The issue of uniqueness of solutions in isotropic and anisotropic least gradient problem

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Motivation

In image processing we encounter problems of minimizing functionals where some summands have only linear growth. The most famous is probably the Rudin-Osher-Fatemi algorithm for denoising a blurred picture, where we have to minimize a functional on $L^2(\Omega)$

$$E(u) = \begin{cases} \int_\Omega |Du| + \int_\Omega \frac{\lambda}{2}(g-u)^2 & \text{if } u \in BV(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Here, we focus on a problem with applications in medical imaging, on which we impose Dirichlet boundary conditions:

$$F(u) = \begin{cases} \int_\Omega |Du| & \text{if } u \in BV(\Omega), Tu = f \\ +\infty & \text{otherwise.} \end{cases}$$

Let us note that the Euler-Lagrange equation corresponding to this functional is the $1-$Laplace equation.

Precise formulation

Consider the variational problem

$$\min\{\int_\Omega |Du| : u \in BV(\Omega), Tu = f\}$$

where $\Omega \subset \mathbb{R}^2$ is an open, nonempty, strictly convex set and $f \in L^1(\partial\Omega)$.

• For $f \in C(\partial\Omega)$ it was proved constructively by Sternberg, Williams, and Ziemer (1992) that there is a unique continuous solution to the above problem.

• For general $f \in L^1(\partial\Omega)$ this problem may have no solutions. An example was first given by Spradlin and Tamasan (2014).

• The functional above is not lower semicontinuous, thus we may not use the usual techniques of the calculus of variations.

Precise formulation

One way to deal with this problem is to find the lower semicontinuous envelope of the functional F, which has minimizers in $L^2(\Omega)$:

$$\overline{F}(u) = \inf\{F(u_n) : u_n \in BV(\Omega), u_n \to u \text{ in } L^1(\Omega)\} =$$

$$= \begin{cases} \int_\Omega |Du| + \int_{\partial\Omega} |Tu - f| d\mathcal{H}^{n-1} & \text{if } u \in BV(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

This approach, using Anzelotti theory was first discussed by Mazón, Rossi, and de León (2004). However, minimizers of $\overline{F}$ are not always minimizers of the original functional F, as they may have trace other than f. This approach only leads to viscosity solutions.

- Question: for which boundary data F attains a minimum?

Existence result

This is partially answered by the following theorem:

Theorem 1 (WG, 2016)

Suppose that $\Omega \subset \mathbb{R}^2$ is an open, nonempty, strictly convex set with C^1 boundary. Let $f \in BV(\partial\Omega)$. Then the minimalization problem

$$\min\{ \int_\Omega |Du| : u \in BV(\Omega), Tu = f \}$$

has at least one solution.

Before we look at the proof, let us see which assumptions are important and which can be relaxed:

• Ω has to be strictly convex. If Ω is only convex, then solutions may not exist even for smooth boundary data. The question of existence is addressed by P. Rybka, A. Sabra and WG (2017).

Existence result

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• The assumptions that $\Omega \subset \mathbb{R}^2$ and $\partial\Omega \in C^1$ allow us to define properly the BV space on $\partial\Omega$ with desired properties, such as approximations by smooth functions in strict topology and that for any set of finite perimeter $E \subset \partial\Omega$ we have $P(E, \partial\Omega) \in \mathbb{N} \cup \{0\}$.

Existence result

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has at least one solution.

- The assumption that $f \in BV(\partial\Omega)$ is somewhat natural; Spradlin and Tamasan (2014) provide an example of a function in $L^\infty(\partial\Omega)$ which is not a trace of a least gradient function. This function is a characteristic function of a fat Cantor set, so it does not lie in $BV(\partial\Omega)$. On the other hand, considering boundary functions with finitely many jumps and intervals of monotonicity, we arrive at the idea of strict convergence of approximations of boundary data.

Existence result

Theorem 1 (WG, 2016)

Suppose that $\Omega \subset \mathbb{R}^2$ is an open, nonempty, strictly convex set with C^1 boundary. Let $f \in BV(\partial\Omega)$. Then the minimalization problem

$$\min\{ \int_\Omega |Du| : u \in BV(\Omega), Tu = f \}$$

has at least one solution.

- The assumption that $f \in BV(\partial\Omega)$ is not a necessary condition for existence of solutions: Sternberg-Williams-Ziemer construction works for all continuous boundary data. There is also an example of a function which has countably many jumps and is not in $BV(\partial\Omega)$, which is a trace of a function of least gradient (WG, 2016).

Existence result

Before we look at the proof, we recall the two underlying principles of dealing with functions of least gradient:

Theorem 2 (Miranda, 1967)

Let u_n be a sequence of least gradient functions in Ω. If $u_n \to u$ in $L^1_{loc}(\Omega)$, then u is also a function of least gradient in Ω.

Theorem 3 (Bombieri, de Giorgi, Giusti, 1969, 1984)

Let u be a function of least gradient in Ω. Then for all t $\chi_{\{u>t\}}$ is also a function of least gradient in Ω, so (in low dimensions) $\partial\{u > t\}$ is an analytical minimal surface.

In dimension two it is exceptionally simple, as the only connected minimal surfaces are intervals.

Existence result

Sketch of proof:

• Find an approximating sequence $f_n \to f$ strictly in $BV(\partial\Omega)$, $f_n \in C^\infty(\partial\Omega) \cap BV(\partial\Omega)$. We apply the Sternberg-Williams-Ziemer construction to f_n and obtain solutions u_n.

• We use the co-area formula to prove that for almost all t we have strict convergence $\chi_{\{f_n \geq t\}} \to \chi_{\{f \geq t\}}$.

• Since $\partial\Omega$ has dimension one, $P(E, \partial\Omega) \in \mathbb{N} \cup \{0\}$ for any set of finite perimeter $E \subset \partial\Omega$. Also $D\chi_E = \pm\delta_{x_i}$.

• In particular, for almost all t the sequence $P(\{f_n \geq t\}, \partial\Omega)$ stabilizes and equals $P(\{f \geq t\}, \partial\Omega) < \infty$ for sufficiently large n; the support of $D\chi_{\{f_n \geq t\}}$ is the finite set $\{x_i^n\} \subset f_n^{-1}(t)$. We prove that $x_i^n \to x_i$.

Existence result

• As $f_n \to f$ in $L^1(\partial\Omega)$, we prove that u_n has a convergent (in $L^1(\Omega)$) subsequence $u_{n_k} \to u$. As u_n were functions of least gradient, by Miranda's theorem u is a function of least gradient.

• We use stabilization of $P(\{f_n \geq t\}, \partial\Omega)$ (and convergence $x_i^n \to x_i$) and the following fact

Proposition 4

Let $\Omega \subset \mathbb{R}^2$ and suppose $u \in BV(\Omega)$ is a function of least gradient. Let $E_t = \{u \geq t\}$. Then for every $t \in \mathbb{R}$ the set ∂E_t is empty or it is a sum of intervals, pairwise disjoint in $\overline{\Omega}$, such that every interval connects two points of $\partial\Omega$.

to prove that $P(\{u_n \geq t\}, \Omega) \to P(\{u \geq t\}, \Omega)$ for almost all t.

Existence result

• We observe that $P(\{u_n \geq t\}, \Omega) \leq P(\Omega, \mathbb{R}^2)$, so we may apply the co-area formula and dominated convergence theorem to prove that $u_{n_k} \to u$ strictly in $BV(\Omega)$.
• The trace operator is continuous with respect to the strict convergence in $BV(\Omega)$, thus

$$Tu = \lim_{k \to \infty} Tu_{n_k} = \lim_{k \to \infty} f_{n_k} = f.$$

As u was (by Miranda's theorem) a function of least gradient, u is a solution of least gradient problem for boundary data $f \in BV(\partial\Omega)$.

Discussion of uniqueness

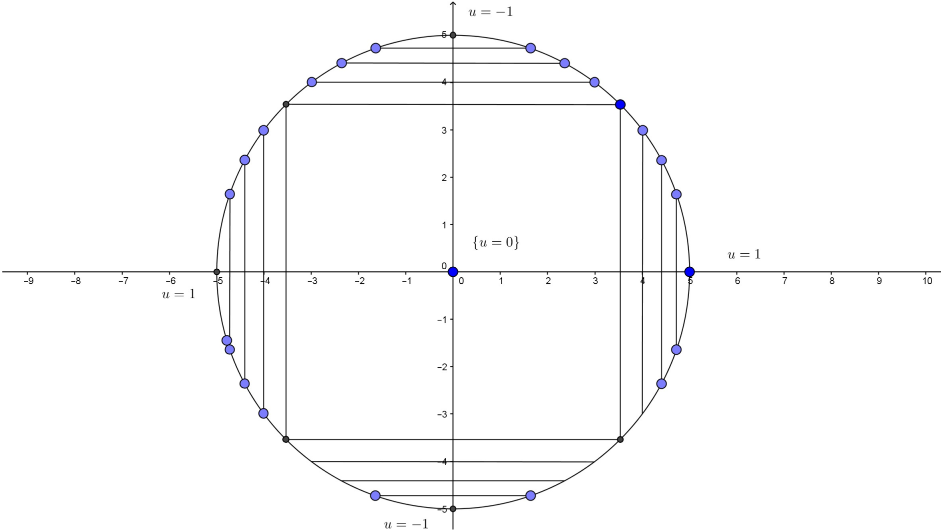
• As proved by Sternberg, Williams and Ziemer (1992), for $f \in C(\partial\Omega)$ the solution u is unique.

• This is not necessarily the case for discontinuous boundary data; the first example was provided by Mazón, Rossi and de León (2004). Consider $\Omega = B(0,1)$ and the function f_0 defined as $f_0(x,y) = x^2 - y^2$. It has zeroes for $(x,y) = (\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$. We add discontinuities at these points, i.e.

$$f(x,y) = \begin{cases} f_0(x,y) + 1 & \text{if } |x| > \frac{1}{\sqrt{2}} \\ f_0(x,y) - 1 & \text{otherwise.} \end{cases}$$

Discussion of uniqueness

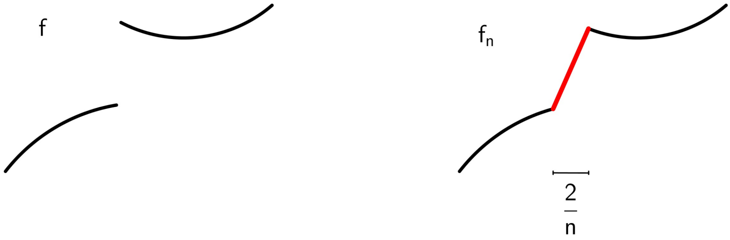
Applying the Sternberg-Williams-Ziemer construction to the continuous function f_0, we obtain the following solution:



Discussion of uniqueness

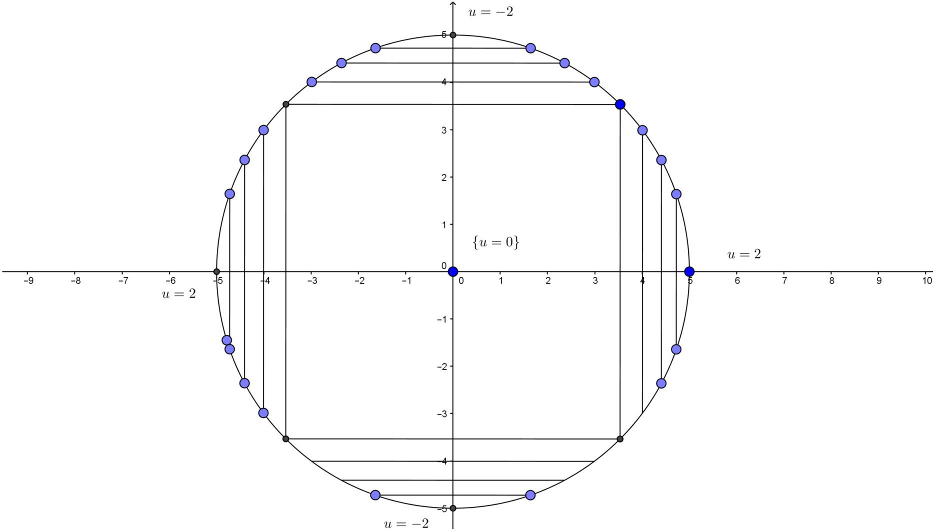
We cannot apply directly the same technique to the discontinuous function f. However, we may find an approximating sequence $f_n \to f$ as in the proof of existence theorem and obtain a solution u_0 as a limit. Let A be the set of discontinuities of f and A_ε its $\varepsilon-$neighbourhood. We choose our sequence f_n in the following way:

$$f_n(x) = \begin{cases} f(x) & \text{if } x \in \partial\Omega \backslash A_{\frac{1}{n}} \\ \text{linear} & \text{if } x \in A_{\frac{1}{n}}. \end{cases}$$



Discussion of uniqueness

The structure of the solution u_0 is as following:



Discussion of uniqueness

Define the functions u_θ in the following way:

$$u_\theta(x,y) = \begin{cases} \theta & \text{if } |x|,|y| < \frac{1}{\sqrt{2}} \\ u_0(x,y) & \text{otherwise.} \end{cases}$$

They have the same trace as u_0 and for $\theta \in [-1,1]$ have the same total variation:

$$|Du_\theta|(\Omega) = |Du_\theta|(\Omega \backslash \square) + 2\sqrt{2}|\theta+1| + 2\sqrt{2}|\theta-1| =$$

$$= |Du_0|(\Omega \backslash \square) + 2\sqrt{2} + 2\sqrt{2} = |Du_0|(\Omega).$$

Thus the functions u_θ are also solutions of least gradient problem for the boundary data f, so the solution is not unique.

Discussion of uniqueness

In general, in dimension two we may prove a following result:

Theorem 5 (WG, 2016)

Let u, v be functions of least gradient in $\Omega \subset \mathbb{R}^2$ such that $Tu = Tv = h$. Assume that we deal with their precise representatives. Then $u = v$ on $\Omega \backslash (C \cup N)$, where both u and v are locally constant on C and N has Hausdorff dimension at most 1.

In particular, it shows that the functions constructed in the previous frame are all solutions to the corresponding least gradient problem. However, the proof relies heavily on the fact that all connected minimal surfaces are intervals, so we need $\Omega \subset \mathbb{R}^2$.

Anisotropic least gradient problem

A continuous function $\phi : \overline{\Omega} \times \mathbb{R}^n \to [0, \infty)$ is called a metric integrand, if it satisfies the following conditions:
(1) ϕ is convex with respect to the second variable for a.e. $x \in \overline{\Omega}$;
(2) ϕ is homogeneous with respect to the second variable, i.e.

$$\forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^n, \quad \forall t \in \mathbb{R} \quad \phi(x, t\xi) = |t|\phi(x, \xi);$$

(3) ϕ is bounded and elliptic in Ω, i.e.

$$\exists \Gamma, \lambda > 0 \quad \forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^n \quad \lambda|\xi| \leq \phi(x, \xi) \leq \Gamma|\xi|.$$

These conditions are sufficient for most of the cases considered in scientific work: they are satisfied for the classical LGP, i.e. $(\phi(x, \xi) = |\xi|)$, as well as for the l_p norms, $p \in [1, \infty]$ and for weighted LGP: $\phi(x, \xi) = g(x)|\xi|$, where $g \geq c > 0$.

Anisotropic least gradient problem

Formal definition of total variation with respect to ϕ is constructed similarly to the usual one, but in case when condition (3) is satisfied we have the following integral representation

Theorem 6 (Amar, Bellettini, 1994)

Let $\varphi : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ be a metric integrand. Then we have an integral representation:

$$\int_\Omega |Du|_\phi = \int_\Omega \phi(x, \nu^u(x))\, |Du|,$$

where ν^u is the Radon-Nikodym derivative $\nu^u = \frac{dDu}{d|Du|}$. In particular, if $E \subset \Omega$ and ∂E is sufficiently smooth, we have a representation

$$P_\phi(E, \Omega) = \int_\Omega \phi(x, \nu_E)\, d\mathcal{H}^{n-1},$$

where ν_E is the external normal to E. □

Anisotropic least gradient problem

The basis of our considerations is the following theorem

Theorem 7 (Jerrard, Moradifam, Nachman, 2015)

Suppose that Ω satisfies the barrier condition with respect to ϕ. Then for $f \in C(\partial\Omega)$ the following problem

$$\min\{\int_\Omega \phi(x, \nu^u(x))\,|Du|,\, u \in BV(\Omega),\ Tu = f\}$$

has at least one solution. If ϕ is smooth away from $\xi = 0$, the solution is unique.

We will consider certain nonsmooth ϕ and discuss uniqueness of solutions.

Nonuniqueness in anisotropic LGP

For $p \in [1, \infty)$ we define the $p-$th norm of a vector on the plane by the formula

$$\|(x, y)\|_p = (|x|^p + |y|^p)^{1/p}$$

and the corresponding metric integrand as

$$\phi_p(x, \xi) = \|\xi\|_p.$$

For $p = \infty$ it is defined as $\phi_\infty(x, \xi) = \|\xi\|_\infty = \sup(|\xi_1|, |\xi_2|)$.

- $\|\cdot\|_1 \geq \|\cdot\|_2 \geq \|\cdot\|_\infty$;
- For $p \in (1, \infty)$ the resulting metric integrand is smooth (except for $\xi = 0$) with respect to the isotropic ($p = 2$) one, whereas for $p = 1, \infty$ is it merely continuous. We will focus on $p = 1$, as $p = \infty$ is analogous.

Nonuniqueness in anisotropic LGP

Proposition 8

[WG, 2016] Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, strictly convex set. Take $\phi(x, Du) = \|Du\|_1$. Let $f \in C(\partial\Omega)$. Denote by u the solution to isotropic LGP for f. Then, if the boundaries of superlevel sets of u are parallel to the axes of the coordinate system, then u is a unique solution of the anisotropic LGP with respect to the l^1 norm.

Let $v \in BV(\Omega)$, $Tv = f$. Then

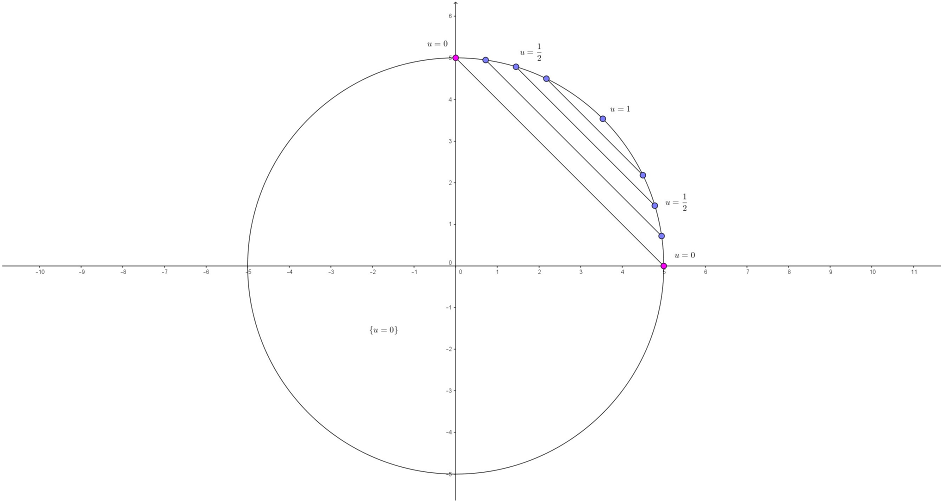
$$\int_\Omega |Dv|_1 \geq \int_\Omega |Dv|_2 \geq \int_\Omega |Du|_2 = \int_\Omega |Du|_1.$$

By uniqueness of solution to Euclidean LGP the second inequality is strict, if only $u \neq v$; thus u is a unique solution to the anisotropic LGP. □

Nonuniqueness in anisotropic LGP

Working with tools such as co-area formula and approximation in strict topology, we may prove the converse: if the boundaries of superlevel sets of u are **not** parallel to the axes of the coordinate system for some t, then the solution to anisotropic LGP is not unique. Instead of a full proof we will focus on an example.

Nonuniqueness in anisotropic LGP



We choose a simple example where $f^{-1}(t)$ consists of only two points. The reason is to have every level set as a curve with boundary point fixed, as $\partial E_t \cap \partial \Omega \subset f^{-1}(t)$.

Nonuniqueness in anisotropic LGP

Sketch of proof:

• Assume we minimize our functional in the domain of C^∞ functions with trace f. If there is a minimizer, by approximation of any BV function in strict topology it is also a minimizer in the space of all BV functions with trace f.

• By Sard theorem for a.e. t the set $\{v = t\}$ is a smooth manifold. We slightly enlarge our domain to all functions such that for a.e. t the set $\{v = t\}$ is a smooth manifold; now we may assume that $\{v = t\}$ contains no closed curves.

• By co-area formula we only have to minimize $P_1(E_t, \Omega)$ with fixed boundary conditions (as $\partial E_t \cap \partial\Omega \subset f^{-1}(t)$) for almost all t.

Nonuniqueness in anisotropic LGP

- Assume we may represent a level set from (x, y) to (z, t) as a graph of a smooth function g, so ∂E_t does not contain vertical intervals. At the point $((s, g(s)))$ the Radon-Nikodym derivative $\nu^{\chi_{E_t}}$ is perpendicular to the level set, so it is a vector $(-\sin\theta, \cos\theta)$, where $g'(s) = \tan\theta$. Thus $\phi(x, \nu^{\chi_{E_t}}) = |\sin\theta| + |\cos\theta|$. As $|D\chi_{E_t}| = \mathcal{H}^{n-1}|_{\partial E_t}$, so in fact we minimize the integral (we may assume that $x < z$):

$$P_1(E_t, \Omega) = \int_\Omega \phi(x, \nu^{\chi_{E_t}})|D\chi_{E_t}| = \int_{\partial E_t} (|\sin\theta| + |\cos\theta|) d\mathcal{H}^{n-1} =$$

$$= \int_x^z (|\sin\theta| + |\cos\theta|)\sqrt{1 + (\tan\theta)^2} dx = \int_x^z (|\sin\theta| + |\cos\theta|)\frac{1}{|\cos\theta|} dx =$$

$$= \int_x^z (1 + |\tan\theta|) dx = |z - x| + \int_x^z |g'| dx \geq |z - x| + |t - y|,$$

where the inequality becomes equality iff g is monotone. Thus there are multiple functions minimizing this integral.

Nonuniqueness in anisotropic LGP

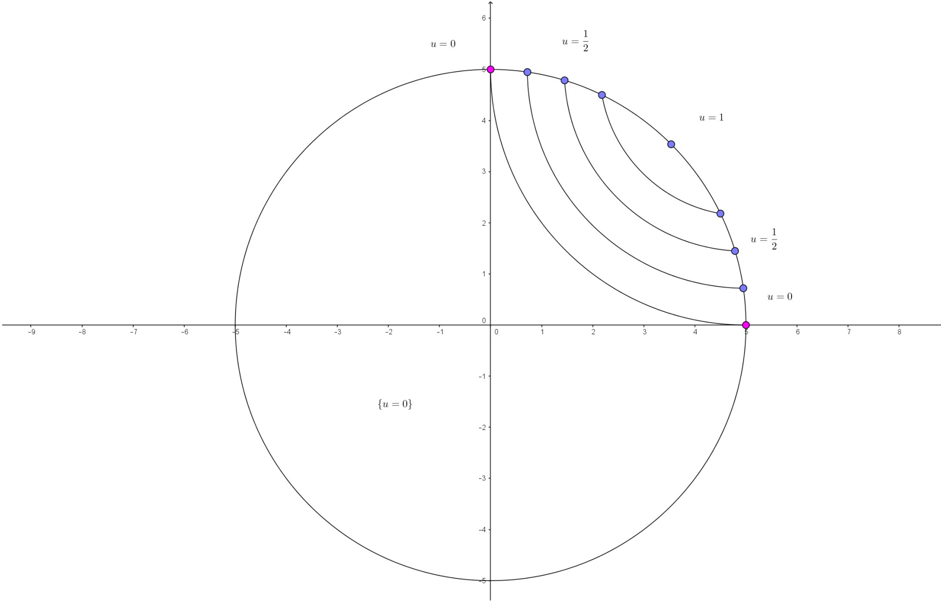
• Modifying the above calculation we cover cases when ∂E_t cannot be represented as a graph.

• Denote by u the solution to Euclidean LGP. Intervals are graphs of monotone functions, so they minimize the above integral. Thus, the value of $\int_\Omega |Dv|_1$ is bounded from below by

$$\int_\Omega |Dv|_1 = \int_{\mathbb{R}} P_1(E_t, \Omega) \geq \int_{\mathbb{R}} P_1(\{u > t\}, \Omega) = \int_\Omega |Du|_1,$$

so by strict approximation this holds for all $v \in BV(\Omega)$ such that $Tv = f$. In particular, the Euclidean solution is a solution to the anisotropic LGP, as are all smooth v s.t. their level sets are represented by graphs of monotone functions.

Nonuniqueness in anisotropic LGP



Conclusions and future work

Conclusions:

• Using suitable approximations of boundary data, we may prove existence of solutions to classical least gradient problem for a larger class of functions; let X be the space of traces of least gradient functions, then

$$C(\partial\Omega) \cup BV(\partial\Omega) \subsetneq X \subsetneq L^1(\Omega)$$

• These solutions have a uniqueness-type property: they can differ only on sets where both solutions are locally constant;

• In the nonsmooth anisotropic case solutions may be not unique even for smooth boundary data.

Future work:

• Characterisation of all solutions to isotropic least gradient problem;

• Analogous existence and uniqueness results in the anisotropic case;

• Extension of these results to $\Omega \subset \mathbb{R}^3$.

Nonuniqueness in anisotropic LGP

Thank you for your attention!

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