Idea of proof. For simplicity, assume that $\partial\Omega \in C^1$. Take a sequence of approximations $f_n \in C(\partial\Omega)$ such that $f_n \to f$ in $L^1(\partial\Omega)$ and which locally preserves the L^∞ bounds of f.

1. By strict convexity of Ω there exist $u_n \in BV(\Omega) \cap C(\overline{\Omega})$, solutions to Problem (LGP) for boundary data f_n (see [8]). Compactness follows from the Poincaré inequality and we obtain $u_{n_k} \to u$ in $L^1(\Omega)$. As the trace operator on $BV(\Omega)$ is not continuous with respect to weak* convergence, we need to prove that $Tu = f$. For $\delta > 0$ and $x_0 \in \partial\Omega$ which is a point of continuity of f, we have

$$f(x_0) - \delta \leq f(x) \leq f(x_0) + \delta \qquad \text{in } B(x_0, r) \cap \partial\Omega$$

and for sufficiently large n we have

$$f(x_0) - \delta \leq f_n(x) \leq f(x_0) + \delta \qquad \text{in } B(x_0, \frac{r}{2}) \cap \partial\Omega.$$

2. Let H be a tangent hyperplane at x_0 and H_- be the halfspace with boundary H which does not contain Ω. Take $s > 0$ small so that $\Omega' = (H_- + s\nu^H) \cap \Omega \subset\subset B(x_0, \frac{r}{2})$. Fix $t > f(x_0) + \delta$ and let $E_t^n = \{u_n \geq t\}$. By continuity of u_n the functions $\chi_{E_t^n}$ and $\chi_{E_t^n \backslash \Omega'}$ have the same trace. But $E_t^n \backslash \Omega'$ equals E_t^n intersected with some halfspace and this operation strictly decreases the perimeter unless $|E_t^n \cap \Omega'| = 0$, which would contradict the minimality of superlevel sets of least gradient functions (see [1]). Hence $|E_t^n \cap \Omega'| = 0$ and

$$f(x_0) - \delta \leq u_n(x) \leq f(x_0) + \delta \qquad \text{a.e. in } B(x_0, r') \cap \Omega \subset \Omega',$$

and passing to the limit with $\delta \to 0$ we obtain $Tu(x_0) = f(x_0)$. □

Problems arising in the unbounded case

When the domain $\Omega \subset \mathbb{R}^N$ is unbounded, our main interest is again to find the conditions we need to impose on the domain and the boundary data to obtain existence and uniqueness of minimisers. We need to modify our notion of a solution, as objects we will construct need not lie in $BV(\Omega)$, but rather in $BV_{loc}(\Omega)$.

Definition. We say that $u \in BV_{loc}(\Omega)$ is a function of least gradient in Ω if for every function $g \in BV(\Omega)$ with compact support $K \subset \Omega$ we have

$$\int_K |Du| \leq \int_K |D(u+g)|.$$

We say that u solves the least gradient problem on Ω with respect to boundary data $f \in L^1_{loc}(\partial\Omega)$, if both of the following conditions hold:

$$u \text{ is a function of least gradient in } \Omega \quad \textit{and} \tag{ULGP}$$

$$\text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega \text{ we have } ⨍_{B(x,r)\cap\Omega} |u(y) - f(x)| dy \to 0 \text{ as } r \to 0.$$

For bounded domains this is equivalent to Problem (LGP). However, the analysis is different, as continuous functions on $\partial\Omega$ need not be bounded and Sternberg-Williams-Ziemer construction does not work. However, using a suitable approximation we prove existence for any continuous boundary data.

Theorem 2. *Let $\Omega \subsetneq \mathbb{R}^N$ be an open unbounded strictly convex set. Let $f \in C(\partial\Omega)$. Then there exists a minimiser $u \in BV_{loc}(\Omega)$ to* (ULGP) *with boundary data f.*

The proof is based on the appropriate choice of approximations Ω_n of the domain and $f_n \in C(\partial\Omega_n)$ of boundary data using the following facts:

- We set Ω_n to be an increasing sequence of strictly convex sets, so that there exist solutions to (LGP) for continuous boundary data;
- Restrictions of least gradient functions on Ω_n to $\Omega_m \subset \Omega_n$ are again least gradient functions on Ω_m. This gives a compactness result for the approximation;
- We use a projection argument as in the proof of Theorem 1 to prove a local comparison principle and ensure that the trace is correct.

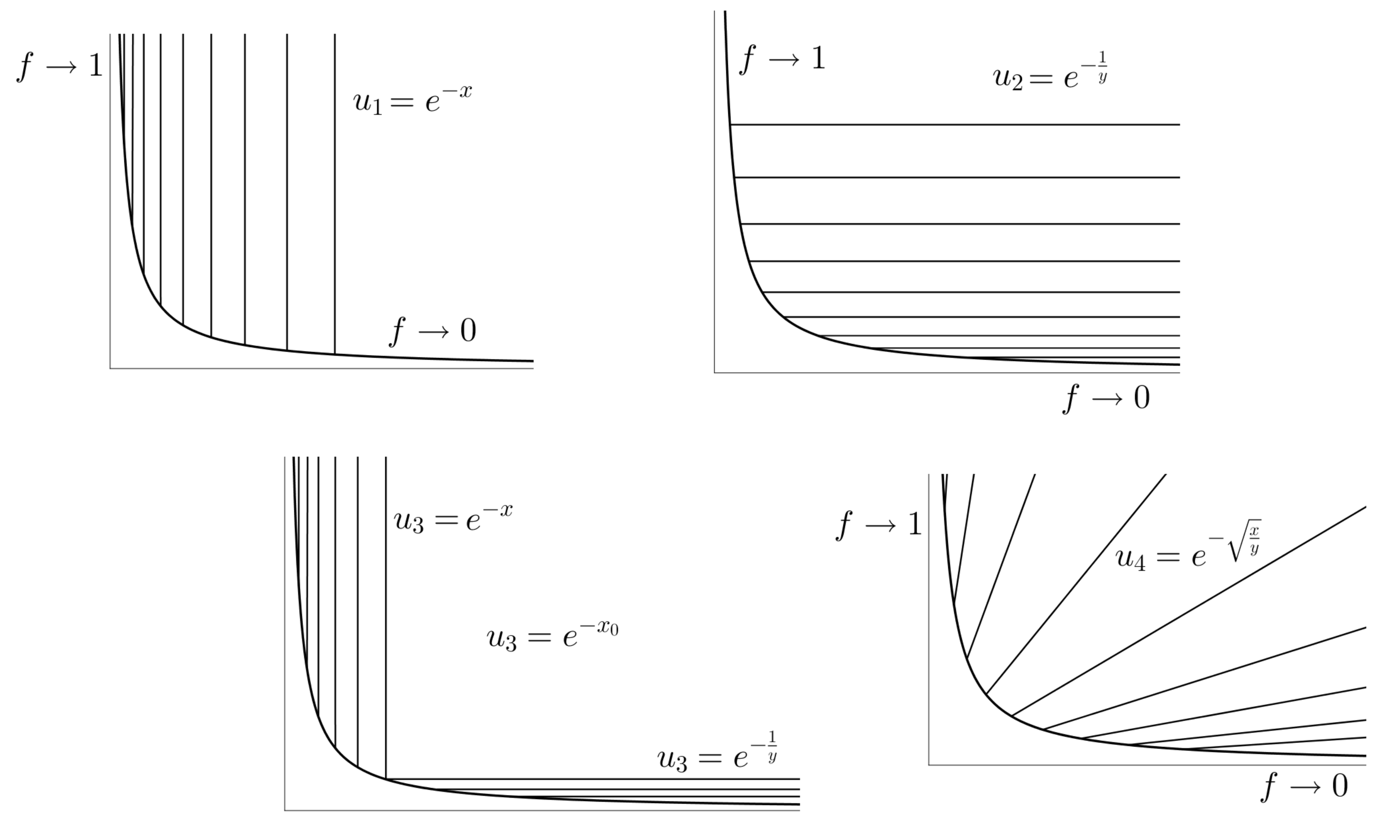
Uniqueness and regularity of unbounded least gradient functions

Uniqueness of minimisers is a more complicated matter and depends on the regularity of the boundary data. When $f \in C_0(\partial\Omega)$, we may obtain uniqueness of minimisers; the key idea is to cut Ω into two parts, so that $\{|u| > \frac{1}{n}\}$ is bounded and $\{|u| \leq \frac{1}{n}\}$ is unbounded, and for $|t| > \frac{1}{n}$ appeal to the uniqueness of the superlevel sets $E_t = \{u \geq t\}$ in the Sternberg-Williams-Ziemer construction. Moreover, a maximum principle for minimal surfaces implies continuity of the solution.

Theorem 3. *Let $\Omega \subsetneq \mathbb{R}^N$ be an open unbounded strictly convex set. Let $f \in C_0(\partial\Omega)$. Then there exists a unique minimiser $u \in BV_{loc}(\Omega) \cap C(\overline{\Omega})$ to* (ULGP) *with boundary data f.*

However, when $f \notin C_0(\partial\Omega)$, then even for monotone boundary data uniqueness of minimisers may fail.

Example. Let $\Omega = \{(x,y) : x > 0, y > 0, xy > 1\} \subset \mathbb{R}^2$. Let $f(x,y) = e^{-x} \in C_b(\partial\Omega)$. Then the following functions are minimisers of (ULGP): $u_1(x,y) = e^{-x}$; $u_2(x,y) = e^{-\frac{1}{y}}$; $u_3(x,y) = e^{-x}\chi_{\{x<x_0\}} + e^{-\frac{1}{y}}\chi_{\{y<\frac{1}{x_0}\}} + e^{-x_0}\chi_{\{\text{else}\}}$; $u_4(x,y) = e^{-\sqrt{\frac{x}{y}}}$.



Anisotropic case

In the classical least gradient problem, when ϕ is the Euclidean norm, existence, regularity, and uniqueness of minimisers depend on the geometry of the set $\Omega \subset \mathbb{R}^N$. In the general case, the situation is slightly more complicated, as we have additionally the interplay between the shape of Ω the structure of ϕ. We are interested in two important cases:

- When ϕ is a strictly convex norm on $\mathbb{R}^N$;
- When ϕ satisfies a Schoen-Simon-Almgren type estimate, which implies that $\phi-$minimal sets have C^2 boundaries apart from a set of dimension $N-3$; see [6]. This assumption is required, as it implies a comparison principle developed in [5]. In particular, this covers the weighted least gradient problem, $\phi(x, Du) = a(x)|Du|$ with $a \in C^2(\overline{\Omega})$.

If ϕ is a strictly convex norm, then we prove Theorems 1 and 2 with a few additional difficulties, such as the need to prove existence of u_n and lack of continuity of u_n. However, due to lack of a maximum principle for ϕ-minimal surfaces, we are unable to prove Theorem 3.

If ϕ depends on location and satisfies a Schoen-Simon-Almgren type estimate, we prove an analogue of Theorem 1 as follows: we may replace the projection argument in the proof of Theorem 1 with a localisation argument and the comparison principle from [5]; however, to obtain existence of u_n, we need a new assumption on Ω instead of strict convexity, called the *barrier condition* (see [5]).

However, as this condition depends on location, we encounter additional problems when we try to prove Theorem 2; the barrier condition boils down to solving a degenerate elliptic equation and we are unable to solve it for general ϕ. It is significantly easier in the case of weighted least gradient problem. Moreover, in the weighted least gradient problem with $a \in C^2(\overline{\Omega})$, we are able to prove an analogue of Theorem 3 using the maximum principle for $\phi-$minimal surfaces developed in [9].

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