Optimal transport methods in the least gradient problem

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The least gradient problem

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It is linked to the study of minimal surfaces, but also (among others) to optimal design, conductivity imaging, and optimal transport.

Classical approach

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- Existence of solutions for f continuous $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$;
- Uniqueness and continuity of solutions for $f \in C(\partial\Omega)$;
- Hölder regularity of solutions for Hölder boundary data (no similar result for Sobolev regularity).

Equivalence with the Beckmann problem

Suppose that $\Omega \subset \mathbb{R}^2$ is convex. Then, the least gradient problem is equivalent to the *Beckmann problem* (Rybka, Sabra, G. 2017):

$$\min \left\{ \int_{\overline{\Omega}} |p| : \quad p \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^2), \quad \mathrm{div}\, p = 0, \quad p \cdot \nu|_{\partial\Omega} = g \right\},$$

where $g = \frac{\partial f}{\partial \tau}$, in the following sense: from a solution to the LGP we may construct a solution to the Beckmann problem, and vice versa if the solution to the Beckmann problem gives no mass to the boundary.

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The equivalence is formally given by $p = R_{-\frac{\pi}{2}} Du$.

Equivalence with an optimal transport problem

Again on convex domains, the Beckmann problem is equivalent to the optimal transport problem with source and target measures on $\partial\Omega$:

$$\min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \, d\gamma : \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}), \, (\Pi_x)_\# \gamma = g^+, (\Pi_y)_\# \gamma = g^- \right\}.$$

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From every solution p to the Beckmann problem we can construct a solution to the OTP with transport density σ_γ (and vice versa) and

$$\sigma_\gamma = |p|.$$

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- Because $\sigma_\gamma = |p| = |Du|$, L^p estimates for σ_γ correspond to $W^{1,p}$ estimates for u. This is (so far) the only way to prove Sobolev regularity of solutions to LGP.

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A modified version of this framework works also in more surprising situations, such as on annuli - which are not contractible.

LGP on annuli

We need to take care of two things. The first one is that when $\partial\Omega$ is not connected, we have equivalence between the Beckmann problem and the following variant of the LGP:

$$\min \left\{ \int_\Omega |Du| : \quad u \in BV(\Omega), \quad \partial_\tau(Tu) = g \right\}.$$

The trace is specified only up to vertical translations on connected components of $\partial\Omega$!

LGP on annuli

The second one is that we need to ensure that a solution to the optimal transport problem gives no mass to the boundary. For this, we need some admissibility conditions for boundary data.

Theorem 1 (Dweik, G.)

Suppose that $\Omega \subset \mathbb{R}^2$ is an annulus. Under certain admissibility conditions on $f \in BV(\partial\Omega)$, there exists a solution $u \in BV(\Omega)$ to the least gradient problem. Moreover:

- *If one of $(\partial_\tau f)^\pm$ is atomless, the solution is unique;*
- *If $f \in W^{1,p}(\partial\Omega)$, then $u \in W^{1,p}(\Omega)$.*

Thank you for your attention!