Weak solutions to gradient flows in metric measure spaces

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p-Laplacian evolution equation

Consider the model problem

$$\begin{cases} u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) & \text{on } (0,T) \times \mathbb{R}^N; \\ u = u_0 & \text{on } \{0\} \times \mathbb{R}^N. \end{cases}$$

How to formulate this in a metric measure space $(\mathbb{X}, d, \nu)$?

Gradient flow of the Dirichlet energy

One possible way is to consider the energy

$$\Phi(u) = \frac{1}{p} \int_\Omega |\nabla u|^p$$

well-defined over $L^2(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ and apply the classical semigroup theory (Brezis, Crandall, Komura, ...) to get existence and uniqueness of solutions to the gradient flow

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(Done in the metric setting by Ambrosio, Gigli and Savaré.)

L. Ambrosio, N. Gigli, G. Savaré, Rev. Mat. Iberoam. **29** (2013).

L. Ambrosio, N. Gigli, G. Savaré, Invent. Math. **195** (2014).

Metric gradient flows

Standard requirements: $(\mathbb{X}, d)$ complete, separable. ν is a nonnegative Borel measure, which is finite on bounded subsets.

We can define the *Cheeger energy*

$$\mathrm{Ch}_p(u) = \begin{cases} \frac{1}{p} \int_{\mathbb{X}} |Du|^p \, d\nu & \text{if } u \in L^2(\mathbb{X}, \nu) \cap W^{1,p}(\mathbb{X}, d, \nu); \\ +\infty & \text{if } u \in L^2(\mathbb{X}, \nu) \setminus W^{1,p}(\mathbb{X}, d, \nu) \end{cases}$$

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The whole machinery works (existence, uniqueness, gradient bounds...).

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- To study asymptotics.

To this end, we will characterise the subdifferential of Ch_p using a first-order differential structure due to Gigli.

Outline of the talk

1. Analysis in metric spaces
2. L^p-normed modules and differential structure
3. p-Laplacian evolution equation
4. Total variation flow
5. Related results

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Derivatives are replaced by upper gradients; we say that g is an upper gradient of u, if for all curves $\gamma : [0,1] \to \mathbb{X}$

$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_0^1 g(\gamma(t)) \, |\dot{\gamma}(t)| \, dt,$$

where

$$|\dot{\gamma}(t)| = \lim_{s \to 0} \frac{\gamma(t+s) - \gamma(t)}{s}.$$

Basics for analysis on metric spaces

We say that $u \in L^p(\mathbb{X}, \nu)$ lies in the Sobolev space $W^{1,p}(\mathbb{X}, d, \nu)$, if it admits an upper gradient g which lies in $L^p(\mathbb{X}, \nu)$.

For every $u \in W^{1,p}(\mathbb{X}, d, \nu)$, there exists a *minimal p-weak upper gradient* $|Du| \in L^p(\mathbb{X}, \nu)$, i.e., a function which satisfies the property

$$|Du| \leq g \qquad \nu - \text{a.e. for any upper gradient } g \in L^p(\mathbb{X}, \nu)$$

and which is an upper gradient of u up to a negligible set of curves. It is defined uniquely up to a set of measure zero.

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(Important difference with the Euclidean case: $|Du|$ may depend on p! But in this talk we always work with fixed p.)

Basics for analysis on metric spaces

The norm in the Sobolev space $W^{1,p}(\mathbb{X}, d, \nu)$ is given by

$$\|u\|_{W^{1,p}(\mathbb{X},d,\nu)} = \left(\int_{\mathbb{X}} |u|^p \, d\nu + \int_{\mathbb{X}} |Du|^p \, d\nu \right)^{1/p}.$$

The space $W^{1,p}(\mathbb{X}, d, \nu)$ contains Lipschitz function with bounded support and thus it is dense in $L^p(\mathbb{X}, \nu)$.

(However, Lipschitz functions with bounded support are not necessarily dense in the norm topology of $W^{1,p}(\mathbb{X}, d, \nu)$. This requires additional assumptions on $(\mathbb{X}, d, \nu)$; more on this later.)

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Gigli's construction aims to create a metric analogue of T^*M and TM.

N. Gigli, Mem. Amer. Math. Soc. **251** (2018).

V. Buffa, G.E. Comi, M. Miranda Jr., Rev. Mat. Iberoam. **38** (2022).

Outline of the talk

L^p-normed modules

A Banach space M is called an L^∞*-module* (over $L^\infty(\mathbb{X},\nu)$) if there exists a bilinear map from $L^\infty(\mathbb{X},\nu) \times M$ to M given by

$$(f, v) \mapsto f \cdot v,$$

called the *pointwise multiplication*, such that

$$(fg) \cdot v = f \cdot (g \cdot v); \quad 1 \cdot v = v; \quad \|f \cdot v\|_M \leq \|f\|_\infty \|v\|_M,$$

which also satisfies *locality* and *gluing* properties.

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We say that M is an L^p*-normed module*, if there is a nonnegative map $|\cdot|_* : M \to L^p(\mathbb{X},\nu)$ such that

$$\||v|_*\|_{L^p(\mathbb{X},\nu)} = \|v\|_M \quad \text{and} \quad |f\cdot v|_* = |f||v|_* \quad \nu-\text{a.e.}$$

for all $f\in L^\infty(\mathbb{X},\nu)$ and $v\in M$. We call $|\cdot|_*$ the *pointwise norm* on M.

L^p-normed modules

A bounded linear map $T : M \to N$ is a module morphism whenever

$$T(f \cdot v) = f \cdot T(v) \qquad \forall v \in M,\ f \in L^\infty(\mathbb{X}, \nu).$$

$\mathrm{HOM}(M, N)$ is the set of all module morphisms between M and N. It has a canonical structure of an L^∞-module, equipped with the operator norm

$$\|T\| = \sup_{v \in M,\, \|v\|_M \leq 1} \|T(v)\|_N.$$

Since $L^1(\mathbb{X}, \nu)$ has a structure of an L^∞-module, one can define a dual module to M in the following sense:

$$M^* = \mathrm{HOM}(M, L^1(\mathbb{X}, \nu)).$$

Gigli differential structure

Define the pre-cotangent module

$$PCM_p = \left\{ \{(f_i, A_i)\} : \quad f_i \in W^{1,p}(\mathbb{X}, d, \nu), \quad \sum_i \|Df_i\|^p_{L^p(A_i,\nu)} < \infty \right\}$$

with A_i a partition of $\mathbb{X}$ into Borel sets.

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Consider the equivalence relation on PCM_p given by

$$\{(f_i, A_i)\} \sim \{(g_j, B_j)\} \Leftrightarrow |D(f_i - g_j)| = 0 \quad \nu - \text{a.e. on } A_i \cap B_j.$$

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The map $|\cdot|_* : PCM_p/\sim \to L^p(\mathbb{X}, \nu)$:

$$|\{(f_i, A_i)\}|_* := |Df_i| \quad \nu-\text{a.e. on } A_i$$

is the pointwise norm on $PCM_p/\sim$.

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The closure of $PCM_p/\sim$ with respect to the norm $\| |\{(f_i, A_i)\}|_* \|_{L^p(\mathbb{X},\nu)}$ is called the *cotangent module* $L^p(T^*\mathbb{X})$. It is an L^p-normed module.

Gigli differential structure

The map $d : W^{1,p}(\mathbb{X}, d, \nu) \to L^p(T^*\mathbb{X})$ given by

$$df := (f, \mathbb{X})$$

is the *differential*. It is linear and continuous.

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The vector fields are defined via duality:

$$L^q(T\mathbb{X}) := (L^p(T^*\mathbb{X}))^*, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

$X \in L^q(T\mathbb{X})$ is a gradient of f, if

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(In the Euclidean case, we have $X = |\nabla u|^{p-2} \nabla u$.)

Divergence of a vector field

$f \in L^r(\mathbb{X}, \nu)$ is the divergence of $X \in L^q(T\mathbb{X})$, if

$$\int_{\mathbb{X}} fg \, d\nu = -\int_{\mathbb{X}} dg(X) \, d\nu$$

for all $g \in W^{1,p}(\mathbb{X}, d, \nu) \cap L^{r'}(\mathbb{X}, \nu)$. We write $f = \mathrm{div}(X)$.

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These objects are a priori nonlocal!

Outline of the talk

The p-Laplacian evolution equation

Recall that we study the gradient flow of the Cheeger energy

$$\mathrm{Ch}_p(u) = \frac{1}{p} \int_{\mathbb{X}} |Du|^p \, d\nu.$$

We use the Gigli structure to provide a characterisation of $\partial \mathrm{Ch}_p$.

Theorem (G.-Mazón, JFA 2022)

Let $1 < p < \infty$. We say that $(u, v) \in \mathcal{A}_p$ iff $u \in L^2(\mathbb{X}, \nu) \cap W^{1,p}(\mathbb{X}, d, \nu)$, $v \in L^2(\mathbb{X}, \nu)$, and there exists $X \in L^q(T\mathbb{X})$ with $\mathrm{div}(X) \in L^2(\mathbb{X}, \nu)$ s.t.

$$-\mathrm{div}(X) = v;$$

$$du(X) = |du|_*^p = |X|^q \quad \nu - a.e.$$

Then, $\partial \mathrm{Ch}_p = \mathcal{A}_p$.

Sketch of proof

It is easy to check that $\mathcal{A}_p \subset \partial \mathrm{Ch}_p$. Since $\partial \mathrm{Ch}_p$ is maximal monotone, we need to show that also $\mathcal{A}_p$ is maximal monotone.

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Minty theorem: a monotone operator $\mathcal{A}$ is maximal iff $R(I + \mathcal{A}) = H$.

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Minty theorem: a monotone operator $\mathcal{A}$ is maximal iff $R(I + \mathcal{A}) = H$.

We need to show that for all $g \in L^2(\mathbb{X}, \nu)$ there exists $u \in D(\mathcal{A}_p)$ and $X \in L^q(T\mathbb{X})$ with $\mathrm{div}(X) \in L^2(\mathbb{X}, \nu)$ such that

$$-\mathrm{div}(X) = g - u;$$

$$du(X) = |du|_*^p = |X|^q \quad \nu - \text{a.e.}$$

We cannot resort to approximations! Instead, we prove this by finding a functional F such that the above is the dual to the minimisation of F.

Sketch of proof

For $u \in W^{1,p}(\mathbb{X}, d, \nu) \cap L^2(\mathbb{X}, \nu)$, we set

$$F(u) = E(du) + G(u),$$

where

$$E(v) = \frac{1}{p} \int_{\mathbb{X}} |v|_*^p \, d\nu$$

and

$$G(u) = \frac{1}{2} \int_{\mathbb{X}} u^2 \, d\nu - \int_{\mathbb{X}} ug \, d\nu.$$

The dual problem to the minimisation of F is

$$\sup_{v^* \in L^q(T\mathbb{X})} \Big\{ - E^*(-v^*) - G^*(d^* v^*) \Big\}.$$

Sketch of proof

Most importantly, the extremality conditions between a minimiser $\overline{u}$ of F and a maximiser $\overline{v}^*$ of the dual problem are

$$E(d\overline{u}) + E^*(-\overline{v}^*) = \langle -\overline{v}^*, d\overline{u} \rangle$$

and

$$G(\overline{u}) + G^*(d^*\overline{v}^*) = \langle \overline{u}, d^*\overline{v}^* \rangle.$$

Once computed, the first condition yields that

$$d\overline{u}(-\overline{v}^*) = |-\overline{v}^*|^q = |d\overline{u}|_*^p \quad \nu - \text{a.e.}$$

and since $d^* = -\mathrm{div}$, the second condition gives

$$-\mathrm{div}(\overline{v}^*) = \overline{u} - g.$$

Thus, the range condition is satisfied once we choose $X = -\overline{v}^*$.

Back to the gradient flow

Theorem (G.-Mazón, JFA 2022)

For any $u_0 \in L^2(\mathbb{X}, \nu)$ and all $T > 0$ there exists a unique weak solution $u(t)$ of the p-Laplacian evolution equation in the following sense:

There exists $u \in C([0,T]; L^2(\mathbb{X},\nu)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\mathbb{X},\nu))$, $u(0,\cdot) = u_0$, for a.e. $t \in (0,T)$ $u(t) \in W^{1,p}(\mathbb{X}, d, \nu)$, and there exist vector fields $X(t) \in L^q(T\mathbb{X})$ with $\mathrm{div}(X(t)) \in L^2(\mathbb{X},\nu)$ such that

$$\mathrm{div}(X(t)) = u_t(t,\cdot) \quad \textit{in } \mathbb{X};$$

$$du(t)(X(t)) = |du(t)|_*^p = |X(t)|^q \quad \nu\textit{-a.e. in } \mathbb{X}.$$

Back to the gradient flow

The characterisation of the subdifferential gives immediately some nice properties of the associated gradient flow:

• It is completely accretive, so we get a contraction estimate;

M. Kell, J. Funct. Anal. **271** (2016).

• It is p-homogeneous, so we may apply the general results of

L. Bungert, M. Burger, J. Evol. Equ. **20** (2020).

to study the asymptotics;

• In some settings, e.g. weighted Euclidean spaces and Finsler manifolds, this definition leads to a pointwise characterisation of the p-Laplacian.

S.I. Otha, K.-T. Sturm, Comm. Pure Appl. Math. **62** (2009).

J.M. Tölle, J. Funct. Anal. **263** (2012).

G. Akagi, K. Ishige, R. Sato, Adv. Calc. Var **13** (2020).

Outline of the talk

The total variation flow

We now study the gradient flow of the 1-Cheeger energy

$$\mathrm{Ch}_1(u) = \int_{\mathbb{X}} |Du|_\nu.$$

L. Ambrosio, S. Di Marino, J. Funct. Anal. **266** (2014).

To provide a characterisation of $\partial \mathrm{Ch}_1$, we need to extend the Gigli structure to functions of bounded variation.

Theorem (G.-Mazón, JFA 2022)

We say that $(u, v) \in \mathcal{A}_1$ iff $u \in L^2(\mathbb{X}, \nu) \cap BV(\mathbb{X}, d, \nu)$, $v \in L^2(\mathbb{X}, \nu)$, and there exists $X \in L^\infty(T\mathbb{X})$ with $\mathrm{div}(X) \in L^2(\mathbb{X}, \nu)$ s.t.

$$-\mathrm{div}(X) = v;$$

$$\|X\|_\infty \leq 1; \quad (X, Du) = |Du|_\nu \quad \textit{as measures.}$$

Then, $\partial \mathrm{Ch}_1 = \mathcal{A}_1$.

BV functions and Anzellotti pairings

The total variation of a function in $L^1(\mathbb{X}, \nu)$ is defined as

$$|Du|_\nu(\mathbb{X}) := \inf \left\{ \liminf_{n \to \infty} \int_\Omega g_{u_n} \, d\nu : \; u_n \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{X}), \; u_n \to u \text{ in } L^1(\mathbb{X}, \nu) \right\},$$

where g_{u_n} is a 1-weak upper gradient of u. Whenever $|Du|_\nu(\mathbb{X}) < \infty$, $|Du|_\nu$ defines a Radon measure, and we set

$$BV(\mathbb{X}, d, \nu) = \{ u \in L^1(\mathbb{X}, \nu) : \; |Du|_\nu(\mathbb{X}) < \infty \}$$

with the norm

$$\|u\|_{BV(\mathbb{X},d,\nu)} = \|u\|_{L^1(\mathbb{X},\nu)} + |Du|_\nu(\mathbb{X}).$$

BV functions and Anzellotti pairings

To have a similar characterisation of the subdifferential for Ch_1, we need to replace $W^{1,p}(\mathbb{X}, d, \nu)$ with $BV(\mathbb{X}, d, \nu)$, and replace the pairing $du(X)$ with the Anzellotti pairing given by

$$\langle (X, Du), f \rangle := -\int_{\mathbb{X}} u \, df(X) \, d\nu - \int_{\mathbb{X}} u \, f \, \mathrm{div}(X) \, d\nu$$

for any $f \in \mathrm{Lip}(\mathbb{X})$ has compact support.

If ν is doubling and $(\mathbb{X}, d, \nu)$ satisfies a weak $(1,1)$-Poincaré inequality, so that we have better approximations by Lipschitz functions, this defines a Radon measure, $(X, Du) \ll |Du|_\nu$ and

$$|(X, Du)| \leq \|X\|_\infty |Du|_\nu.$$

Back to the gradient flow

Theorem (G.-Mazón, JFA 2022)

For any $u_0 \in L^2(\mathbb{X}, \nu)$ and all $T > 0$ there exists a unique weak solution $u(t)$ to the total variation flow in the following sense:

There exists $u \in C([0,T]; L^2(\mathbb{X}, \nu)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\mathbb{X}, \nu))$, $u(0,\cdot) = u_0$, for a.e. $t \in (0,T)$ $u(t) \in BV(\mathbb{X}, d, \nu)$, and there exist vector fields $X(t) \in L^\infty(T\mathbb{X})$ with $\mathrm{div}(X(t)) \in L^2(\mathbb{X}, \nu)$ such that

$$\mathrm{div}(X(t)) = u_t(t, \cdot) \quad \textit{in } \mathbb{X};$$

$$\|X(t)\|_\infty \leq 1; \quad (X(t), Du(t)) = |Du(t)|_\nu \quad \textit{as measures.}$$

+ asymptotics, contraction estimates, ...

Outline of the talk

Extensions: L^1 initial data

If $\nu(\mathbb{X}) < \infty$, then for any $u_0 \in L^1(\mathbb{X}, \nu)$, there exists a unique *entropy solution* of the total variation flow in the following sense:

• $u \in C([0, T]; L^1(\mathbb{X}, \nu)) \cap W^{1,1}_{\rm loc}([0, T]; L^1(\mathbb{X}, \nu))$;

• $u(0, \cdot) = u_0$;

• For a.e. $t \in [0, T]$ and all $k > 0$ we have $T_k u(t) \in BV(\mathbb{X}, d, \nu)$;

• There exist vector fields $X(t) \in L^\infty(T\mathbb{X})$ with $\mathrm{div}(X(t)) \in L^1(\mathbb{X}, \nu)$ and $\|X(t)\|_\infty \leq 1$ s.t.

$$\mathrm{div}(X(t)) = u_t(t, \cdot) \quad \text{in } \mathbb{X};$$

$$(X(t), DT_k u(t)) = |DT_k u(t)|_\nu \quad \text{as measures for all } k > 0.$$

Other applications

The techniques introduced to study the operators $\Delta_{p,\nu}$ and $\Delta_{1,\nu}$ proved to be effective tools in the study of several related problems in MMS:

1. Total variation flow on bounded domains;
2. Gradient flows of functionals with inhomogeneous growth;
3. Cheeger cut problem;
4. Characterisation of the Cheeger constant;
5. Least gradient problem.

W. Górny, J.M. Mazón, JFA/ACV/CCM, 2022-23.

W. Górny, J.M. Mazón, Weak solutions to metric gradient flows, forthcoming book.