Weak solutions to the total variation flow in metric measure spaces

Wojciech Górny

University of Vienna, University of Warsaw

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p-Laplacian evolution equation

Consider the model problem

$$\begin{cases} u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) & \text{on } (0,T) \times \mathbb{R}^N; \\ u = u_0 & \text{on } \{0\} \times \mathbb{R}^N. \end{cases}$$

How to formulate this in a metric measure space?

Gradient flow of the Dirichlet energy

One possible way is to consider the energy

$$\Phi(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p$$

well-defined over $L^2(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ and apply the classical semigroup theory (Brezis, Crandall, Komura, ...) to get existence and uniqueness of solutions to the gradient flow

$$u_t + \partial\Phi(u) \ni 0,$$

where $\partial\Phi(u)$ is the subdifferential of Φ.

Metric gradient flows

$(\mathbb{X}, d)$ - complete and separable, ν - Radon measure.

We can define the *Cheeger energy*

$$\mathrm{Ch}_p(u) = \frac{1}{p} \int_{\mathbb{X}} |Du|^p \, d\nu$$

for $u \in L^2(\mathbb{X}, \nu) \cap W^{1,p}(\mathbb{X}, d, \nu)$ and view the p-Laplace equation as its gradient flow in $L^2(\mathbb{X}, \nu)$.

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The whole machinery works (existence, uniqueness, gradient bounds...).

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To study asymptotics.

Basics for analysis on metric spaces

Standard requirements: $(\mathbb{X}, d)$ complete, separable. ν is a Radon measure, which is finite on bounded subsets.

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Derivatives are replaced by upper gradients; we say that g is an upper gradient of u, if for all curves $\gamma : [0, l_\gamma] \to \mathbb{X}$

$$|u(\gamma(l_\gamma)) - u(\gamma(0))| \leq \int_0^{l_\gamma} g(\gamma(t)) \, |\dot{\gamma}(t)| \, dt.$$

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"Minimal g" will be denoted $|Du|$.

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Gigli's construction aims to create a metric analogue of T^*M and TM.

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Define the pre-cotangent module

$$PCM_p = \left\{ \{(f_i, A_i)\} : \quad f_i \in W^{1,p}(\mathbb{X}, d, \nu), \quad \sum_i \|Df_i\|_{L^p(A_i,\nu)} < \infty \right\}$$

with A_i a partition of $\mathbb{X}$ into Borel sets.

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Consider the equivalence relation on PCM_p given by

$$\{(f_i, A_i)\} \sim \{(g_j, B_j)\} \Leftrightarrow |D(f_i - g_j)| = 0 \quad \nu - \text{a.e. on } A_i \cap B_j.$$

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We call the map $|\cdot|_* : PCM_p/\sim \to L^p(\mathbb{X}, \nu)$:

$$|\{(f_i, A_i)\}|_* := |Df_i| \quad \nu - \text{a.e. on } A_i$$

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The closure of $PCM_p/\sim$ with respect to the norm $\| |\{(f_i, A_i)\}|_* \|_{L^p(\mathbb{X},\nu)}$ is called the *cotangent module* $L^p(T^*\mathbb{X})$. It is an L^∞-normed module.

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The map $d : W^{1,p}(\mathbb{X}, d, \nu) \to L^p(T^*\mathbb{X})$ given by

$$df := (f, \mathbb{X})$$

is the *differential*. It is linear and continuous.

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The vector fields are defined via duality:

$$L^q(T\mathbb{X}) := (L^p(T^*\mathbb{X}))^*, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

$X \in L^q(T\mathbb{X})$ is a gradient of f, if

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(In the Euclidean case, we have $X = |\nabla u|^{p-2} \nabla u$.)

Divergence of a vector field

$f \in L^r(\mathbb{X},\nu)$ is the divergence of $X \in L^q(T\mathbb{X})$, if

$$\int_{\mathbb{X}} fg\, d\nu = -\int_{\mathbb{X}} dg(X)\, d\nu$$

for all $g \in W^{1,p}(\mathbb{X},d,\nu) \cap L^{r'}(\mathbb{X},\nu)$. We write $f = \mathrm{div}(X)$.

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These objects are a priori nonlocal!

The p-Laplacian evolution equation

Recall that we study the gradient flow of the Cheeger energy

$$\mathrm{Ch}_p(u) = \frac{1}{p} \int_{\mathbb{X}} |Du|^p \, d\nu.$$

We use the Gigli structure to provide a characterisation of $\partial \mathrm{Ch}_p$.

Theorem (G.-Mazón, JFA 2022)

Let $1 < p < \infty$. We say that $(u, v) \in \mathcal{A}_p$ iff $u \in L^2(\mathbb{X}, \nu) \cap W^{1,p}(\mathbb{X}, d, \nu)$, $v \in L^2(\mathbb{X}, \nu)$, and there exists $X \in L^q(T\mathbb{X})$ with $\mathrm{div}(X) \in L^2(\mathbb{X}, \nu)$ s.t.

$$-\mathrm{div}(X) = v;$$

$$du(X) = |du|_*^p = |X|^q \quad \nu - a.e.$$

Then, $\partial \mathrm{Ch}_p = \mathcal{A}_p$.

Sketch of proof

It is easy to check that $\mathcal{A}_p \subset \partial \mathrm{Ch}_p$. Since $\partial \mathrm{Ch}_p$ is maximal monotone, we need to show that also $\mathcal{A}_p$ is maximal monotone.

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Minty theorem: a monotone operator $\mathcal{A}$ is maximal iff $R(I + \mathcal{A}) = H$.

We need to show that for all $g \in L^2(\mathbb{X}, \nu)$ there exists $u \in D(\mathcal{A}_p)$ and $X \in L^q(T\mathbb{X})$ with $\mathrm{div}(X) \in L^2(\mathbb{X}, \nu)$ such that

$$-\mathrm{div}(X) = g - u;$$

$$du(X) = |du|_*^p = |X|^q \quad \nu - \text{a.e.}$$

We cannot resort to approximations! Instead, we prove this by finding a functional G such that the above is the dual to the minimisation of G.

Back to the gradient flow

Theorem (G.-Mazón, JFA 2022)

For any $u_0 \in L^2(\mathbb{X}, \nu)$ and all $T > 0$ there exists a unique weak solution $u(t)$ of the p-Laplacian evolution equation in the following sense:

There exists $u \in C([0,T]; L^2(\mathbb{X},\nu)) \cap W^{1,2}_{loc}(0,T; L^2(\mathbb{X},\nu))$, $u(0,\cdot) = u_0$, for a.e. $t \in (0,T)$ $u(t) \in W^{1,p}(\mathbb{X}, d, \nu)$, and there exist vector fields $X(t) \in L^q(T\mathbb{X})$ with $\mathrm{div}(X(t)) \in L^2(\mathbb{X},\nu)$ such that

$$\mathrm{div}(X(t)) = u_t(t,\cdot) \quad \textit{in } \mathbb{X};$$

$$du(t)(X(t)) = |du(t)|_*^p = |X(t)|^q \quad \nu\textit{-a.e. in } \mathbb{X}.$$

How to do this for $p = 1$?

We need to replace $W^{1,p}(\mathbb{X}, d, \nu)$ with $BV(\mathbb{X}, d, \nu)$, and the pairing $du(X)$ with the Anzellotti pairing given by

$$\langle (X, Du), f \rangle := -\int_{\mathbb{X}} u \, df(X) \, d\nu - \int_{\mathbb{X}} u \, f \, \mathrm{div}(X) \, d\nu.$$

Here, $f \in \mathrm{Lip}(\mathbb{X})$ has compact support. This formula defines a Radon measure and the condition $du(X) = |du|_*^p = |X|^q$ is replaced with $\|X\|_\infty \leq 1$ and $(X, Du) = |Du|_\nu$.

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The pairing (X, Du) agrees with $du(X)$ for Lipschitz functions and satisfies the 'expected' properties such as the validity of a Gauss-Green formula.

The total variation flow

We understand the TV flow as the gradient flow of the 1-Cheeger energy

$$\mathsf{Ch}_1(u) = \int_{\mathbb{X}} |Du|_\nu.$$

Under a bit more restrictive assumptions on ν, we use the Gigli structure and the new Anzellotti pairing to provide a characterisation of $\partial \mathsf{Ch}_1$.

Theorem (G.-Mazón, JFA 2022)

We say that $(u, v) \in \mathcal{A}_1$ iff $u \in L^2(\mathbb{X}, \nu) \cap BV(\mathbb{X}, d, \nu)$, $v \in L^2(\mathbb{X}, \nu)$, and there exists $X \in L^\infty(T\mathbb{X})$ with $\|X\|_\infty \leq 1$ and $\mathrm{div}(X) \in L^2(\mathbb{X}, \nu)$ s.t.

$$-\mathrm{div}(X) = v;$$

$$(X, Du) = |Du|_\nu \quad \text{as measures.}$$

Then, $\partial \mathsf{Ch}_1 = \mathcal{A}_1$.

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For any $u_0 \in L^2(\mathbb{X}, \nu)$ and all $T > 0$ there exists a unique weak solution $u(t)$ of the total variation flow in the following sense:

There exists $u \in C([0,T]; L^2(\mathbb{X},\nu)) \cap W^{1,2}_{loc}(0,T; L^2(\mathbb{X},\nu))$, $u(0,\cdot) = u_0$, for a.e. $t \in (0,T)$ $u(t) \in BV(\mathbb{X}, d, \nu)$, and there exist vector fields $X(t) \in L^\infty(T\mathbb{X})$ with $\|X\|_\infty \leq 1$ and $\mathrm{div}(X(t)) \in L^2(\mathbb{X},\nu)$ such that

$$\mathrm{div}(X(t)) = u_t(t,\cdot) \quad \textit{in } \mathbb{X};$$

$$(X(t), Du(t)) = |Du(t)|_\nu \quad \textit{as measures on } \mathbb{X}.$$

Extensions: bounded domains

We need to work under assumptions that guarantee existence of a linear and continuous trace operator.

$(u, v) \in \mathcal{A}_{\mathcal{N}}$ (resp. $\mathcal{A}_f$) if and only if $u, v \in L^2(\Omega, \nu)$, $u \in BV(\Omega, d, \nu)$ and there exists a vector field $X \in L^\infty(T\Omega)$ with $\|X\|_\infty \leq 1$ s.t.

$$-\mathrm{div}_0(X) = v \quad \text{in } \Omega;$$

$$(X, Du) = |Du|_\nu \quad \text{as measures;}$$

$$(X \cdot \nu_\Omega)^- = 0 \quad (\text{resp. } (X \cdot \nu_\Omega)^- \in \mathrm{sign}(T_\Omega u - f)) \quad |D\chi_\Omega|_\nu - a.e. \text{ on } \partial\Omega.$$

Extensions: L^1 initial data

If $\nu(\mathbb{X}) < \infty$, then for any $u_0 \in L^1(\mathbb{X}, \nu)$, there exists a unique *entropy solution* of the total variation flow in the following sense:

- $u \in C([0,T]; L^1(\mathbb{X},\nu)) \cap W^{1,2}_{loc}([0,T]; L^1(\mathbb{X},\nu))$;
- $u(0,\cdot) = u_0$;
- For a.e. $t \in [0,T]$ and all $k > 0$ we have $T_k u(t) \in BV(\mathbb{X}, d, \nu)$;
- There exist vector fields $X(t) \in L^\infty(T\mathbb{X})$ with $\mathrm{div}(X(t)) \in L^1(\mathbb{X},\nu)$ and $\|X(t)\|_\infty \le 1$ s.t.

$$\mathrm{div}(X(t)) = u_t(t,\cdot) \quad \text{in } \mathbb{X};$$

$$(X(t), DT_k u(t)) = |DT_k u(t)|_\nu \quad \text{as measures for all } k > 0.$$