Hölder regularity of anisotropic least gradient functions

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Variational Problems in Optical Engineering and Free Material Design,
8 June 2018

Least gradient problem

The least gradient problem is a minimalization problem

$$\min\{\int_\Omega |Du|, \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = g\}.$$

This minimalization problem may be viewed as a formal limit of $p-$Laplace problems for $p \to 1$. Furthermore, in dimension two, for convex domains Ω it is equivalent to the Beckmann problem

$$\min\{\int_\Omega |p| : \quad p \in \mathcal{M}(\Omega; \mathbb{R}^2), \quad \operatorname{div} p = 0, \quad p \cdot n = \frac{\partial g}{\partial \tau}\}.$$

The boundary condition

In the least gradient problem, we may take two approaches to the boundary condition. Firstly, we may consider the relaxation of the total variation functional, namely

$$F(u) = \int_\Omega |Du| + \int_{\partial\Omega} |Tu - g| d\mathcal{H}^{n-1}$$

and search for the minimizers of F. Existence of minimizers has been proved (for $f \in L^1(\partial\Omega)$) by Mazon-Rossi-de Leon (2014) in the isotropic case and Mazon (2016) in the anisotropic case. However, the solution is defined in terms of Anzelotti pairings and the boundary condition is understood in a weaker sense.

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We will assume that $\Omega \subset \mathbb{R}^2$ and that Ω is strictly convex.

Known results in the isotropic case

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• Sternberg-Williams-Ziemer 1992: If Ω is uniformly convex, $\partial\Omega \in C^2$ and $g \in C^{0,\alpha}(\partial\Omega)$, then $u \in C^{0,\alpha/2}(\Omega)$.

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• WG 2016: If $\partial\Omega \in C^1$ and $g \in BV(\partial\Omega)$, then there exists a (not necessarily unique) solution to the least gradient problem with boundary data g.

Anisotropic least gradient problem

In this talk, we are mainly interested in the anisotropic least gradient problem

$$\min\{ \int_\Omega \phi(Du), \quad u \in BV(\Omega), \quad Tu = g\},$$

where ϕ is a norm on $\mathbb{R}^2$ and $g \in C(\partial\Omega)$. Let us stress that $\Omega \subset \mathbb{R}^2$ and Ω is strictly convex. Our goal is to prove a general existence result independent of the regularity of ϕ and some regularity estimates with weaker than usual assumptions on the regularity of $\partial\Omega$.

Barrier condition

For continuous boundary data, the *barrier condition* is sufficient for existence of minimizers (Jerrard-Nachman-Tamasan 2013):

Definition

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Suppose that ϕ is an elliptic metric integrand. We say that Ω satisfies the barrier condition if for every $x_0 \in \partial\Omega$ and sufficiently small $\varepsilon > 0$, if V minimizes $P_\phi(\cdot\,; \mathbb{R}^N)$ in

$$\{W \subset \Omega : W \backslash B(x_0, \varepsilon) = \Omega \backslash B(x_0, \varepsilon)\}$$

then

$$\partial V \cap \partial\Omega \cap B(x_0, \varepsilon) = \emptyset.$$

In the isotropic case $\phi(x, \xi) = \|\xi\|_2$ in dimension two this is equivalent to strict convexity of Ω.

Conditions for uniqueness

Known conditions for uniqueness are more complicated and involve:

• uniform convexity of ϕ;

• high regularity of ϕ (slightly weaker than $W^{3,\infty}$ away from the origin).

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Known results on regularity:

- Dweik-Santambrogio 2018, for uniformly convex Ω with C^2 boundary, then for $p \leq 2$ we have $g \in W^{1,p}(\Omega) \Rightarrow u \in W^{1,p}(\Omega)$. This result is valid for any strictly convex ϕ regardless of its regularity.

Sketch of the reasoning

• Prove that if the unit ball $B_\phi(0,1)$ is strictly convex, we have a similar result to Bombieri-de Giorgi-Giusti, i.e. boundaries of superlevel sets of u are minimal surfaces (unions of line segments);

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- Introduce a proper notion of uniform convexity of Ω and prove a regularity estimate for u in terms of modulus of continuity of g;
- If $B_\phi(0,1)$ is not strictly convex, justify a limit passage with $\phi_n \to \phi$ to prove existence and regularity of a single minimizer of the anisotropic least gradient problem;
- In this case, look at the properties of the other solutions.

Shape of superlevel sets

We use an anisotropic version of the classical BGG theorem:

Proposition

(Mazon, 2016) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Assume that the metric integrand ϕ has a continuous extension to $\mathbb{R}^N$. Take $u \in BV_\phi(\Omega)$. Then u is a function of $\phi-$least gradient in Ω if and only if $\chi_{\{u>t\}}$ is a function of $\phi-$least gradient for almost all $t \in \mathbb{R}$.

Thus, we only need to look at characteristic functions and prove that a boundary of a minimal set (a set such that it is of least gradient relative to some boundary conditions) is a union of line segments.

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Step 3. We use Step 2 to conclude that every connected component of the boundary of a minimal set is a line segment.

Existence and uniqueness

When we know that the line segment is the only minimal set relative to its own boundary data (we do not need the full regularity result), the barrier condition is satisfied. Hence, there exist solutions to the anisotropic least gradient problem for continuous boundary data.

As for uniqueness, when we know that the boundary of a minimal set is a union of line segments, we may imitate the proof of the comparison principle in Sternberg-Williams-Ziemer (1992) and obtain uniqueness of minimizers.

Theorem

Suppose that $\Omega \subset \mathbb{R}^2$ and $B_\phi(0,1)$ are strictly convex. Let $g \in C(\partial\Omega)$. Then there exists a unique solution $u \in C(\overline{\Omega})$ of the anisotropic least gradient problem.

Uniform convexity

Even in the isotropic case, it is known that some form of uniform convexity is required to obtain regularity of minimizers to the least gradient problem. However, the requirements in the literature involve C^2 regularity of $\partial\Omega$. We want to get rid of this assumption.

Definition

We say that the set an open bounded convex set Ω is uniformly convex, if the following condition is satisfied: let $P = \{y \geq ax^2\}$, where $a > 0$. Let $x_0 \in \partial\Omega$ and let l be a supporting line at x_0. Then there exists P', an isometric image of P, tangent to l at x_0 such that $\overline{\Omega} \subset P'$ and $\partial P' \cap \overline{\Omega} = \{x_0\}$.

Regularity of minimizers

Using our line of reasoning, a typical regularity result is of such form:

Proposition

Suppose that $\Omega \subset \mathbb{R}^2$ is uniformly convex and $B_\phi(0,1)$ is strictly convex. Let $g \in C(\partial\Omega)$ and take ω to be its modulus of continuity. Let u be the solution of the anisotropic least gradient problem with boundary data g. Then $u \in C(\overline{\Omega})$ and it is continuous with modulus of continuity

$$\overline{\omega}(|x-y|) = \omega(c(\Omega)|x-y|^{1/2}).$$

In particular, $g \in C^{0,\alpha}(\partial\Omega) \Rightarrow u \in C^{0,\alpha/2}(\Omega)$.

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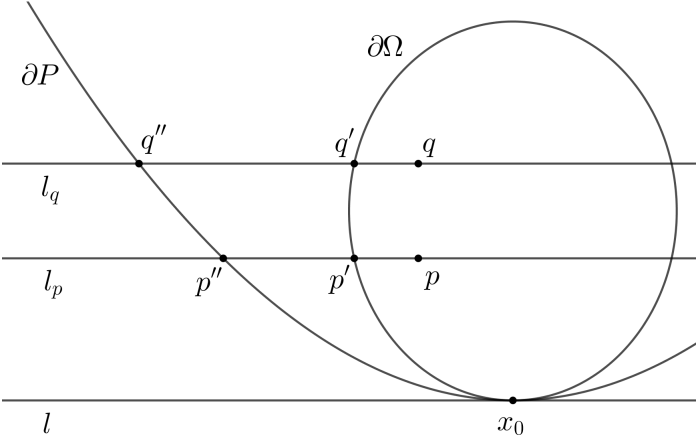
Step 2. We consider the following case:



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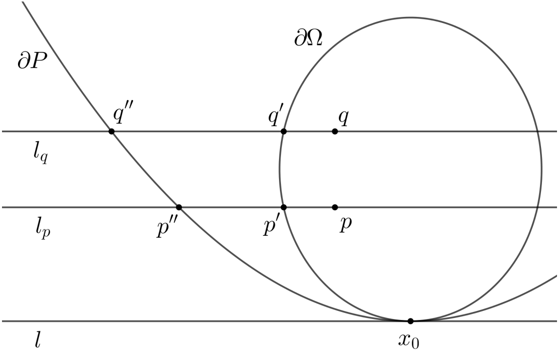

Step 2. We estimate $|u(q) - u(p)| = |u(q') - u(p')| \leq \omega(|q' - p'|) \leq \omega(|q'' - p''|) \leq \omega(c(\Omega)|q - p|^{1/2}) = \overline{\omega}(|q - p|)$.

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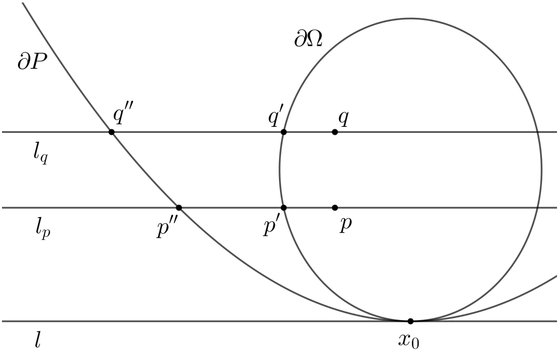

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Step 3. We justify the reduction of the general case to the one in Step 2.

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• Direct expressions for the constant $c(\Omega)$ in terms of the diameter and the lower bound on mean curvature of Ω;

• Works well with slightly different conditions on $\partial\Omega$.

The greatest disadvantage is:

• This approach does not suffice to prove the implication $g \in C^{1,\alpha}(\partial\Omega) \Rightarrow u \in C^{0,(1+\alpha)/2}(\Omega)$.

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• The barrier condition is empty for sets with C^1 boundary, so we may not use it to claim existence of minimizers;

• We may use the regularity results for the strictly convex case and pass to the limit, but the regularity estimates are only valid for one minimizer; there may be multiple minimizers with weaker regularity.

Stability via $\Gamma-$convergence

Take the following functional (a relaxation of the anisotropic total variation functional with respect to Dirichlet boundary data, see Mazon (2016)):

$$F_\phi(v) = \int_\Omega |Dv|_\phi + \int_{\partial\Omega} \phi(\nu^\Omega)|Tv - g| d\mathcal{H}^{n-1}.$$

Proposition

Let ϕ and ϕ_n be anisotropic norms on $\mathbb{R}^n$ such that $\phi_n \to \phi$ pointwise. Then the sequence of functionals F_{ϕ_n} $\Gamma-$converges (with respect to the L^1 convergence) to the functional F_ϕ.

In particular, minimizers of F_{ϕ_n} converge to minimizers of F_ϕ.

Existence, approach 1

Theorem

Let $\Omega \subset \mathbb{R}^2$ be an open bounded uniformly convex set. Suppose that $B_\phi(0,1)$ is not strictly convex and let $g \in C(\partial\Omega)$. Then there exists a solution $u \in C(\overline{\Omega})$ to the anisotropic least gradient problem. Additionally, if ω is the modulus of continuity of g, then u has the same modulus of continuity $\overline{\omega}$ as if $B_\phi(0,1)$ was strictly convex.

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The proof of this fact is very simple: let us take a sequence $\phi_n = \phi + \frac{1}{n} l^2$. We construct minimizers u_n for the anisotropic norm ϕ_n with boundary data g. All the functions u_n admit the same modulus of continuity, so we may pass to the limit in $\overline{\Omega}$; in particular, the trace condition is satisfied.

Existence, approach 2

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The idea of proof is the following: let us take a sequence $\phi_n = \phi + \frac{1}{n} l^2$. Construct minimizers u_n for ϕ_n and g. We obtain uniform bounds in $BV(\Omega)$ for u_n and use the stability theorem to pass to the limit $u_n \to u$. Now, we only have to check that $Tu = g$.

Sketch of proof

Step 1. The set T of such $x \in \partial\Omega$ such that

$$\fint_{B(x,r)\Omega} |u(y) - Tu(x)| dy \to 0$$

when $r \to 0$ is of $\mathcal{H}^1-$full measure. Fix $x \in T$.

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We cut off $\Delta \subset \Omega$ such that $\partial\Delta$ is a line segment connecting the two points of $\partial B(x, \delta_1) \cap \partial\Omega$. We prove a version of a comparison principle to get

$$g(x) - \varepsilon \leq u_n(y) \leq g(x) + \varepsilon \qquad \text{in } B(x, \delta_2) \cap \Omega$$

for some $B(x, \delta_2) \cap \Omega \subset \Delta$. We pass to the pointwise limit with $u_{n_k} \to u$.

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Step 3. We check the mean integral condition in the limit.

Nonuniqueness of minimizers

Let $I \subset \partial B_\phi(0,1)$ be a line segment. We notice that if $\alpha\nu_0 \in \operatorname{int} I$, then there exists a neighbourhood $N \subset S^1$ of ν_0 such that for each $\nu \in N$ a positive multiple of ν, namely $(\nu_0 \cdot \nu)^{-1}\alpha\nu$, lies in I.

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Proposition

Let F be a set such that its boundary is a piecewise C^1 curve from $p_1 \in \partial\Omega$ to $p_2 \in \partial\Omega$ such that the normal vector to ∂F at each point lies in N. Then F is a $\phi-$minimal set.

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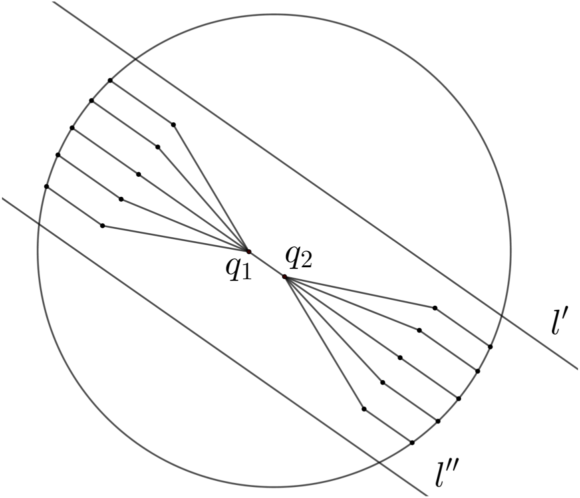
Firstly, we prove it for polygonal chains and then approximate C^1 curves in the strict topology by polygonal chains.

In particular, the barrier condition is not satisfied by any set with C^1 boundary.

Examples of low regularity

$u \notin W^{1,1}(\Omega)$



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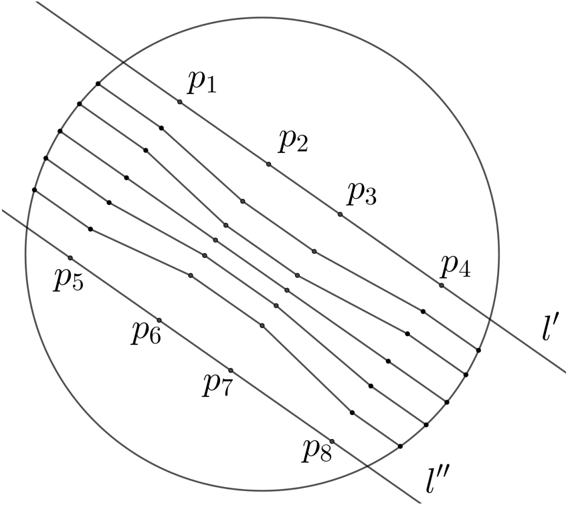

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Thank you for your attention!

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