The least gradient problem with respect to a non-smooth or non-strictly convex norm

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Anisotropic least gradient problem

The anisotropic least gradient problem is a following minimization problem

$$\min\{ \int_\Omega \phi(Du), \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = g \},$$

where ϕ is a norm on $\mathbb{R}^N$. This problem arises (in dimension two) as a dimensional reduction of the Beckmann problem

$$\min\{ \int_\Omega \phi(R_{-\frac{\pi}{2}} p) : \quad p \in \mathcal{M}(\Omega; \mathbb{R}^2), \quad \operatorname{div} p = 0, \quad p \cdot n = \frac{\partial g}{\partial \tau} \}.$$

Anisotropic least gradient problem

We highlight the following two issues:

$$\min\{\int_\Omega \phi(Du), \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = g\}.$$

In general ϕ may additionally depend on location; however, this approach requires C^3 regularity of ϕ and we want to discuss the low-regularity situation.

The boundary condition

The second issue is the boundary condition:

$$\min\{\int_\Omega \phi(Du), \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = g\}.$$

In the least gradient problem, we may take two approaches to the boundary condition. Firstly, we may consider the relaxation of the total variation functional, namely

$$F(u) = \int_\Omega \phi(Du) + \int_{\partial\Omega} \phi(\nu^\Omega)|Tu - g| d\mathcal{H}^{N-1}$$

and search for the minimizers of F. Existence of minimizers has been proved (for $f \in L^1(\partial\Omega)$) by Mazón-Rossi-Segura de León (2014) in the isotropic case and Mazón (2016) in the anisotropic case. However, the solution is defined in terms of Anzelotti pairings and the boundary condition is understood in a weaker sense.

The boundary condition

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We will assume that Ω is strictly convex.

Known results in the isotropic case

A least gradient function is a minimizer of the least gradient problem for some boundary data f.

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• Sternberg-Williams-Ziemer 1992: If Ω is uniformly convex, $\partial\Omega \in C^2$ and $g \in C^{0,\alpha}(\partial\Omega)$, then $u \in C^{0,\alpha/2}(\Omega)$.

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• WG 2016: If $\partial\Omega \in C^1$ and $g \in BV(\partial\Omega)$, then there exists a (not necessarily unique) solution to the least gradient problem with boundary data g.

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In this talk, we consider separately two situations:

Case 1. ϕ is a strictly convex norm.

Case 2. ϕ is a non-strictly convex norm.

In both cases we assume nothing about regularity of ϕ. We are interested mainly in existence of minimizers for discontinuous boundary data. We additionally study regularity of solutions and comment on the structure of minimizers, which in general are not unique.

Barrier condition

For continuous boundary data, the *barrier condition* is sufficient for existence of minimizers (Jerrard-Nachman-Tamasan 2013):

Definition

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Suppose that ϕ is an elliptic metric integrand. We say that Ω satisfies the barrier condition if for every $x_0 \in \partial\Omega$ and sufficiently small $\varepsilon > 0$, if V minimizes $P_\phi(\cdot\,; \mathbb{R}^N)$ in

$$\{W \subset \Omega : W \backslash B(x_0, \varepsilon) = \Omega \backslash B(x_0, \varepsilon)\}$$

then

$$\partial V \cap \partial\Omega \cap B(x_0, \varepsilon) = \emptyset.$$

In the isotropic case $\phi(x, \xi) = \|\xi\|_2$ in dimension two this is equivalent to strict convexity of Ω.

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Let $\Omega \subset \mathbb{R}^N$ be an open bounded strictly convex set. Let $f \in C(\partial\Omega)$. Then for any strictly convex norm ϕ there exists a minimizer to the anisotropic least gradient problem.

2. If ϕ is not strictly convex, then (on the plane) no set with C^1 boundary satisfies the barrier condition - we have to use a different approach.

Existence for discontinuous boundary data

From this, using a projection-based technique we can prove existence of minimizers also for discontinuous boundary data, provided that the discontinuity set is small.

Theorem

Let $\Omega \subset \mathbb{R}^N$ be an open bounded strictly convex set. Let $f \in L^1(\partial\Omega)$ be such that the set of its continuity points is of $\mathcal{H}^{N-1}$-full measure. Then for any strictly convex norm ϕ there exists a minimizer to the anisotropic least gradient problem.

Sketch of proof

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- Take $u_n \in BV(\Omega)$, solutions for boundary data f_n;
- Use Miranda's theorem to prove convergence of u_{n_k} to u in $L^1(\Omega)$, where u is a function of ϕ-least gradient;
- Show that the trace of u equals f. We recall that if u is a function of ϕ-least gradient, then $\chi_{\{u \geq t\}}$ is as well (Mazón 2016).

Sketch of proof that $Tu = f$

- Take a point of continuity $x_0 \in \partial\Omega$ and $r > 0$ such that

$$f(x_0) - \delta \le f(x) \le f(x_0) + \delta \qquad \text{in } B(x_0, r) \cap \partial\Omega.$$

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- If we chose the approximation f_n properly, then

$$f(x_0) - \delta \leq f_n(x) \leq f(x_0) + \delta \qquad \text{in } B(x_0, \frac{r}{2}) \cap \partial\Omega.$$

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- Take a hyperplane H tangent to $\partial\Omega$ at x_0; there exists a parallel hyperplane H' such that

$$H' \cap \Omega \subset B(x_0, \frac{r}{2}) \cap \Omega.$$

Using the barrier condition and the fact that projection decreases the $\phi-$perimeter, we prove that the sets $\partial\{u \geq t\}$ does not intersect H' for $t > f(x_0) + \delta$ and $t < f(x_0) - \delta$.

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• Then, in a neighbourhood of x_0 in Ω which does not depend on n, we have

$$f(x_0) - \delta \leq u_n(x) \leq f(x_0) + \delta$$

and we pass with $n \to \infty$. Hence there exists a ball $B(x_0, \rho)$ such that

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• As for any $\delta > 0$ there exists $B(x_0, \rho)$ as above, we see that

$$\lim_{\rho \to 0} \text{ess sup}_{y \in B(x_0,\rho)\cap\Omega} |u(y) - f(x_0)| = 0,$$

so $Tu(x_0) = f(x_0)$.

Existence for non-strictly convex norm

Using a similar technique, the Theorem can be extended to the non-strictly convex case:

Theorem

Let $\Omega \subset \mathbb{R}^N$ be an open bounded strictly convex set. Let $f \in L^1(\partial\Omega)$ be such that the set of its continuity points is of $\mathcal{H}^{N-1}$-full measure. Then for any norm ϕ there exists a minimizer to the anisotropic least gradient problem.

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We proceed similarly to the proof above; we rely on existence of solutions for discontinuous boundary data for a strictly convex norm $\phi_n = \phi + \frac{1}{n} l_2$. We set $f_n = f$ and u_n to be a solution for the norm ϕ_n.

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We proceed similarly to the proof above; we rely on existence of solutions for discontinuous boundary data for a strictly convex norm $\phi_n = \phi + \frac{1}{n} l_2$. We set $f_n = f$ and u_n to be a solution for the norm ϕ_n.

This is not possible if ϕ is a norm, as the barrier condition changes with n!

Conditions for uniqueness

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We intend to work around this conditions when proving or disproving uniqueness of minimizers.

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We intend to work around this conditions when proving or disproving uniqueness of minimizers.

From now on, we focus on the planar case $\Omega \subset \mathbb{R}^2$.

Uniqueness of minimizers in two dimensions

The situation is much different in our usual two cases:

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2. If ϕ is not strictly convex, there are infinitely many ϕ-minimal surfaces (which we can precisely determine).

Moreover, even for $\Omega = B(0,1)$ there exist boundary data $f \in C^\infty(\partial\Omega)$ such that the solutions are not unique.

Regularity of minimizers

Known results on regularity:

• Dweik-Santambrogio 2018, for uniformly convex Ω with C^2 boundary, then for $p \leq 2$ we have $g \in W^{1,p}(\Omega) \Rightarrow u \in W^{1,p}(\Omega)$. This result is valid for any strictly convex ϕ regardless of its regularity.

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It turns out we can prove an anisotropic analogue of the classical regularity estimate:

Proposition

Suppose that $\Omega \subset \mathbb{R}^2$ is uniformly convex and ϕ is strictly convex. Let $g \in C(\partial\Omega)$ and take ω to be its modulus of continuity. Let u be the solution of the anisotropic least gradient problem with boundary data g. Then $u \in C(\overline{\Omega})$ and it is continuous with modulus of continuity

$$\overline{\omega}(|x-y|) = \omega(c(\Omega)|x-y|^{1/2}).$$

In particular, $g \in C^{0,\alpha}(\partial\Omega) \Rightarrow u \in C^{0,\alpha/2}(\overline{\Omega})$.

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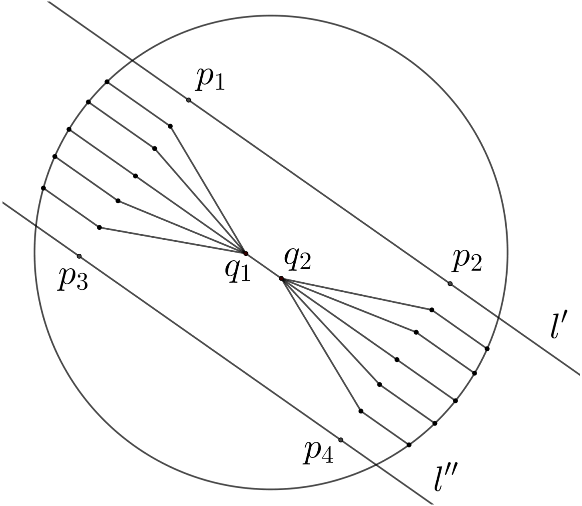
If ϕ is not strictly convex, our regularity estimates are only valid for one minimizer; there may be multiple minimizers with weaker regularity.

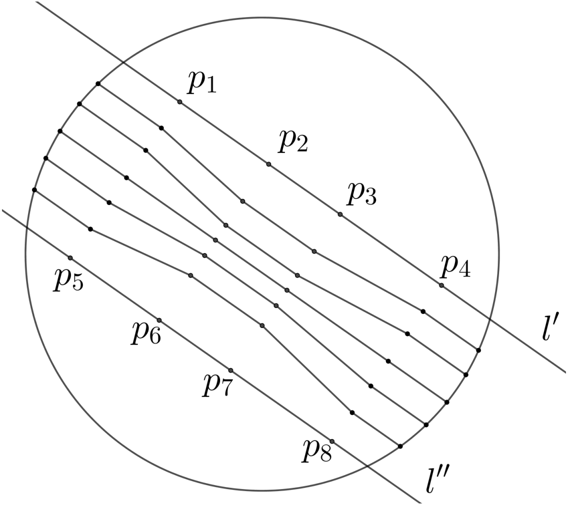
Examples of low regularity

$u \notin W^{1,1}(\Omega)$



Examples of low regularity

$u \notin SBV(\Omega)$



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Thank you for your attention!

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