Geometric aspects of the 1-Laplacian

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The least gradient problem

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Take $f \in L^1(\partial\Omega)$. Consider the following minimisation problem:

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Its Euler-Lagrange equation is the 1-Laplace equation

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Relationship to minimal surfaces

$$\int_\Omega |Du| \leq \int_\Omega |D(u+g)| \text{ for all } g \in BV_0(\Omega)$$

$$\Leftrightarrow$$

$$\text{for all } t \in \mathbb{R} \ \int_\Omega |D\chi_{\{u>t\}}| \leq \int_\Omega |D(\chi_{\{u>t\}} + g)| \text{ for all } g \in BV_0(\Omega).$$

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So, $\partial\{u > t\}$ are area-minimising $\rightarrow$ regularity theory for $\partial\{u > t\}$.

Geometry of the domain

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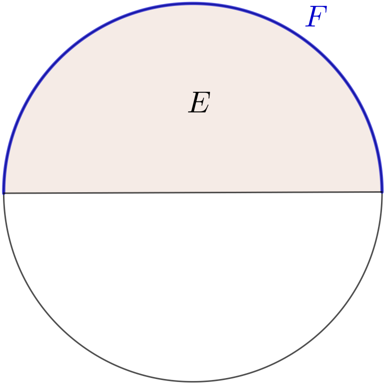
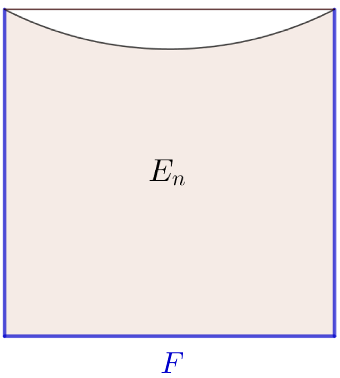
If $u = \chi_E$ and $f = \chi_F$, the problem has a clear geometrical meaning:





It heavily depends on the geometry of the domain!

Classical approach

$$\min\left\{\int_{\Omega}|Du|,\quad u\in BV(\Omega),\quad u|_{\partial\Omega}=f\right\}$$

The main difficulty is that the trace operator is not continuous with respect to weak* convergence in BV; hence, we need to prove estimates on u near the boundary using the tools of geometric measure theory.

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Out of range of classical methods:

- Weak differentiability of solutions;
- Stability of families of solutions.

Equivalence with the Beckmann problem

Suppose that $\Omega \subset \mathbb{R}^2$ is convex. Then, the least gradient problem is equivalent to the *Beckmann problem* (Rybka, Sabra, G. 2017):

$$\min \bigg\{ \int_{\overline{\Omega}} |p| : \quad p \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^2), \quad \mathrm{div}\, p = 0, \quad p \cdot \nu|_{\partial\Omega} = g \bigg\},$$

where $g = \frac{\partial f}{\partial \tau}$, in the following sense: from a solution to the LGP we may construct a solution to the Beckmann problem, and vice versa if the solution to the Beckmann problem gives no mass to the boundary.

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The equivalence is formally given by $p = R_{-\frac{\pi}{2}} Du$.

Equivalence with an optimal transport problem

Again on convex domains, the Beckmann problem is equivalent to the optimal transport problem with source and target measures on $\partial\Omega$:

$$\min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \, d\gamma : \, \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}), \, (\Pi_x)_\# \gamma = g^+, (\Pi_y)_\# \gamma = g^- \right\}.$$

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From every solution p to the Beckmann problem we can construct a solution to the OTP with transport density σ_γ (and vice versa) and

$$\sigma_\gamma = |p|.$$

Use of transport techniques in LGP

These two results were put together by Dweik and Santambrogio (2019).

Least gradient problem	Optimal transport
$f \in BV(\partial\Omega)$	$(\partial_\tau f)^\pm \in \mathcal{M}^+(\partial\Omega)$
$\partial\{u > t\}$	transport rays
$Tu = f$	$\sigma_\gamma(\partial\Omega) = 0$
$f \in C(\partial\Omega)$	$(\partial_\tau f)^\pm$ is atomless
$u \in W^{1,p}(\partial\Omega)$	$(\partial_\tau f)^\pm \in L^p(\partial\Omega)$
$u \in W^{1,p}(\Omega)$	$\sigma_\gamma \in L^p(\Omega)$

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- Its L^p norm can be estimated in an intrinsic way, and contribution of each point q_n depends on the L^p norm of f^+ on $\Delta_{q_n} \cap \partial\Omega$;
- We sum up these estimates and get

$$\|\sigma_\gamma\|_{L^p(\Omega)} \leq C\|(\partial_\tau f)^+\|_{L^p(\partial\Omega)}.$$

If $(\partial_\tau f)^-$ is not finitely atomic, we use approximations.

I. Extension to dual problems

Theorem 1 (G. 2021)

Let $\Omega \subset \mathbb{R}^2$ be convex. The dual of the least gradient problem

$$\sup \left\{ \int_{\partial\Omega} [z, \nu] \, f \, d\mathcal{H}^1 : z \in \mathcal{Z} \right\},$$

where $f \in BV(\partial\Omega)$ and

$$\mathcal{Z} = \left\{ z \in L^\infty(\Omega; \mathbb{R}^2), \quad \mathrm{div}(z) = 0, \quad \|z\|_\infty \leq 1 \textit{ a.e. in } \Omega \right\},$$

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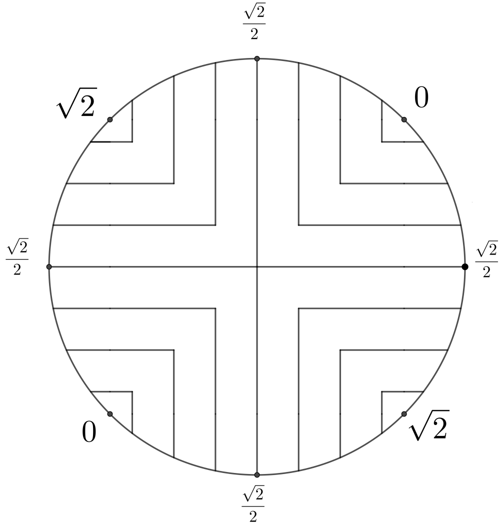
$$f(\theta) = \cos(2\theta); \qquad g(\theta) = -2\sin(2\theta)$$



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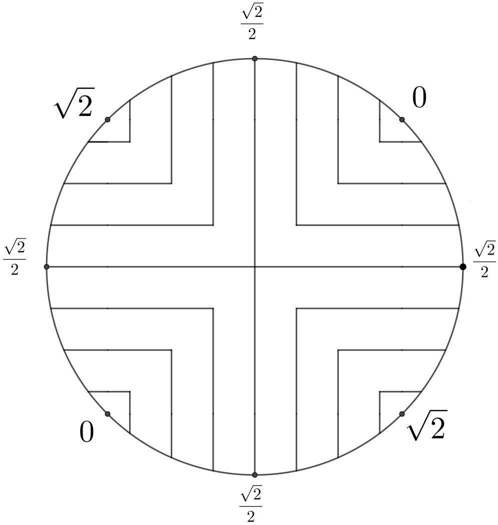

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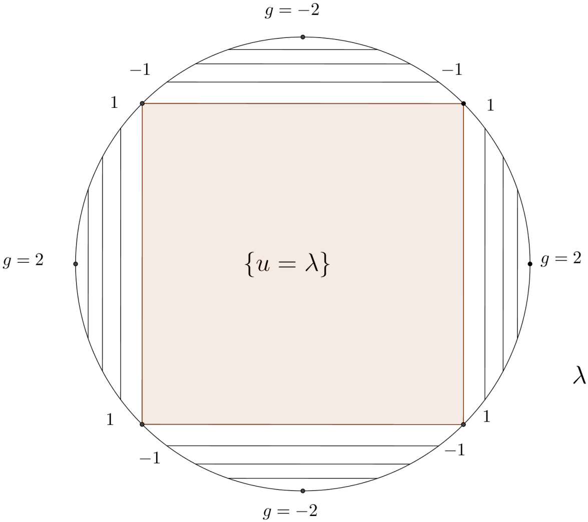

$\lambda \in [-1, 1]$

II. Stability of solutions

Theorem 2 (G. 2021)

Let $\Omega \subset \mathbb{R}^2$ be strictly convex. Suppose that $g_n \to g$ strictly in $BV(\partial\Omega)$. Let $u_n \in BV(\Omega)$ be solutions to LGP with boundary data g_n. Then, there exists $u \in BV(\Omega)$, a solution to problem LGP with boundary data g, such that $u_{n_k} \to u$ strictly in $BV(\Omega)$.

Sketch of proof: Renormalise the sequence g_n to make $(\partial_\tau g_n)^\pm$ probability measures. Use Prokhorov's theorem (on $\overline{\Omega} \times \overline{\Omega}$) for γ_n, optimal transport plans corresponding to u_n, and show that no mass escapes to $\partial\Omega$.

III. SBV regularity

Theorem 3 (G. 2021)

Let $\Omega \subset \mathbb{R}^2$ be uniformly convex. Let $g \in SBV(\partial\Omega)$. If $u \in BV(\Omega)$ is a solution to the least gradient problem, then $u \in SBV(\Omega)$.

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Idea of proof: g corresponds to some optimal transport plan $\overline{\gamma}$.
Split $\overline{\gamma}$ into several parts γ_i and use a similar reasoning as Dweik and Santambrogio for each γ_i. Sum up these estimates and go back to g.

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$$A_2 := \left(\left(\bigcup_{p \in D^+} \Delta_p \right) \setminus A_1 \right) \cup D^+.$$

$$A_3 := \left(\left(\bigcup_{q \in D^-} \Delta_q \right) \setminus A_1 \right) \cup D^-.$$

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For $i = 1, 2, 3$, set

$$B_i := (A_i \cap \partial\Omega) \cap (A_i \cap \partial\Omega).$$

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$$B_4 := (\partial\Omega \times \partial\Omega) \backslash (B_1 \cup B_2 \cup B_3).$$

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do not charge points, so they are absolutely continuous. Hence (DS2019):

$$\sigma_{\gamma_2}, \sigma_{\gamma_3}, \sigma_{\gamma_4} \in L^1(\Omega).$$

Finally, σ_{γ_1} is supported on a set of Hausdorff dimension one, hence σ (so also Du) has no Cantor part.