Evolution equations on two overlapping random walk structures

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Recent Progress in PDEs
Rome, 20-21 February 2025

Nonlocal PDEs in $\mathbb{R}^N$

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radially symmetric and continuous function with $\int_{\mathbb{R}^N} J(z)\, dz = 1$. Nonlocal evolution problems of the type

$$u_t(x,t) = \int_{\mathbb{R}^N} J(y-x)\,(u(y,t) - u(x,t))\, dy$$

appear in relation to phase transition or image processing models.

G. Alberti and G. Bellettini, Math. Ann. **310** (1998).

S. Kindermann, S. Osher and P. Jones, SIAM J. Multiscale Model. Simul. **4** (2005).

PDEs in graphs

Consider a locally finite weighted discrete graph G with vertices $V(G)$ and edges $E(G)$. If $(x, y) \in E(G)$, we assign to this edge a positive weight $w_{xy} = w_{yx}$; otherwise, $w_{xy} = 0$.

One may study PDEs in this setting by introducing the *weighted gradient*

$$(\nabla_w f)(x, y) = \sqrt{w(x, y)}\,(f(y) - f(x))$$

and the weighted divergence

$$(\mathrm{div}_w F)(x) = \frac{1}{2} \sum_{(x,y)\in E} \sqrt{w(x, y)}\,(F(x, y) - F(y, x)).$$

With this definition, both operators are linear, and $\mathrm{div}_w = -\nabla_w^*$.

PDEs in graphs

The theory for PDEs in weighted graphs was developed primarily in the 90s and 00s, and a common framework may be found in

G. Gilboa and S. Osher, SIAM J. Multiscale Model. Simul. **7** (2008).

The PDEs in weighted graphs have many applications in machine learning and image processing. As a simple example of a second-order differential operator in this setting, the graph Laplacian is defined as

$$(\Delta_w f)(x) := (\mathrm{div}_w(\nabla_w f))(x) \\ = \sum_{(x,y)\in E} w(x,y)(f(y) - f(x)),$$

and it corresponds to the energy functional

$$\mathcal{E}(f) := \frac{1}{2}\|\nabla_w f\|^2_{L^2(E(G))}.$$

Looking for a joint framework

In both examples, the 'nonlocal gradient'

$$u(y) - u(x)$$

is 'integrated' with respect to some 'kernel'. Other common features are

→ lack of singularities;

→ existence of invariant measures;

→ symmetry of interactions.

A joint framework including these features is called a *random walk space*.

Y. Ollivier, J. Funct. Anal. **256** (2009).

J.M. Mazón, M. Solera, J. Toledo, Variational and Diffusion Problems in Random Walk Spaces, Birkhäuser, 2023.

Outline of the talk

1. Random walk spaces
2. Nonlocal differential operators
3. Two random walk structures
4. Partition of the random walk

W. Górny, J.M. Mazón, J. Toledo, arXiv:2410.15203.

Outline of the talk

1. Random walk spaces
2. Nonlocal differential operators
3. Two random walk structures
4. Partition of the random walk

Random walk spaces

Basic ingredients:

• $(X, \mathcal{B})$ - a measurable space with a countably generated σ-field;

• A random walk m on $(X, \mathcal{B})$, i.e., a family of probability measures $(m_x)_{x \in X}$ on $\mathcal{B}$ such that

$$x \mapsto m_x(B)$$

is a measurable function on X for each fixed $B \in \mathcal{B}$.

The probability measure m_x acts as a replacement of a ball around $x \in X$.

Random walk spaces

Definition

Let m be a random walk on $(X, \mathcal{B})$ and ν a σ-finite measure on X. The convolution of ν with m on X is the measure

$$\nu * m(A) := \int_X m_x(A)\, d\nu(x) \quad \forall A \in \mathcal{B}.$$

Definition

If m is a random walk on $(X, \mathcal{B})$, a σ-finite measure ν on X is *invariant* with respect to the random walk m if

$$\nu * m = \nu.$$

The measure ν is said to be *reversible* if moreover

$$dm_x(y)\, d\nu(x) = dm_y(x)\, d\nu(y).$$

In fact, reversibility of ν implies its invariance.

Random walk spaces

Definition

Let $(X, \mathcal{B})$ be a measurable space with a countably generated σ-field. Let m be a random walk on $(X, \mathcal{B})$ and ν a σ-finite measure which is invariant and reversible with respect to m. Then, we call the quadruple $[X, \mathcal{B}, m, \nu]$ a random walk space.

Random walk spaces

Definition

Let $(X, \mathcal{B})$ be a measurable space with a countably generated σ-field. Let m be a random walk on $(X, \mathcal{B})$ and ν a σ-finite measure which is invariant and reversible with respect to m. Then, we call the quadruple $[X, \mathcal{B}, m, \nu]$ a random walk space.

$\rightarrow$ Sometimes reversibility is omitted (but it is crucial for PDEs!);

$\rightarrow$ Sometimes a requirement that $\mathcal{B}$ is generated by a metric d is added; then, $[X, d, m, \nu]$ is called a metric random walk space.

Example 1: Euclidean spaces

Example

Consider the metric measure space $(\mathbb{R}^N, d_{\mathrm{Eucl}}, \mathcal{L}^N)$ and let $\mathcal{B}$ be the Borel σ-algebra. Let $J : \mathbb{R}^N \to [0, +\infty)$ be a measurable, nonnegative and radially symmetric function verifying $\int_{\mathbb{R}^N} J(x)\, dx = 1$. Let m^J be the following random walk on $(\mathbb{R}^N, \mathcal{B})$:

$$m^J_x(A) := \int_A J(x-y)\, dy \quad \text{for } x \in \mathbb{R}^N \text{ and Borel } A \subset \mathbb{R}^N.$$

Applying the Fubini theorem, it is easy to see that $\mathcal{L}^N$ is reversible with respect to m^J. Therefore, $[\mathbb{R}^N, \mathcal{B}, m^J, \mathcal{L}^N]$ is a random walk space.

Example 2: Weighted graphs

Example

Consider a locally finite weighted discrete graph G with vertices $V(G)$ and edges $E(G)$. If $(x, y) \in E(G)$, we assign to this edge a positive weight $w_{xy} = w_{yx}$; otherwise, $w_{xy} = 0$.

For $x \in V(G)$ we define

$$d_x := \sum_{(x,y)\in E(G)} w_{xy}; \qquad m_x := \frac{1}{d_x} \sum_{(x,y)\in E(G)} w_{xy}\, \delta_y.$$

It is not difficult to see that the measure ν defined as

$$\nu(A) := \sum_{x\in A} d_x \quad \text{for } A \subset V(G)$$

is reversible with respect to m. Therefore, $[V(G), \mathcal{B}, m, \nu]$ is a random walk space, where $\mathcal{B}$ is the σ-algebra of all subsets of $V(G)$.

Outline of the talk

Gradient and divergence

Given $u : X \to \mathbb{R}$, we define its *nonlocal gradient* $\nabla u : X \times X \to \mathbb{R}$ as

$$\nabla u(x,y) := u(y) - u(x) \quad \forall\, x, y \in X.$$

For $\mathbf{z} : X \times X \to \mathbb{R}$, its *m-divergence* $\mathrm{div}_m \mathbf{z} : X \to \mathbb{R}$ is defined as

$$(\mathrm{div}_m \mathbf{z})(x) := \frac{1}{2} \int_X (\mathbf{z}(x,y) - \mathbf{z}(y,x))\, dm_x(y).$$

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They are connected by the following *integration by parts formula*.

Theorem (integration by parts)

If $v \in L^p(X,\nu)$ and $\mathbf{z} \in L^{p'}(X \times X, \nu \otimes m_x)$, then

$$\int_X v(x) \, \mathrm{div}_m(\mathbf{z})(x) \, d\nu(x) = -\frac{1}{2} \int_{X \times X} \mathbf{z}(x,y) \, \nabla v(x,y) \, d(\nu \otimes m_x)(x,y).$$

Memo: subdifferential

Definition

Let $\mathcal{F} : E \to (-\infty, +\infty]$ be proper (i.e. $\mathcal{F} \not\equiv +\infty$) and convex. The *subdifferential* (or *subgradient*) $\partial\mathcal{F}$ of the functional $\mathcal{F}$ is defined as

$$\partial\mathcal{F}(x) = \Big\{ x^* \in E^* : \mathcal{F}(y) - \mathcal{F}(x) \geq \langle x^*, y - x \rangle \quad \forall\, y \in E \Big\},$$

where E^* denotes the dual of E. Equivalently, if we identify a multivalued operator with its graph, it is a subset of $E \times E^*$ defined by

$$\partial\mathcal{F} = \Big\{ (x, x^*) \in E \times E^* : \mathcal{F}(y) - \mathcal{F}(x) \geq \langle x^*, y - x \rangle \quad \forall\, y \in E \Big\}.$$

Example

Let $E = \mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}$ be differentiable. Then, $\partial f(x) = \{\nabla f(x)\}$.

Example

Let Ω be an open bounded subset of $\mathbb{R}^N$ with smooth boundary. Let $\mathcal{F} : L^2(\Omega) \to [0, +\infty]$ be given by

$$\mathcal{F}(u) = \begin{cases} \int_\Omega |\nabla u|^2 \, dx & \text{if } u \in W_0^{1,2}(\Omega); \\ +\infty & \text{if } u \in L^2(\Omega) \setminus W_0^{1,2}(\Omega). \end{cases}$$

Then, $\partial\mathcal{F}(u) = -\Delta u$ and $D(\partial\mathcal{F}) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

The subdifferentials of convex functions in Banach spaces are important in the optimization theory due to the following fact: observe that

$$0 \in \partial\mathcal{F}(x) \iff \mathcal{F}(y) \geq \mathcal{F}(x) \quad \forall\, y \in E,$$

so $0 \in \partial\mathcal{F}(x)$ is the *Euler-Lagrange equation* of the variational problem

$$\mathcal{F}(x) = \min_{y \in E} \mathcal{F}(y).$$

Memo: evolution equations in Hilbert spaces

Definition

If E is a Hilbert space H equipped with a scalar product $(\cdot,\cdot)$ and a norm

$$\|x\|_H := \sqrt{(x,x)},$$

we will say that an operator A in H is *monotone* if

$$(x-\hat{x}, y-\hat{y}) \geq 0 \quad \text{for all } (x,y),(\hat{x},\hat{y}) \in A.$$

If $\mathcal{F}$ is defined on a Hilbert space H, $\partial\mathcal{F}$ is a *monotone operator* in H.

Moreover, if $\mathcal{F}$ is lower semicontinuous, then the subdifferential $\partial\mathcal{F}$ has a dense domain and is *maximal monotone*, i.e., it is maximal with respect to inclusion among monotone operators.

Memo: evolution equations in Hilbert spaces

Consider the abstract Cauchy problem

$$\begin{cases} \frac{du}{dt} + \partial\mathcal{F}(u(t)) \ni f(t,\cdot), & t \in (0,T), \\ u(0) = u_0, & u_0 \in H. \end{cases} \tag{P}$$

Definition

We say that $u \in C([0,T];H)$ is a *strong solution* of problem (P), if the following conditions hold: $u \in W^{1,2}_{\mathrm{loc}}(0,T;H)$; for almost all $t \in (0,T)$ we have $u(t) \in D(\partial\mathcal{F})$; and it satisfies (P).

Theorem (Brezis-Kōmura theorem)

Let $\mathcal{F} : H \to (-\infty,\infty]$ be a proper, convex, and lower semicontinuous functional. Given $u_0 \in \overline{D(\partial\mathcal{F})}$ and $f \in L^2(0,T;H)$, there exists a unique strong solution $u(t)$ of the abstract Cauchy problem (P).

Nonlocal p-Laplacian

For $p > 1$, we consider the functional

$$\mathcal{F}_{p,m} : L^2(X,\nu) \to (-\infty, +\infty]$$

defined by

$$\mathcal{F}_{p,m}(u) := \frac{1}{2p} \int_{X \times X} |u(y) - u(x)|^p \, d(\nu \otimes m_x)(x,y)$$

if $\nabla u \in L^p(X \times X, \nu \otimes m_x)$ and $+\infty$ otherwise. Observe that

$$L^p(X,\nu) \cap L^2(X,\nu) \subset D(\mathcal{F}_{p,m}).$$

Nonlocal p-Laplacian

Since $\mathcal{F}_{p,m}$ is convex and lower semicontinuous, the subdifferential

$$\partial_{L^2(X,\nu)}\mathcal{F}_{p,m}$$

is a maximal monotone operator with a dense domain.

To have a definition consistent with the standard case, we *define* the (multivalued) nonlocal p-Laplacian operator Δ_p^m by

$$(u,v) \in \Delta_p^m \iff (u,-v) \in \partial_{L^2(X,\nu)}\mathcal{F}_{p,m}.$$

Nonlocal p-Laplacian

Theorem (G.-Mazón-Toledo 2024)

Let $p > 1$. $(u, v) \in \Delta_p^m$ if and only if the following conditions hold:

- $u, v \in L^2(X, \nu)$;
- $\nabla u \in L^p(X \times X, \nu \otimes m_x)$;
- $v(x) = \mathrm{div}_m(|\nabla u|^{p-2}\nabla u)(x) = \int_X |\nabla u(x,y)|^{p-2} \nabla u(x,y)\, dm_x(y)$.

This result was known already for $p = 2$; a proof will be presented below.

J.M. Mazón, M. Solera, J. Toledo, J. Math. Anal. Appl. **483** (2020).

Proof of the characterisation

Proof. For every $(u, -v)$, $(\hat{u}, -\hat{v}) \in \Delta_p^m$, by the integration by parts formula, we have

$$
\begin{aligned}
&\int_X (u - \hat{u})(v - \hat{v})\, d\nu \\
&\quad = -\int_X (u - \hat{u})(\Delta_p^m u - \Delta_p^m \hat{u})\, d\nu \\
&\quad = -\int_X (u - \hat{u}) \cdot \Delta_p^m u\, d\nu + \int_X (u - \hat{u}) \cdot \Delta_p^m \hat{u}\, d\nu \\
&\quad = \frac{1}{2} \int_{X \times X} |\nabla u|^{p-2}\, \nabla u\, \nabla(u - \hat{u})\, d(\nu \otimes m_x) \\
&\qquad\qquad - \frac{1}{2} \int_{X \times X} |\nabla \hat{u}|^{p-2}\, \nabla \hat{u}\, \nabla(u - \hat{u})\, d(\nu \otimes m_x) \\
&\quad = \frac{1}{2} \int_{X \times X} \left(|\nabla u|^{p-2}\, \nabla u - |\nabla \hat{u}|^{p-2}\, \nabla \hat{u} \right) \nabla(u - \hat{u})\, d(\nu \otimes m_x) \geq 0,
\end{aligned}
$$

so the operator $-\Delta_p^m$ is monotone.

Proof of the characterisation

Since $\partial\mathcal{F}_{p,m}$ is maximal monotone, it suffices to show that

$$\partial\mathcal{F}_{p,m} \subset -\Delta_p^m.$$

Let $(u,v) \in \partial\mathcal{F}_{p,m}$. Then, for every $w \in L^1(X,\nu) \cap L^\infty(X,\nu)$ and $t > 0$, we have

$$\frac{\mathcal{F}_{p,m}(u+tw) - \mathcal{F}_{p,m}(u)}{t} \geq \int_X vw \, d\nu.$$

Then, taking limit as $t \to 0^+$, we obtain that

$$\frac{1}{2}\int_{X\times X} |\nabla u(x,y)|^{p-2}\nabla u(x,y)\nabla w(x,y)\, dm_x(y)\, d\nu(x) \geq \int_X vw\, d\nu.$$

Proof of the characterisation

Since this inequality is also true for $-w$, we have

$$\frac{1}{2}\int_{X\times X}|\nabla u(x,y)|^{p-2}\nabla u(x,y)\nabla w(x,y)\,dm_x(y)\,d\nu(x)=\int_X vw\,d\nu.$$

Then, applying again the integration by parts formula, we get

$$-\int_X \Delta_p^m u(x)\,w(x)\,d\nu(x)=\int_X vw\,d\nu \quad \forall\, w\in L^1(X,\nu)\cap L^\infty(X,\nu).$$

From here, we deduce that $v=-\Delta_p^m u$, and consequently $(u,-v)\in\Delta_p^m$.

Nonlocal 1-Laplacian

We define the space of *functions of bounded variation* in $[X, \mathcal{B}, m, \nu]$ as

$$BV_m(X,\nu) := \left\{ u : X \to \mathbb{R} : \int_{X \times X} |\nabla u(x,y)| \, dm_x(y) \, d\nu(x) < \infty \right\}.$$

The total variation functional $\mathcal{F}_{1,m} : L^2(X,\nu) \to (-\infty, +\infty]$ is defined by

$$\mathcal{F}_{1,m}(u) := \frac{1}{2} \int_{X \times X} |u(y) - u(x)| \, d(\nu \otimes m_x)(x,y)$$

if $u \in BV_m(X,\nu)$ and $+\infty$ otherwise. Observe that

$$L^1(X,\nu) \cap L^2(X,\nu) \subset D(\mathcal{F}_{1,m}).$$

Nonlocal 1-Laplacian

To have a definition consistent with the standard case, we *define* the (multivalued) nonlocal 1-Laplacian operator Δ_1^m by

$$(u, v) \in \Delta_1^m \iff (u, -v) \in \partial_{L^2(X,\nu)} \mathcal{F}_{1,m}.$$

An equivalent characterisation is the following: there exists an antisymmetric function $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ such that

$$\|\mathbf{g}\|_{L^\infty(X \times X, \nu \otimes m_x)} \leq 1;$$

$$v(x) = \int_X \mathbf{g}(x, y) \, dm_x^1(y) \quad \text{for} \quad \nu\text{-a.e. } x \in X;$$

$$\mathbf{g}(x, y) \in \operatorname{sign}(u(y) - u(x)) \quad \text{for} \quad (\nu \otimes m_x)\text{-a.e. } (x, y) \in X \times X.$$

J.M. Mazón, M. Solera, J. Toledo, Calc. Var. PDE **59** (2020).

Outline of the talk

Nonlocal equations with inhomogeneous growth

Our goal is to propose a framework to study evolution problems with inhomogeneous growth on random walk spaces. We consider two cases:

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Nonlocal equations with inhomogeneous growth

Our goal is to propose a framework to study evolution problems with inhomogeneous growth on random walk spaces. We consider two cases:

$\rightarrow$ The measurable space $(X, \mathcal{B})$ supports two random walk structures m^1 and m^2 (with invariant measures ν_1 and ν_2), which may overlap, and the functional has different growth on the two structures;

$\rightarrow$ We have a single random walk space $[X, \mathcal{B}, m, \nu]$ and a partition of m, where again the functional has different growth on the two pieces.

Two random walk structures

Let $[X, \mathcal{B}, m^1, \nu_1]$ and $[X, \mathcal{B}, m^2, \nu_2]$ are two random walk spaces defined on the same measurable space. We assume that

$$\nu_2 \ll \nu_1$$

and

$$\mu := \frac{d\nu_2}{d\nu_1} \in L^\infty(X, \nu_1),$$

where $\mu > 0$ ν_1-a.e. Due to these assumptions, we may consider the evolution in a joint Hilbert space, denoted by

$$H := L^2(X, \nu_1).$$

(This is satisfied by our most of the standard examples.)

Two random walk structures

For $1 \leq q \leq p$, consider the functionals $\mathcal{F}_{q,m^1} : L^2(X,\nu_1) \to (-\infty,+\infty]$ and $\mathcal{F}_{p,m^2} : L^2(X,\nu_1) \to (-\infty,+\infty]$ given by

$$\mathcal{F}_{q,m^1}(u) := \frac{1}{2q} \int_{X \times X} |u(y) - u(x)|^q \; d(\nu_1 \otimes m^1_x)(x,y)$$

if $|\nabla u|^q \in L^1(X \times X, \nu_1 \otimes m^1_x)$ and $+\infty$ otherwise, and

$$\mathcal{F}_{p,m^2}(u) := \frac{1}{2p} \int_{X \times X} |u(y) - u(x)|^p \; d(\nu_2 \otimes m^2_x)(x,y)$$

if $|\nabla u|^p \in L^1(X \times X, \nu_2 \otimes m^2_x)$ and $+\infty$ otherwise. Both functionals are convex and lower semicontinuous in H.

Two random walk structures

Theorem (G.-Mazón-Toledo 2024)

Let $1 \leq q \leq p$. Assume that

$$\mu := \frac{d\nu_2}{d\nu_1} \in L^\infty(X, \nu_1),$$

and there exists $c > 0$ such that $\mu \geq c\ \nu_1$-a.e.

Suppose that one of the following conditions holds:

(a) $\nu_1(X) < \infty$ *and* $q \leq 2$;

(b) $\nu_1(X) = +\infty$ *and* $q \leq \frac{p}{p-1} \leq 2 \leq p$.

Then, we have

$$\partial_H \left(\mathcal{F}_{q,m^1} + \mathcal{F}_{p,m^2}\right) = -\Delta_q^{m^1} - \mu\Delta_p^{m^2}.$$

Moreover, this operator has a dense domain in H.

Two random walk structures

Under these conditions, we get the following existence result.

Theorem (G.-Mazón-Toledo 2024)

Let $T > 0$. For any $u_0 \in L^2(X, \nu_1)$ and $f \in L^2(0, T; L^2(X, \nu_1))$, the following problem has a unique strong solution:

$$\begin{cases} u_t - \Delta_q^{m^1} u - \mu \Delta_p^{m^2} u \ni f \quad \text{on } [0, T] \\ u(0) = u_0. \end{cases} \tag{1}$$

Two random walk structures: asymptotics

In the case $f \equiv 0$, we can get more information concerning the asymptotic behaviour of solutions to the problem

$$\begin{cases} u_t - \Delta_q^{m^1} u - \mu \Delta_p^{m^2} u \ni 0 \quad \text{on } [0, T] \\ u(0) = u_0. \end{cases} \tag{2RW}$$

For this, we need to assume a structural condition on the random walk space. Let $\nu_1(X) < \infty$. We say that $\mathcal{F}_{q,m^1}$ satisfies a $(q, 2)$-Poincaré inequality, if there us a constant $\lambda_2(\mathcal{F}_{q,m^1}) > 0$ such that

$$\lambda_2(\mathcal{F}_{q,m^1}) \, \|u - \overline{u}\|^q_{L^2(X,\nu_1)} \leq \mathcal{F}_{q,m^1}(u) \quad \forall \, u \in L^2(X, \nu_1),$$

where

$$\overline{u} := \frac{1}{\nu_1(X)} \int_X u \, d\nu_1.$$

Two random walk structures: asymptotics

Theorem (G.-Mazón-Toledo 2024)

Assume that $\nu_1(X) < \infty$ and $\mathcal{F}_{q,m^1}$ satisfies a $(q,2)$-Poincaré inequality. For $u_0 \in L^2(X,\nu_1)$, let $u(t)$ be the solution of (2RW) with $q < 2$. Then,

$$\|u(t) - \overline{u_0}\|^{2-q}_{L^2(X,\nu_1)} \leq \Big(\|u_0 - \overline{u_0}\|^{2-q}_{L^2(X,\nu_1)} - \lambda_2(\mathcal{F}_{q,m^1})\, t \Big)^+ \quad \forall t > 0.$$

In particular, if we denote by

$$T_{\rm ex}(u_0) := \inf\{ T > 0 : \; u(t) = \overline{u_0} \;\; \forall t \geq T \}$$

the extinction time, it is finite and we have the following bound

$$T_{\rm ex}(u_0) \leq \frac{\|u_0 - \overline{u_0}\|^{2-q}_{L^2(X,\nu_1)}}{(2-q)\lambda_2(\mathcal{F}_{q,m^1})}.$$

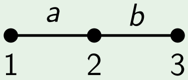
Results of this type hold also for $\nu_1(X) = +\infty$ and $q = 2$.

The $(1,2)$-Laplace equation on a linear graph

Example

Consider a linear graph $G = (V, E)$ with three vertices $V = \{1, 2, 3\}$, two edges $E = \{(1,2), (2,3)\}$, and with positive weights

$$w_{1,2} = a, \quad w_{2,3} = b.$$



We have

$$\nu(\{1\}) = a, \quad \nu(\{2\}) = a + b, \quad \nu(\{3\}) = b,$$

and the random walk m is given by

$$m_1 = \delta_2, \quad m_2 = \frac{a}{a+b}\delta_1 + \frac{b}{a+b}\delta_3, \quad m_3 = \delta_2.$$

Two random walk structures: asymptotics

Example

Consider the evolution problem

$$u_t = \Delta_1^m u + \Delta_2^m u \quad \text{in } V.$$

Let us call $x(t) := u(1,t)$, $y(t) = u(2,t)$ and $z(t) = u(3,t)$. Then, the above equation can be written as the following system of ODEs

$$\begin{cases} x'(t) = \mathbf{g}_t(1,2) + y(t) - x(t); \\ y'(t) = -\frac{a}{a+b}\mathbf{g}_t(1,2) + \frac{b}{a+b}\mathbf{g}_t(2,3) \\ \qquad\qquad + \frac{a}{a+b}(x(t) - y(t)) + \frac{b}{a+b}(z(t) - y(t)); \\ z'(t) = -\mathbf{g}_t(2,3) + y(t) - z(t). \end{cases}$$

Two random walk structures: asymptotics

Example

The antisymmetric functions $\mathbf{g}_t(1,2), \mathbf{g}_t(2,3)$ satisfy

$$\mathbf{g}_t(1,2) \in \mathrm{sign}(y(t) - x(t)), \quad \mathbf{g}_t(2,3) \in \mathrm{sign}(z(t) - y(t)).$$

We add the initial condition $u(0) = c\chi_{\{1\}}$, or equivalently

$$x(0) = c, \quad y(0) = 0, \quad z(0) = 0.$$

We now examine the behaviour of this system in three special cases.

Two random walk structures: asymptotics

Example

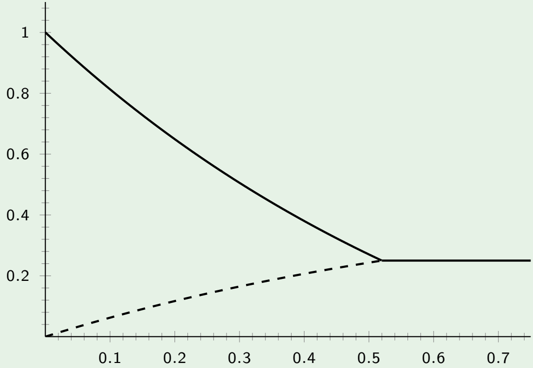


Figure: Case A. $a = b = 1$, $c = 1$. $x(t)$ continuous line; $y(t) = z(t)$ dashed line. After $t \approx 0.51986$, $x(t) = y(t) = z(t)$.

Two random walk structures: asymptotics

Example

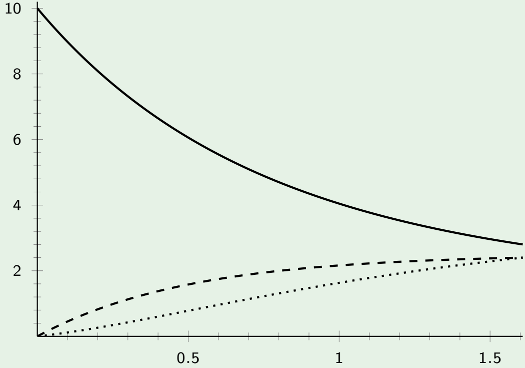


Figure: Case B. $a = b = 1$, $c = 10$. $x(t)$ continuous line; $y(t)$ dashed line; $z(t)$ dotted line. Valid for $0 \leq t \lesssim 1.609438$.

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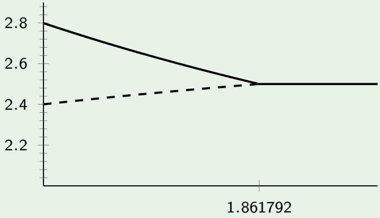

Figure: Case B. $a = b = 1$, $c = 10$. $x(t)$ continuous line; $y(t) = z(t)$ dashed line. Valid for $t \gtrsim 1.609438$. After $t \approx 1.861792$, $x(t) = y(t) = z(t)$.

Two random walk structures: asymptotics

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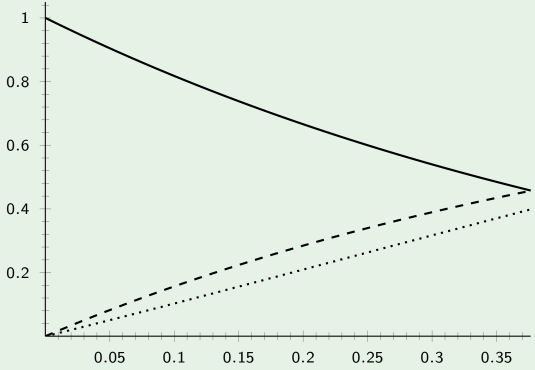

Figure: Case C. $a = 10$, $b = 1$, $c = 1$. $x(t)$ continuous line; $y(t)$ dashed line; $z(t)$ dotted line. Valid for $0 \leq t \lesssim 0.376844$.

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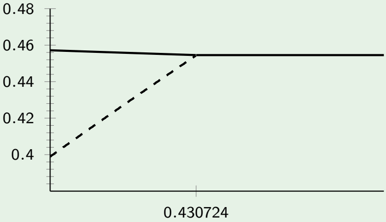

Figure: Case C. $a = 10$, $b = 1$, $c = 1$. $x(t) = y(t)$ continuous line; $z(t)$ dotted line. Valid for $t \gtrsim 0.376844$. After $t \approx 0.430724$, $x(t) = y(t) = z(t)$.

Two random walk structures: asymptotics

Example

The solution behaves much different depending in the three cases:

$\rightarrow$ Case A: The value of u is at all times equal in the vertices 2 and 3;

$\rightarrow$ Case B: The value of u is larger in the vertex 2 than 3, until at some point $u(1) > u(2) = u(3)$;

$\rightarrow$ Case C: The value of u is larger in the vertex 2 than 3, until at some point $u(1) = u(2) > u(3)$.

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Still, there are two important shared properties:

→ There is a finite extinction time;

→ The mean of the initial data (with respect to ν) is preserved.

Outline of the talk

Partition of a random walk

Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. Fix measurable sets A_x, B_x with

$$\operatorname{supp}(m_x) = A_x \cup B_x.$$

The sets A_x and B_x may overlap. Consider the energy functional

$$\mathcal{F}(u) = \int_X \left(\frac{1}{2q} \int_{A_x} |u(y) - u(x)|^q dm_x(y) + \frac{1}{2p} \int_{B_x} |u(y) - u(x)|^p dm_x(y) \right)$$

where $\mathcal{F}(u) = +\infty$ if the integral is not finite. By reversibility of ν with respect to m, we have that

$$\begin{aligned} \mathcal{F}(u) = \frac{1}{2q} \int_X \int_X |u(y) - u(x)|^q \frac{\chi_{A_x}(y) + \chi_{A_y}(x)}{2} \, dm_x(y) \, d\nu(x) \\ + \frac{1}{2p} \int_X \int_X |u(y) - u(x)|^p \frac{\chi_{B_x}(y) + \chi_{B_y}(x)}{2} \, dm_x(y) \, d\nu(x). \end{aligned}$$

Partition of a random walk

Consider the symmetric functions $K_A, K_B : X \to \mathbb{R}$ defined by

$$K_A(x,y) := \frac{\chi_{A_x}(y) + \chi_{A_y}(x)}{2} \quad \text{and} \quad K_B(x,y) := \frac{\chi_{B_x}(y) + \chi_{B_y}(x)}{2}.$$

Then, we define $\mathcal{F}_{A,q,m}, \mathcal{F}_{B,p,m} : L^2(X,\nu) \to (-\infty, +\infty]$ as

$$\mathcal{F}_{A,q,m}(u) := \frac{1}{2q} \int_{X\times X} |u(y) - u(x)|^q \, K_A(x,y) \, d(\nu \otimes m_x)(x,y)$$

and

$$\mathcal{F}_{B,p,m}(u) := \frac{1}{2p} \int_{X\times X} |u(y) - u(x)|^p \, K_B(x,y) \, d(\nu \otimes m_x)(x,y)$$

Both functionals are convex and lower semicontinuous with respect to convergence in $L^2(X,\nu)$.

Partition of the random walk

For $p \geq 1$, we define the *m-p-B-Laplacian operator* $\Delta^m_{p,B}$ in $[X, \mathcal{B}, m, \nu]$ as

$$(u, v) \in \Delta^m_{p,B} \iff (u, -v) \in \partial_{L^2(X,\nu)} \mathcal{F}_{B,p,m}. \tag{2}$$

Theorem

For $p > 1$, we have

$$(u, v) \in \Delta^m_{p,B} \iff u, v \in L^2(X, \nu),\ |\nabla u|^{p-1} \in L^1(X \times X, \nu \otimes m_x) \text{ and}$$

$$v(x) = \mathrm{div}_m(K_B |\nabla u|^{p-2} \nabla u)(x)$$

$$= \int_X K_B(x, y) |\nabla u(x, y)|^{p-2} \nabla u(x, y)\, dm_x(y).$$

Partition of the random walk

We have a similar characterisation of the *m-1-A-Laplacian operator* $\Delta^m_{p,A}$.

Theorem

We have

$$(u,v) \in \partial_{L^2(X,\nu)} \mathcal{F}_{A,1,m} \iff u, v \in L^2(X,\nu)$$

and there exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ *antisymmetric with*

$$\|\mathbf{g}\|_{L^\infty(X \times X, \nu \otimes m_x)} \leq 1;$$

$$v(x) = -\int_X \mathbf{g}(x,y) K_A(x,y)\, dm_x(y) \quad \textit{for } \nu\textit{-a.e. } x \in X;$$

and

$$\mathbf{g}(x,y) K_A(x,y) \in \operatorname{sign}(u(y) - u(x)) K_A(x,y) \quad (\nu \otimes m_x)\textit{-a.e.}$$

Partition of the random walk

Theorem (G.-Mazón-Toledo 2024)

Let $1 \leq q \leq p$. Suppose that one of the following conditions holds:

(a) $\nu(X) < \infty$, $q \leq 2$;

(b) $\nu(X) = +\infty$ *and* $q \leq \frac{p}{p-1} \leq 2 \leq p$.

Then, we have

$$\partial_{L^2(X,\nu)} \left(\mathcal{F}_{A,q,m} + \mathcal{F}_{B,p,m}\right) = -\Delta^m_{q,A} - \Delta^m_{p,B}.$$

Furthermore, this operator has a dense domain in $L^2(X, \nu)$.

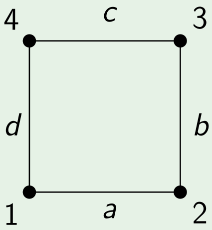
We immediately obtain the corresponding existence and uniqueness result.

Partition of the random walk: asymptotics

Example

Consider the graph $G = (V, E)$ with vertices $V = \{1, 2, 3, 4\}$ and edges $E = \{(1,4), (1,2), (2,3), (3,4)\}$. We assign to the edges positive weights

$$w_{1,2} = a, \; w_{2,3} = b, \; w_{3,4} = c, \; w_{4,1} = d.$$

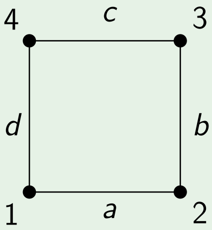


The invariant measure ν is

$$\nu(\{1\}) = a + d, \quad \nu(\{2\}) = a + b, \quad \nu(\{3\}) = b + c, \quad \nu(\{4\}) = c + d.$$

Partition of the random walk: asymptotics

Example



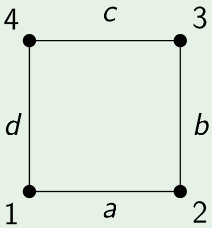
The random walk m is given by

$$m_1 = \frac{a}{a+d}\delta_2 + \frac{d}{a+d}\delta_4, \quad m_2 = \frac{a}{a+b}\delta_1 + \frac{b}{a+b}\delta_3,$$

$$m_3 = \frac{b}{b+c}\delta_2 + \frac{c}{b+c}\delta_4, \quad m_4 = \frac{c}{c+d}\delta_3 + \frac{d}{c+d}\delta_1.$$

Partition of the random walk: asymptotics

Example



We make the following partition on the random walk:

$$A_1 = \{4\}, \quad A_2 = \{3\}, \quad A_3 = \{2\}, \quad A_4 = \{1\}$$

and

$$B_1 = \{2\}, \quad B_2 = \{1\}, \quad B_3 = \{4\}, \quad B_4 = \{3\}.$$

This corresponds to the 1-Laplacian in the edges $(1, 4)$ and $(2, 3)$, and the Laplacian in the edges $(1, 2)$ and $(3, 4)$.

Partition of the random walk: asymptotics

Example

We now consider the equation

$$u_t - \Delta^m_{1,A}(u) - \Delta^m_{2,B}(u) \ni 0.$$

We denote

$$x(t) := u(t,1), \quad y(t) := u(t,2), \quad z(t) := u(t,3), \quad w(t) := u(t,4),$$

and see how the evolution differs from the previous case.

Partition of the random walk: asymptotics

Example

The equation then becomes the following ODE

$$\begin{cases} x'(t) = \frac{d}{a+d}\mathbf{g}_t(1,4) + \frac{a}{a+d}(y(t) - x(t)); \\ y'(t) = \frac{b}{a+b}\mathbf{g}_t(2,3) + \frac{b}{a+b}(x(t) - y(t)); \\ z'(t) = -\frac{b}{b+c}\mathbf{g}_t(2,3) + \frac{c}{b+c}(w(t) - z(t))); \\ w'(t) = -\frac{d}{c+d}\mathbf{g}_t(1,4) + \frac{c}{c+d}(z(t) - w(t)) \end{cases}$$

for antisymmetric functions $\mathbf{g}_t$ satisfying

$$\mathbf{g}_t(1,4) \in \mathrm{sign}(w(t) - x(t)),\ \mathbf{g}_t(2,3) \in \mathrm{sign}(z(t) - y(t)).$$

We take equal weights $a = b = c = d = 1$ and the initial datum

$$x(0) = 2, \quad y(0) = 0, \quad z(0) = 1, \quad w(0) = 0.$$

Partition of the random walk: asymptotics

Example

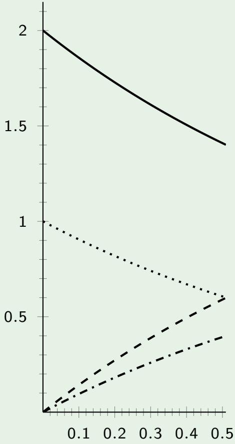


Figure: $x(t)$: continuous line; $y(t)$: dashed line; $z(t)$: dotted line; $w(t)$: dashed-dotted line. Valid for $0 \leq t \lesssim 0.510826$.

Example

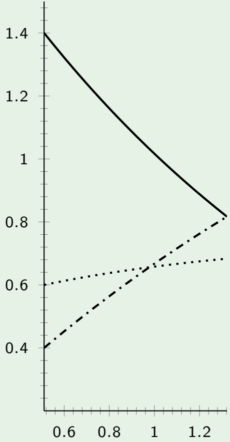


Figure: $x(t)$: continuous line; $y(t) = z(t)$: dotted line; $w(t)$: dashed-dotted line. Valid for $0.510826 \lesssim t \lesssim 1.32176$.

Partition of the random walk: asymptotics

Example

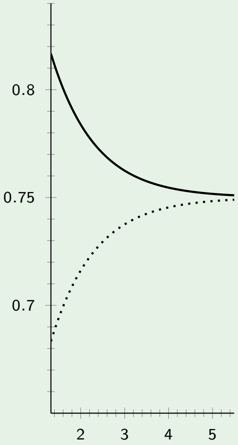


Figure: $x(t) = z(t)$: continuous line; $y(t) = z(t)$: dotted line.
Valid for $t \gtrsim 1.32176$.

Partition of the random walk: asymptotics

Example

There are two main differences with respect to the previous example:

$\rightarrow$ The solution converges to the mean of the initial data, but has an infinite extinction time.

$\rightarrow$ The graph effectively splits into two pieces; the sets $\{1, 4\}$ and $\{2, 3\}$. The evolution within them is primarily governed by the 1-Laplacian (and has a finite extinction time within the smaller set).

Partition of the random walk: asymptotics

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Due to the fact that the partition of the random walk in general bears no relation to the invariant measure, validity of a Poincaré inequality in this setting does not imply finite extinction time.