A new notion of solutions to gradient flows in metric measure spaces

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Nonuniformly elliptic problems
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p-Laplacian evolution equation

Consider the model problem

$$\begin{cases} u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) & \text{on } (0,T) \times \mathbb{R}^N; \\ u = u_0 & \text{on } \{0\} \times \mathbb{R}^N. \end{cases}$$

How to formulate this in a metric measure space?

Gradient flow of the Dirichlet energy

One possible way is to consider the energy

$$\Phi_p(u) = \begin{cases} \frac{1}{p} \int_\Omega |\nabla u|^p & \text{if } u \in W^{1,p}(\mathbb{R}^N); \\ +\infty & \text{otherwise} \end{cases}$$

well-defined over $L^2(\mathbb{R}^N)$ and apply the classical semigroup theory to get existence and uniqueness of solutions to the gradient flow

$$0 \in u_t + \partial \Phi_p(u),$$

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$$0 \in u_t + \partial \Phi_p(u),$$

where $\partial\Phi$ is the subdifferential of a convex and lower semicontinuous functional Φ, i.e.,

$$\partial\Phi(u) = \Big\{ v \in L^2(\mathbb{R}^N) : \Phi(w) - \Phi(u) \geq v \cdot (w-u) \text{ for all } w \in L^2(\mathbb{R}^N) \Big\}.$$

Metric gradient flows

$(\mathbb{X}, d)$ - complete and separable; ν - Radon, finite on bounded sets.

For $u \in L^2(\mathbb{X}, \nu)$, we can define its *Cheeger energy*

$$\mathrm{Ch}_p(u) = \begin{cases} \frac{1}{p} \int_{\mathbb{X}} |Du|^p \, d\nu & \text{if } u \in W^{1,p}(\mathbb{X}, d, \nu) \\ +\infty & \text{otherwise} \end{cases}$$

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The whole machinery works (existence, uniqueness, gradient bounds...).

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To study asymptotics.

Small detour: 1-Laplacian

For $p = 1$, this equation has the form

$$\begin{cases} u_t = \mathrm{div}\left(\frac{Du}{|Du|}\right) & \text{on } (0,T) \times \mathbb{R}^N; \\ u = u_0 & \text{on } \{0\} \times \mathbb{R}^N. \end{cases}$$

For $p > 1$, we would define weak solutions using test functions.

For $p = 1$, this is not possible because test functions might fail to detect discontinuities of u; we need to define solutions using the subdifferential of the 1-Dirichlet energy (i.e., the total variation).

Small detour: 1-Laplacian

For $p > 1$ and $u \in W^{1,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, we have

$$v \in \partial\Phi_p(u) \quad \Leftrightarrow \quad v = -\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

For $p = 1$ and $u \in BV(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, we have

$$v \in \partial\Phi_1(u) \quad \Leftrightarrow \quad \exists\, \mathbf{z} \in L^\infty(\mathbb{R}^N;\mathbb{R}^N) \text{ s.t. } \|\mathbf{z}\|_\infty \leq 1,$$
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Here, $(\mathbf{z}, Du)$ is the Radon measure defined by

$$\langle (\mathbf{z}, Du), \varphi \rangle := -\int_{\mathbb{R}^N} u\, \varphi\, \mathrm{div}(\mathbf{z})\, dx - \int_{\mathbb{R}^N} u\, \mathbf{z} \cdot \nabla \varphi\, dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$. For $u \in W^{1,1}(\mathbb{R}^N)$, it agrees with $\mathbf{z} \cdot \nabla u\, d\mathcal{L}^N$.

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In order to characterise the subdifferential in the metric setting, we will introduce a multivalued operator similar to the one above (even for $p > 1$).

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"Minimal g" will be denoted $|Du|$.

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Gigli's construction aims to create a metric analogue of T^*M and TM.

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Define the pre-cotangent module

$$PCM_p = \left\{ \{(f_i, A_i)\} : \quad f_i \in W^{1,p}(\mathbb{X}, d, \nu), \quad \sum_i \|Df_i\|^p_{L^p(A_i,\nu)} < \infty \right\}$$

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Consider the equivalence relation on PCM_p given by

$$\{(f_i, A_i)\} \sim \{(g_j, B_j)\} \Leftrightarrow |D(f_i - g_j)| = 0 \quad \nu - \text{a.e. on } A_i \cap B_j.$$

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We call the map $|\cdot|_* : PCM_p/\sim \to L^p(\mathbb{X}, \nu)$:

$$|\{(f_i, A_i)\}|_* := |Df_i| \quad \nu - \text{a.e. on } A_i$$

the pointwise norm on $PCM_p/\sim$.

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The closure of $PCM_p/\sim$ with respect to the norm $\| |\{(f_i, A_i)\}|_* \|_{L^p(\mathbb{X},\nu)}$ is called the *cotangent module* $L^p(T^*\mathbb{X})$. It is an L^∞-normed module.

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The map $d : W^{1,p}(\mathbb{X}, d, \nu) \to L^p(T^*\mathbb{X})$ given by

$$df := (f, \mathbb{X})$$

is the *differential*. It is linear and continuous.

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The vector fields are defined via duality:

$$L^q(T\mathbb{X}) := (L^p(T^*\mathbb{X}))^*, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

$X \in L^q(T\mathbb{X})$ is a p-gradient of f, if

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(In the Euclidean case, we have $X = |\nabla u|^{p-2}\nabla u$.)

Divergence of a vector field

$f \in L^r(\mathbb{X}, \nu)$ is the divergence of $X \in L^q(T\mathbb{X})$, if

$$\int_{\mathbb{X}} fg \, d\nu = - \int_{\mathbb{X}} dg(X) \, d\nu$$

for all $g \in W^{1,p}(\mathbb{X}, d, \nu) \cap L^{r'}(\mathbb{X}, \nu)$. We write $f = \mathrm{div}(X)$.

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These objects are a priori nonlocal! (But, for sufficiently regular spaces, they can be expressed pointwise using the Gromov-Hausdorff tangents.)

The p-Laplacian evolution equation

Recall that we study the gradient flow of the Cheeger energy

$$\mathrm{Ch}_p(u) = \begin{cases} \frac{1}{p}\int_{\mathbb{X}} |Du|^p \, d\nu & \text{if } u \in W^{1,p}(\mathbb{X}, d, \nu); \\ +\infty & \text{otherwise.} \end{cases}$$

We use the Gigli structure to provide a characterisation of $\partial \mathrm{Ch}_p$.

Theorem (G.-Mazón, JFA 2022)

Let $1 < p < \infty$. We say that $(u, v) \in \mathcal{A}_p$ iff $u \in L^2(\mathbb{X}, \nu) \cap W^{1,p}(\mathbb{X}, d, \nu)$, $v \in L^2(\mathbb{X}, \nu)$, and there exists $X \in L^q(T\mathbb{X})$ with $\mathrm{div}(X) \in L^2(\mathbb{X}, \nu)$ s.t.

$$v = -\mathrm{div}(X);$$

$$du(X) = |du|_*^p = |X|^q.$$

Then, $\partial \mathrm{Ch}_p = \mathcal{A}_p$.

Sketch of proof

It is easy to check that $\mathcal{A}_p \subset \partial \mathsf{Ch}_p$. Since $\partial \mathsf{Ch}_p$ is maximal monotone, we need to show that also $\mathcal{A}_p$ is maximal monotone.

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Minty theorem: a monotone operator $\mathcal{A}$ is maximal iff $R(I + \mathcal{A}) = H$.

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Minty theorem: a monotone operator $\mathcal{A}$ is maximal iff $R(I + \mathcal{A}) = H$.

We need to show that for all $g \in L^2(\mathbb{X}, \nu)$ there exists $u \in D(\mathcal{A}_p)$ and $X \in L^q(T\mathbb{X})$ with $\mathrm{div}(X) \in L^2(\mathbb{X}, \nu)$ such that

$$-\mathrm{div}(X) = g - u;$$

$$du(X) = |du|_*^p = |X|^q.$$

We cannot resort to approximations! Instead, we prove this by finding a functional F such that the above is the dual to the minimisation of F.

Sketch of proof

Idea: use the Fenchel-Rockafellar duality theorem.

Let U, V be two Banach spaces and let $A : U \to V$ be a continuous linear operator. Denote by $A^* : V^* \to U^*$ its dual. Then, if the primal problem is of the form

$$\inf_{u \in U} \Big\{ E(Au) + G(u) \Big\}, \tag{P}$$

then the dual problem is defined as the maximisation problem

$$\sup_{p^* \in V^*} \Big\{ - E^*(-p^*) - G^*(A^* p^*) \Big\}, \tag{P*}$$

where E^* and G^* are the Legendre–Fenchel transformations (conjugate functions) of E and G respectively, i.e.,

$$E^*(u^*) := \sup_{u \in U} \left\{ \langle u, u^* \rangle - E(u) \right\}.$$

Sketch of proof

Assume that E and G are proper, convex and lower semi-continuous. If there exists $u_0 \in U$ such that $E(Au_0) < \infty$, $G(u_0) < \infty$ and E is continuous at Au_0, then

$$\inf\,(\mathrm{P}) = \sup\,(\mathrm{P}^*)$$

and the dual problem (P*) admits at least one solution. Moreover, the optimality condition of these two problems is given by

$$A^* \overline{v}^* \in \partial G(\overline{u}), \quad -\overline{v}^* \in \partial E(A\overline{u}),$$

where $\overline{u}$ is solution of (P) and $\overline{v}^*$ is solution of (P*).

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We need to identify the spaces U, V, the functionals E, G and the operator A so that our auxiliary problem fits into this framework.

Sketch of proof

We set

$$U = W^{1,p}(\mathbb{X}, d, \nu) \cap L^2(\mathbb{X}, \nu), \qquad V = L^p(T^*\mathbb{X}),$$

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We set $E : L^p(T^*\mathbb{X}) \to \mathbb{R}$ by the formula

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and $G : W^{1,p}(\mathbb{X}, d, \nu) \cap L^2(\mathbb{X}, \nu) \to \mathbb{R}$ by

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We have $E(0) = 0$, $G(0) = 0$, and E is continuous at 0, so the dual problem has at least one solution $\overline{v}^*$.

Sketch of proof

The first optimality condition is

$$E(A\overline{u}) + E^*(-\overline{v}^*) = \langle -\overline{v}^*, A\overline{u} \rangle$$

which translates to

$$\frac{1}{p}\int_{\mathbb{X}} |d\overline{u}|_*^p \, d\nu + \frac{1}{q}\int_{\mathbb{X}} |-\overline{v}^*|^q \, d\nu = \int_{\mathbb{X}} d\overline{u}(-\overline{v}^*) \, d\nu,$$

so $d\overline{u}(-\overline{v}^*) = |d\overline{u}|_*^p = |-\overline{v}^*|^q$ ν-a.e.

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In particular, $\mathrm{div}(v^*) \in L^2(\mathbb{X}, \nu)$. Since $\partial G(\overline{u}) = \{\overline{u} - g\}$, we get

$$-\mathrm{div}(-\overline{v}^*) = g - \overline{u}.$$

Thus, the pair $(\overline{u}, -\overline{v}^*)$ satisfies the desired conditions, so $\mathcal{A}_p$ satisfies the range condition. Hence, it is maximal monotone, and $\mathcal{A}_p = \partial \mathrm{Ch}_p$.

Back to the gradient flow

Theorem (G.-Mazón, JFA 2022)

For any $u_0 \in L^2(\mathbb{X}, \nu)$ *and all* $T > 0$ *there exists a unique weak solution* $u(t)$ *of the p-Laplacian evolution equation in the following sense:*

- $u \in C([0, T]; L^2(\mathbb{X}, \nu)) \cap W^{1,2}_{loc}(0, T; L^2(\mathbb{X}, \nu))$;
- $u(0, \cdot) = u_0$;
- *For a.e.* $t \in (0, T)$, *we have* $u(t) \in W^{1,p}(\mathbb{X}, d, \nu)$;
- *For a.e.* $t \in (0, T)$, *there exists a vector field* $X(t) \in L^q(T\mathbb{X})$ *with* $\mathrm{div}(X(t)) \in L^2(\mathbb{X}, \nu)$ *such that*

$$\mathrm{div}(X(t)) = u_t(t, \cdot);$$

$$du(t)(X(t)) = |du(t)|_*^p = |X(t)|^q.$$

How to do this for $p = 1$?

We need to replace $W^{1,p}(\mathbb{X}, d, \nu)$ with $BV(\mathbb{X}, d, \nu)$. Assume that ν is doubling and the space satisfies a Poincaré inequality. Given a function $u \in L^1(\mathbb{X}, \nu)$, we set

$$|Du|_\nu(A) = \inf \left\{ \liminf_{n\to\infty} \int_A |Du_n| \, d\nu : \; u_n \in \mathrm{Lip}_{loc}(A), \; u_n \to u \text{ in } L^1(A, \nu) \right\}$$

for any open set $A \subset \mathbb{X}$.

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for any open set $A \subset \mathbb{X}$.

We need to recover a linear structure on the metric measure space in this case. To this end, we define a metric analogue of the Anzellotti pairing as

$$\langle (X, Du), f \rangle := - \int_{\mathbb{X}} u \, df(X) \, d\nu - \int_{\mathbb{X}} u \, f \, \mathrm{div}(X) \, d\nu.$$

Here, $f \in \mathrm{Lip}(\mathbb{X})$ has compact support. This formula defines a Radon measure; (X, Du) agrees with $du(X)$ for Lipschitz functions and satisfies the 'expected' properties such as the validity of a Gauss-Green formula.

The total variation flow

We understand the TV flow as the gradient flow of the 1-Cheeger energy

$$\mathrm{Ch}_1(u) = \begin{cases} \int_{\mathbb{X}} |Du|_\nu & \text{if } u \in BV(\mathbb{X}, d, \nu); \\ +\infty & \text{otherwise.} \end{cases}$$

Under the more restrictive assumptions on ν, we use the Gigli structure and the new Anzellotti pairing to provide a characterisation of $\partial \mathrm{Ch}_1$.

Theorem (G.-Mazón, JFA 2022)

We say that $(u, v) \in \mathcal{A}_1$ iff $u \in L^2(\mathbb{X}, \nu) \cap BV(\mathbb{X}, d, \nu)$, $v \in L^2(\mathbb{X}, \nu)$, and there exists $X \in L^\infty(T\mathbb{X})$ with $\|X\|_\infty \leq 1$ and $\mathrm{div}(X) \in L^2(\mathbb{X}, \nu)$ s.t.

$$-\mathrm{div}(X) = v;$$

$$(X, Du) = |Du|_\nu.$$

Then, $\partial \mathrm{Ch}_1 = \mathcal{A}_1$.

Sketch of proof

Again, we only need to check that $\mathcal{A}_1$ is maximal monotone, i.e. show that for all $g \in L^2(\mathbb{X}, \nu)$ there exists $u \in D(\mathcal{A}_1)$ and $X \in L^\infty(T\mathbb{X})$ with $\|X\|_\infty \leq 1$ and $\mathrm{div}(X) \in L^2(\mathbb{X}, \nu)$ such that

$$-\mathrm{div}(X) = g - u;$$

$$(X, Du) = |Du|_\nu.$$

We again use the Fenchel-Rockafellar duality theorem; we cannot work directly with the BV space, because we need to use the differential d.

Sketch of proof

We need to find a problem of the form

$$\inf_{u \in U} \left\{ E(Au) + G(u) \right\}$$

relevant to our case. We set

$$U = W^{1,1}(\mathbb{X}, d, \nu) \cap L^2(\mathbb{X}, \nu), \qquad V = L^1(T^*\mathbb{X}),$$

and the operator $A : U \to V$ is the differential d.

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We set $E : L^1(T^*\mathbb{X}) \to \mathbb{R}$ by the formula

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and $G : W^{1,1}(\mathbb{X}, d, \nu) \cap L^2(\mathbb{X}, \nu) \to \mathbb{R}$ by

$$G(u) := \frac{1}{2} \int_{\mathbb{X}} u^2 \, d\nu - \int_{\mathbb{X}} ug \, d\nu.$$

As before, the dual problem has at least one solution $\overline{v}^*$.

Sketch of proof

Now, the primal problem does not necessarily have a solution, so we cannot use the optimality conditions. Instead, we use that for any minimising sequence u_n in the primal problem satisfies

$$0 \leq E(Au_n) + E^*(-\overline{v}^*) - \langle -\overline{v}^*, Au_n \rangle \leq \varepsilon_n$$

and

$$0 \leq G(u_n) + G^*(-A^*\overline{v}^*) - \langle -A^*\overline{v}^*, u_n \rangle \leq \varepsilon_n.$$

We find a solution $\overline{u} \in BV(\mathbb{X}, d, \nu) \cap L^2(\mathbb{X}, \nu)$ of the relaxation of the primal problem and use a suitably chosen approximation u_n to deduce that

$$(-\overline{v}^*, Du) = |Du|_\nu \quad \text{and} \quad \| - \overline{v}^*\|_\infty \leq 1$$

and

$$\mathrm{div}(-\overline{v}^*) = g - u.$$

The total variation flow

Theorem (G.-Mazón, JFA 2022)

For any $u_0 \in L^2(\mathbb{X}, \nu)$ *and all* $T > 0$ *there exists a unique weak solution* $u(t)$ *of the total variation flow in the following sense:*

- $u \in C([0, T]; L^2(\mathbb{X}, \nu)) \cap W^{1,2}_{loc}(0, T; L^2(\mathbb{X}, \nu))$;
- $u(0, \cdot) = u_0$;
- *For a.e.* $t \in (0, T)$, *we have* $u(t) \in BV(\mathbb{X}, d, \nu)$;
- *For a.e.* $t \in (0, T)$, *there exists a vector field* $X(t) \in L^\infty(T\mathbb{X})$ *with* $\|X\|_\infty \leq 1$ *and* $\mathrm{div}(X(t)) \in L^2(\mathbb{X}, \nu)$ *such that*

$$\mathrm{div}(X(t)) = u_t(t, \cdot) \quad \text{in } \mathbb{X};$$

$$(X(t), Du(t)) = |Du(t)|_\nu \quad \text{as measures on } \mathbb{X}.$$

L^1 initial data

Theorem (G.-Mazón, ACV 2022)

If $\nu(\mathbb{X}) < \infty$, then for any $u_0 \in L^1(\mathbb{X}, \nu)$, there exists a unique entropy solution $u(t)$ of the total variation flow in the following sense:

- $u \in C([0, T]; L^1(\mathbb{X}, \nu)) \cap W^{1,2}_{loc}(0, T; L^1(\mathbb{X}, \nu))$;
- $u(0, \cdot) = u_0$;
- *For a.e.* $t \in (0, T)$ *and all* $k > 0$, *we have* $T_k u(t) \in BV(\mathbb{X}, d, \nu)$;
- *For a.e.* $t \in (0, T)$, *there exists a vector field* $X(t) \in L^\infty(T\mathbb{X})$ *with* $\mathrm{div}(X(t)) \in L^1(\mathbb{X}, \nu)$ *and* $\|X(t)\|_\infty \leq 1$ *s.t.*

$$\mathrm{div}(X(t)) = u_t(t, \cdot) \quad \textit{in } \mathbb{X};$$

$$(X(t), DT_k u(t)) = |DT_k u(t)|_\nu \quad \textit{as measures for all } k > 0.$$

Comparison principle

Theorem (G.-Mazón, ACV/JFA 2022)

The operators $\mathcal{A}_p$ are completely accretive for $p \in [1, \infty)$. In particular, if u_1 and u_2 are weak solutions to the gradient flow of Ch_p with initial data $u_{1,0}$ and $u_{2,0}$ respectively. For all $r \in [1, \infty]$, if

$$u_{1,0}, u_{2,0} \in L^2(\mathbb{X}, \nu) \cap L^r(\mathbb{X}, \nu),$$

then

$$\|(u_1(t) - u_2(t))^+\|_r \leq \|(u_{1,0} - u_{2,0})^+\|_r.$$

(If $\nu(\mathbb{X}) < \infty$, a similar result is also valid for entropy solutions.)

Asymptotic behaviour

Theorem (G.-Mazón, JFA 2022)

Let $\nu(\mathbb{X}) < \infty$. Assume that a Poincaré inequality holds. Fix $u_0 \in L^2(\mathbb{X}, \nu)$ and let $u(t)$ be the weak solution to the gradient flow of Ch_p. Then:

- *(Finite extinction time) For $1 \leq p < 2$, we have*

$$T_{\mathrm{ex}}(u_0) := \inf\{ T > 0 : \; u(t) = \overline{u_0} \; \forall \, t \geq T \} \\ = T_{\mathrm{ex}}(u_0)(\mathbb{X}, p, \|u_0\|_{L^2(\mathbb{X},\nu)}) < \infty.$$

- *(Infinite extinction time) For $p \geq 2$,*

$$T_{\mathrm{ex}}(u_0) = +\infty.$$

- *Bounds for the L^2 norm of the solution;*
- *Characterisation of asymptotic profiles.*