Duality methods for gradient flows of linear growth functionals

Wojciech Górny

University of Vienna, University of Warsaw

(joint work with J.M. Mazón)

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Problem 1: time-dependent minimal surface equation

Consider the model problem

$$\begin{cases} u_t(t,x) = \operatorname{div}\left(\frac{Du(t,x)}{\sqrt{1+|Du(t,x)|^2}}\right) & \text{in } (0,T)\times\Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

This corresponds to the gradient flow of the functional

$$F(u) = \int_\Omega \sqrt{1+|Du|^2},$$

which has linear growth; how to define a notion of solutions?

Problem 1: time-dependent minimal surface equation

Due to the linear growth of the Lagrangian, the natural energy space for the right-hand side (for fixed time) is $BV(\Omega)$, i.e.

$$BV(\Omega) = \left\{ u \in L^1(\Omega) : \ Du \text{ is a Radon measure} \right\},$$

where Du denotes the distributional gradient of u. It is only a measure, so we need to give meaning to the expression on the right-hand side of

$$u_t(t,x) = \mathrm{div} \left(\frac{Du(t,x)}{\sqrt{1 + |Du(t,x)|^2}} \right)$$

and the expression $\int_\Omega \sqrt{1 + |Du|^2}$ in the functional F.

Problem 1: time-dependent minimal surface equation

For $u \in BV(\Omega)$, write

$$Du = \nabla u \, \mathcal{L}^N + D^s u,$$

i.e. ∇u is the absolutely continuous part of Du and $D^s u$ is its singular part. In place of the functional F, we consider its relaxation $\mathcal{F}$, i.e.

$$\mathcal{F}(u) = \inf \left\{ \liminf_{n \to \infty} F(u_n) : \quad u_n \in W^{1,1}(\Omega), \quad u_n \to u \text{ in } L^1(\Omega) \right\}.$$

A direct computation shows that

$$\mathcal{F}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} \, dx + \int_\Omega |D^s u|.$$

Problem 1: time-dependent minimal surface equation

We now denote

$$a(t,x) = \frac{\nabla u(t,x)}{\sqrt{1 + |\nabla u(t,x)|^2}},$$

which defines $\mathcal{L}^N$-a.e. in Ω the vector field a. Therefore, $\|a(\cdot,t)\|_\infty \leq 1$ and we need $\mathrm{div}(a(\cdot,t)) \in L^2(\Omega)$. In other words, if we denote

$$X_2(\Omega) = \left\{ \mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N) : \mathrm{div}(\mathbf{z}) \in L^2(\Omega) \right\},$$

we necessarily have $a(\cdot,t) \in X_2(\Omega)$.

Problem 1: time-dependent minimal surface equation

Therefore, we understand the equation

$$u_t(t,x) = \mathrm{div}\left(\frac{Du(t,x)}{\sqrt{1+|Du(t,x)|^2}}\right)$$

in the following sense: for $a(t,x) = \frac{\nabla u(t,x)}{\sqrt{1+|\nabla u(t,x)|^2}}$, we have

$$u_t(t) = \mathrm{div}(a(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$a(t) \cdot D^s u(t) = |D^s u(t)| \quad \text{as measures.}$$

The last equation is understood in a suitable weak sense due to Anzellotti. We also need to add the boundary and initial conditions and apply the classical theory of semigroup solutions to get existence of solutions.

Problem 1: time-dependent minimal surface equation

Then, given any $u_0 \in L^2(\Omega)$, there exists a unique weak solution u to the time-dependent MSE in the following sense:

$$u \in C([0,T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega));$$

$$u(0,\cdot) = u_0;$$

and for almost all $t \in (0,T)$ we have $u(t) \in BV(\Omega)$ and there exist vector fields $a(t) \in X_2(\Omega)$ such that the following conditions hold:

$$a(t) = \frac{\nabla u(t,x)}{\sqrt{1 + |\nabla u(t,x)|^2}} \quad \mathcal{L}^N - \text{a.e. in } \Omega;$$

$$u_t(t) = \mathrm{div}(a(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$a(t) \cdot D^s u(t) = |D^s u(t)| \quad \text{as measures};$$

$$[a(t), \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Problem 1: time-dependent minimal surface equation

This definition is based on the ideas of Demengel and Temam (1984), and it was proved by Andreu, Ballester, Caselles and Mazón (1999-2004) that it can be extended to other functionals of linear growth in the following way. For $u_0 \in L^2(\Omega)$, we consider the equation

$$\begin{cases} u_t(t,x) = \mathrm{div}\, a(x, Du(t,x)) & \text{in } \ (0,T) \times \Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \ (0,T) \times \partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \ \Omega, \end{cases}$$

where a is the gradient of a differentiable function with linear growth, i.e. $f \in C^1(\overline{\Omega} \times \mathbb{R}^N)$ and $a(x,\xi) = \partial_\xi f(x,\xi)$. Then, one can give a similar definition of solutions and prove their existence and uniqueness.

Problem 2: total variation flow

Now, consider a different problem

$$\begin{cases} u_t(t,x) = \mathrm{div}\left(\frac{Du(t,x)}{|Du(t,x)|}\right) & \text{in } (0,T)\times\Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

appearing in relation to image processing. It corresponds to the gradient flow of the functional

$$\mathcal{F}(u) = \int_\Omega |Du|.$$

Observe that the corresponding integrand $f(x,\xi) = |\xi|$ is not differentiable, so we cannot apply the previous definition. How to define solutions?

Problem 2: total variation flow

Again, the natural energy space for the right-hand side is $BV(\Omega)$, and we need to give meaning to the expression on the right hand side of

$$u_t(t,x) = \operatorname{div}\left(\frac{Du(t,x)}{|Du(t,x)|}\right).$$

We will replace the expression $\frac{Du}{|Du|}$ by a non-uniquely defined vector field. To be exact, we require that there exists $\mathbf{z} \in X_2(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$ such that

$$u_t(t) = \operatorname{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$(\mathbf{z}(t), Du(t)) = |Du(t)| \quad \text{as measures.}$$

The last equation is again understood in a suitable weak sense due to Anzellotti. We also add the initial and boundary condition.

Problem 2: total variation flow

Then, given any $u_0 \in L^2(\Omega)$, there exists a unique weak solution u to the Neumann problem for the total variation flow in $[0, T]$, i.e.

$$u \in C([0, T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0, T; L^2(\Omega));$$

$$u(0, \cdot) = u_0;$$

and for almost all $t \in (0, T)$ we have $u(t) \in BV(\Omega)$ and there exist vector fields $\mathbf{z}(t) \in X_2(\Omega)$ with $\|\mathbf{z}(t)\|_\infty \leq 1$ such that

$$u_t(t) = \mathrm{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$(\mathbf{z}(t), Du(t)) = |Du(t)| \quad \text{as measures};$$

$$[\mathbf{z}(t), \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Comparison between the two approaches

We can write both problems as gradient flows of

$$\int_\Omega f(x, Du)$$

with f of linear growth (with given initial and boundary conditions).

Time-dependent MSE	Total variation flow
$f(\xi) = \sqrt{1+\|\xi\|^2}$	$f(\xi) = \|\xi\|$
f is differentiable	f is not differentiable at 0
f is "1-homogeneous at infinity"	f is 1-homogeneous
$a = \nabla f$	$\mathbf{z}$ is not explicit (and nonunique)
separate conditions on the absolutely continuous and singular parts	a joint condition

Can we make a joint framework to study both problems?

Working assumptions

For simplicity, we consider the Neumann case.

$$\begin{cases} u_t(t,x) = \mathrm{div}(\partial_\xi f(x, Du(t,x))) & \text{in } (0,T)\times\Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

We assume the following two (quite general) conditions on the integrand.

(A1) $f \in C(\overline{\Omega} \times \mathbb{R}^N)$ is convex in the second variable and has linear growth, i.e. there exists $M > 0$ such that

$$|f(x,\xi)| \leq M(1+|\xi|) \qquad \text{for all } (x,\xi) \in \overline{\Omega}\times\mathbb{R}^N;$$

(A2) The following limit exists:

$$f^0(x,\xi) = \lim_{t\to 0^+} tf(x,\xi/t)$$

and it defines a *recession function* which is jointly continuous in (x,ξ).

Function of a measure

For a function f with linear growth, one may define its action on a Radon measure, which is itself a Radon measure. In the particular case $\mu = Du$, where $u \in BV(\Omega)$, we define the measure $f(x, Du)$ by

$$\int_B f(x, Du) = \int_B f(x, \nabla u(x))\, dx + \int_B f^0\left(x, \frac{dD^s u}{d|D^s u|}\right) d|D^s u|$$

for all Borel sets $B \subset \Omega$ (and similarly for $\mu = D^s u$).

Under the assumptions (A1)-(A2), the functional

$$\mathcal{F}(u) = \int_\Omega f(x, Du)$$

is lower semicontinuous with respect to convergence in $L^1(\Omega)$.

Anzellotti pairings

Definition

For $\mathbf{z} \in X_2(\Omega)$ and $u \in BV(\Omega) \cap L^2(\Omega)$, define the functional $(\mathbf{z}, Du) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (\mathbf{z}, Du), \varphi \rangle := -\int_\Omega u\,\varphi\,\mathrm{div}(\mathbf{z})\,dx - \int_\Omega u\,\mathbf{z} \cdot \nabla\varphi\,dx.$$

The distribution $(\mathbf{z}, Du)$ is a Radon measure, $(\mathbf{z}, Du) \ll |Du|$ and

$$|(\mathbf{z}, Du)| \leq \|\mathbf{z}\|_\infty |Du|.$$

Anzellotti pairings

One can verify that

$$\int_\Omega (\mathbf{z}, Du) = \int_\Omega \mathbf{z} \cdot \nabla u \, dx \qquad \text{for all } w \in W^{1,1}(\Omega),$$

so $(\mathbf{z}, Du)$ agrees on Sobolev functions with the dot product of $\mathbf{z}$ and ∇u.

Moreover, if we set

$$\mathbf{z} \cdot D^s u := (\mathbf{z}, Du) - (\mathbf{z} \cdot \nabla u) \, d\mathcal{L}^N,$$

we have that $\mathbf{z} \cdot D^s u$ is a bounded measure, $\mathbf{z} \cdot D^s u \ll |D^s u|$ and

$$|\mathbf{z} \cdot D^s u| \leq \|\mathbf{z}\|_\infty |D^s u|.$$

Our goal: characterisation of solutions

Theorem

Given $u_0 \in L^2(\Omega)$, there exists a *weak solution* u of the Neumann problem in $[0,T]$, i.e. $u \in C([0,T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega))$, $u(0,\cdot) = u_0$, and for a.e. $t \in (0,T)$ we have $u(t) \in BV(\Omega)$ and there exist vector fields $\mathbf{z}(t) \in X_2(\Omega)$ such that:

$$u_t(t) = \mathrm{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$\mathbf{z}(t) \in \partial_\xi f(x, \nabla u(t)) \quad \mathcal{L}^N - \text{a.e. in } \Omega;$$

$$\mathbf{z}(t) \cdot D^s u(t) = f^0(x, D^s u(t)) \quad \text{as measures;}$$

$$[\mathbf{z}(t), \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Outline of proof

We first introduce a multivalued operator $\mathcal{A}$ on $L^2(\Omega)$ which describes the desired characterisation (details on the next slide). We will prove that it coincides with the subdifferential of $\mathcal{F}$, where

$$\mathcal{F}(u) = \int_\Omega f(x, Du).$$

To this end, we check that we have the inclusion

$$\mathcal{A} \subset \partial \mathcal{F},$$

and in particular $\mathcal{A}$ is monotone. We then show that the range condition holds, i.e.

$$\text{Given } g \in L^2(\Omega),\ \exists\, u \in D(\mathcal{A}) \ s.t. \ \ g \in u + \mathcal{A}(u),$$

so by the Minty theorem $\mathcal{A}$ is maximal monotone. Hence, $\mathcal{A} = \partial \mathcal{F}$. Applying the Brezis-Komura semigroup theory, we get the desired result.

Auxiliary operator

We first introduce the following operator.

Definition

We say that $(u, v) \in \mathcal{A}$ if and only if $u, v \in L^2(\Omega)$, $u \in BV(\Omega)$ and there exists a vector field $\mathbf{z} \in X_2(\Omega)$ such that:

$$-\mathrm{div}(\mathbf{z}) = v \quad \text{in } \mathcal{D}'(\Omega);$$

$$\mathbf{z} \in \partial_\xi f(x, \nabla u) \quad \mathcal{L}^N - \text{a.e. in } \Omega;$$

$$\mathbf{z} \cdot D^s u = f^0(x, D^s u) \quad \text{as measures};$$

$$[\mathbf{z}, \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

We show that it coincides with $\partial\mathcal{F}$. The proof of the inclusion $\mathcal{A} \subset \partial\mathcal{F}$ is simple and follows using Green's formula; we focus on the range condition.

Range condition

The range condition states that

$$\text{For every} \quad g \in L^2(\Omega), \ \exists\, u \in D(\mathcal{A}) \ s.t. \ \ g \in u + \mathcal{A}(u),$$

or equivalently there exists a bounded vector field $\mathbf{z} \in X_2(\Omega)$ such that

$$-\mathrm{div}(\mathbf{z}) = g - u \quad \text{in } \Omega;$$

$$\mathbf{z} \in \partial_\xi f(x, \nabla u) \quad \mathcal{L}^N - \text{a.e. in } \Omega;$$

$$\mathbf{z} \cdot D^s u = f^0(x, D^s u) \quad \text{as measures;}$$

$$[\mathbf{z}, \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

We find such u and $\mathbf{z}$ using the Fenchel-Rockafellar duality theorem for a suitably defined functional.

Sketch of proof

We set $U = W^{1,1}(\Omega) \cap L^2(\Omega)$, $V = L^1(\partial\Omega, \mathcal{H}^{N-1}) \times L^1(\Omega; \mathbb{R}^N)$, and

$$Au = (u|_{\partial\Omega}, \nabla u).$$

Clearly, $A : U \to V$ is linear and continuous. We denote $p = (p_0, \overline{p}) \in V$ and define $E : L^1(\partial\Omega, \mathcal{H}^{N-1}) \times L^1(\Omega; \mathbb{R}^N) \to \mathbb{R}$ as

$$E(p_0, \overline{p}) = E_0(p_0) + E_1(\overline{p}), \quad E_0(p_0) = 0, \quad E_1(\overline{p}) = \int_\Omega f(x, \overline{p})\, dx.$$

We also define $G : W^{1,1}(\Omega) \cap L^2(\Omega) \to \mathbb{R}$ as

$$G(u) := \frac{1}{2} \int_\Omega u^2\, dx - \int_\Omega ug\, dx.$$

Sketch of proof

We compute the objects required to apply the Fenchel-Rockafellar duality theorem. Since the dual of the gradient is minus divergence, we get

$$A^* p^* = -\mathrm{div}(\overline{p}^*).$$

(Hence, $\overline{p}^* \in X_2(\Omega)$.) The functional $E_0^* : L^\infty(\partial\Omega, \mathcal{H}^{N-1}) \to [0, +\infty]$ is given by

$$E_0^*(p_0^*) = \begin{cases} 0 & \text{if } \; p_0^* = 0; \\ +\infty & \text{if } \; p_0^* \neq 0 \end{cases}$$

and $E_1^* : L^\infty(\Omega; \mathbb{R}^N) \to (-\infty, +\infty]$ is given by

$$E_1^*(\overline{p}^*) = \int_\Omega f^*(x, \overline{p}^*) \, dx.$$

Here, E^* denotes the convex conjugate of E, i.e., for any $u^* \in X^*$

$$E^*(u^*) := \sup_{u \in X} \left\{ \langle u, u^* \rangle - E(u) \right\}.$$

Sketch of proof

Consider the energy functional $\mathcal{G} : L^2(\Omega) \to (-\infty, +\infty]$ defined by

$$\mathcal{G}(v) := \begin{cases} \mathcal{F}(v) + G(v) & \text{if } v \in BV(\Omega) \cap L^2(\Omega); \\ +\infty & \text{if } v \in L^2(\Omega) \setminus BV(\Omega). \end{cases}$$

This is an extension of the functional $E \circ A + G$, which is well-defined for functions in $W^{1,1}(\Omega) \cap L^2(\Omega)$, to the whole $L^2(\Omega)$. Since $\mathcal{G}$ is coercive, convex and lower semicontinuous, the primal minimisation problem

$$\min_{v \in L^2(\Omega)} \mathcal{G}(u) = \inf_{v \in W^{1,1}(\Omega) \cap L^2(\Omega)} \left\{ E(Av) + G(v) \right\} \tag{P}$$

admits a solution u.

Sketch of proof

Now, for $u_0 \equiv 0$ we have $E(Au_0) = G(u_0) = 0 < \infty$ and E is continuous at u_0, so by the Fenchel-Rockafellar duality theorem there is no duality gap and the dual problem

$$\sup_{p^* \in L^\infty(\partial\Omega, \mathcal{H}^{N-1}) \times L^\infty(\Omega; \mathbb{R}^N)} \left\{ - E_0^*(-p_0^*) - E_1^*(-\overline{p}^*) - G^*(A^* p^*) \right\} \quad \text{(P*)}$$

admits at least one solution. Moreover, for any minimising sequence u_n for (P) and a maximiser p^* of (P*), we have

$$0 \leq E(Au_n) + E^*(-p^*) - \langle -p^*, Au_n \rangle \leq \varepsilon_n$$

$$0 \leq G(u_n) + G^*(A^* p^*) - \langle u_n, A^* p^* \rangle \leq \varepsilon_n$$

with $\varepsilon_n \to 0$.

Back to the range condition

Since E_0^* takes only values 0 and $+\infty$, for the maximiser we have $E_0^*(-p_0^*) = 0$, from which we infer that

$$[p_0^*, \nu_\Omega] = [-\overline{p}^*, \nu_\Omega] = 0.$$

Moreover, the condition

$$0 \leq G(u_n) + G^*(A^* p^*) - \langle u_n, A^* p^* \rangle \leq \varepsilon_n$$

on the minimising sequences implies that

$$-\mathrm{div}(\overline{p}^*) = A^* p^* \in \partial G(u) = \{u - g\}.$$

Back to the range condition

Finally, the condition

$$0 \leq E(Au_n) + E^*(-p^*) - \langle -p^*, Au_n \rangle \leq \varepsilon_n$$

coupled with the Reshetnyak continuity theorem yields that

$$\int_\Omega f(\cdot, \nabla u)\, dx + \int_\Omega f^*(x, \overline{p}^*)\, dx = \int_\Omega -\overline{p}^* \cdot \nabla u\, dx,$$

so $-\overline{p}^* \in \partial_\xi f(x, \nabla u)$, and

$$\int_\Omega f^0\left(\cdot, \frac{dD^s u}{d|D^s u|}\right) d|D^s u| = \int_\Omega (-\overline{p}^*, Du)^s,$$

so $-\overline{p}^* \cdot D^s u = f^0(x, D^s u)$. Therefore, the range condition is satisfied for the pair $(u, -\overline{p}^*)$, where u is a minimiser of $\mathcal{G}$ and p^* is a solution of the dual problem.

Definition of solutions

Definition

Given $u_0 \in L^2(\Omega)$, we say that u is a *weak solution* of the Neumann problem in $[0, T]$, i.e. $u \in C([0, T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0, T; L^2(\Omega))$, $u(0, \cdot) = u_0$, and for a.e. $t \in (0, T)$ we have

$$0 \in u_t(t, \cdot) + \mathcal{A}u(t, \cdot).$$

In other words, we have $u(t) \in BV(\Omega)$ and there exist vector fields $\mathbf{z}(t) \in X_2(\Omega)$ such that:

$$u_t(t) = \mathrm{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$\mathbf{z}(t) \in \partial_\xi f(x, \nabla u(t)) \quad \mathcal{L}^N - \text{a.e. in } \Omega;$$

$$\mathbf{z}(t) \cdot D^s u(t) = f^0(x, D^s u(t)) \quad \text{as measures;}$$

$$[\mathbf{z}(t), \nu_\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Existence and uniqueness

Since $\mathcal{A} = \partial \mathcal{F}$, we can apply the classical theory of gradient flows of maximal monotone operators and get the following result.

Theorem

For any $u_0 \in L^2(\Omega)$ and all $T > 0$ there exists a unique weak solution of the Neumann problem

$$\begin{cases} u_t(t,x) = \mathrm{div}(\partial_\xi f(x, Du(t,x))) & \text{in } (0,T) \times \Omega; \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } (0,T) \times \partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

A similar result holds for the Dirichlet and Cauchy problems.

Highlight of used techniques

1. The Green formula:

$$\int_\Omega u \,\mathrm{div}(\mathbf{z})\, dx + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u\, [\mathbf{z}, \nu_\Omega]\, d\mathcal{H}^{N-1}.$$

2. Pointwise estimates for the normal trace: the formula

$$[\mathbf{z}, \nu_\Omega](x) = \lim_{\rho\to 0^+} \lim_{r\to 0^+} \frac{1}{2r\omega_{N-1}\rho^{N-1}} \int_{C_{r,\rho}(x,\nu_\Omega(x))} \mathbf{z}(y)\cdot \nu_\Omega(x)\, dy$$

holds for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, where

$$C_{r,\rho}(x,\alpha) := \{\xi \in \mathbb{R}^N \;:\; |(\xi - x)\cdot\alpha| < r,\ |(\xi - x) - [(\xi - x)\cdot\alpha]\alpha| < \rho\}.$$

A similar formula holds for the Radon-Nikodym derivative $\frac{d(\mathbf{z},Du)}{d|Du|}$.

3. Reshetnyak continuity theorem.
4. Fenchel-Rockafellar duality theorem.