A double-bubble problem for the ℓ_1 anisotropy

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Classical double-bubble problem

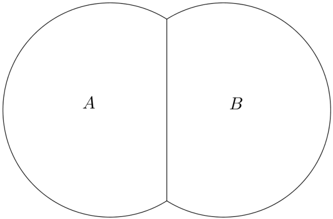
To each configuration (A, B) consisting of two planar finite perimeter sets, we associate the energy

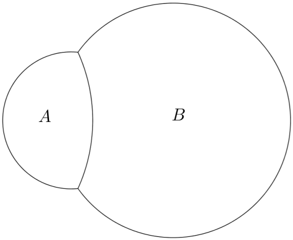
$$E_{\mathrm{db}}(A,B) := \mathcal{H}^1(\partial^* A) + \mathcal{H}^1(\partial^* B) - \mathcal{H}^1(\partial^* A \cap \partial^* B).$$

Given two *volumes* V_A, $V_B > 0$, the *double-bubble problem* is

$$\min \Big\{ E_{\mathrm{db}}(A,B) : \; A \cap B = \emptyset, \; \mathcal{L}^2(A) = V_A \text{ and } \mathcal{L}^2(B) = V_B \Big\}.$$

In 2D, the unique solution are three circle arcs, meeting at angles of $2\pi/3$ at the two meeting points.





ℓ_1 double-bubble problem

To each configuration (A, B) consisting of two planar finite perimeter sets, we associate the energy

$$E(A, B) := \ell_1(\partial^* A) + \ell_1(\partial^* B) + (\eta - 2)\,\ell_1(\partial^* A \cap \partial^* B),$$

where for all $F \subset \partial^* A \cup \partial^* B$ we set

$$\ell_1(F) = \int_F (|\nu_1| + |\nu_2|)\,\mathrm{d}\mathcal{H}^1.$$

Given two *volumes* V_A, $V_B > 0$ and an *interaction intensity* $\eta \in (0, 2)$, the *double-bubble problem* for the ℓ_1 norm is

$$\min \Big\{ E(A, B) : \ A \cap B = \emptyset, \ \mathcal{L}^2(A) = V_A \text{ and } \mathcal{L}^2(B) = V_B \Big\}.$$

What are the possible shapes of minimisers?

Types of minimisers

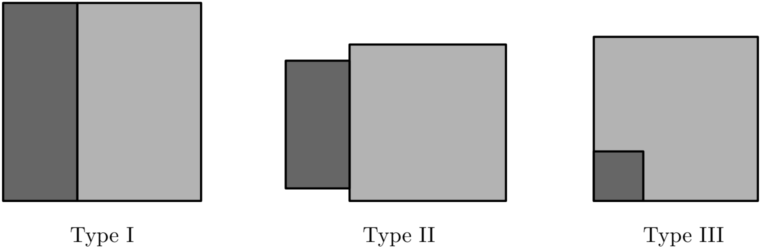


Figure: Types of minimisers for the double-bubble problem

The following shapes were conjectured to be minimisers for $\eta = 1$ in

F. Morgan, C. French, S. Greenleaf, J. Geom. Anal. **8** (1998)

and shown to be the only minimisers (roughly, again for $\eta = 1$) in

P. Duncan, R. O'Dwyer, E. B. Procaccia, J. Geom. Anal. **33** (2023).

Types of minimisers

If we simply compare the energies of the three types and check which of them is smallest, we get the following diagram:

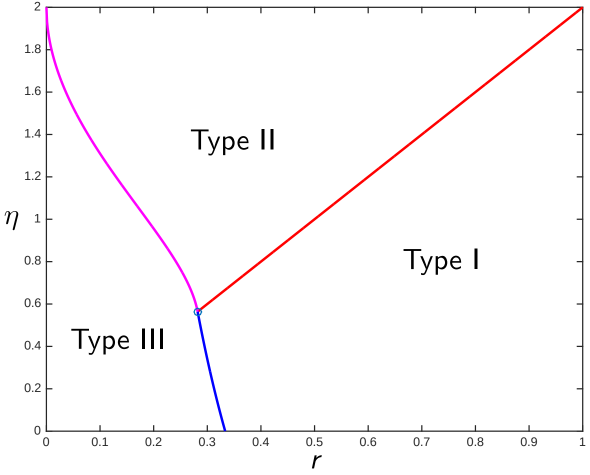


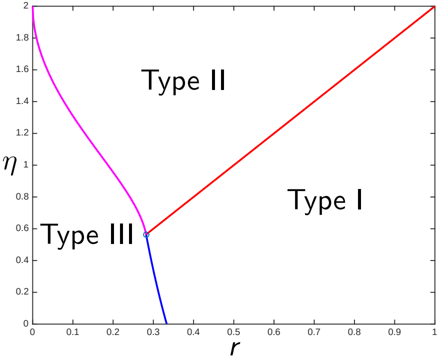
Figure: Types of minimisers for $(r, \eta) \in (0, 1) \times (0, 2)$.

Main result

M. Friedrich, W. Górny, U. Stefanelli, *A characterization of ℓ_1 double bubbles with general interface interaction*, arXiv:2311.07782.

Theorem

For all $r = \frac{V_B}{V_A} \in (0,1]$ and $\eta \in (0,2)$, this picture describes all possible minimisers even if we allow for A and B to be just sets of finite perimeter.



Proof - slicing

Step 1: The 1D energy of each nonempty horizontal slice is at least 2 (for pure slices) or $2 + \eta$ (for mixed slices);

Step 2: By slicing, this allows to give a lower bound in terms of the sizes of vertical and horizontal projections of A and B;

Step 3: An exhaustive check of all possible values of r, η and relative sizes of the projections in coordinate directions yields the result.

Rough idea behind the proof: We need to minimise the 1D energy in every slice, and then figure out how to adjust their sizes and stack them on top of each other in a minimal way.