Functions of bounded variation and their applications

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Overview of the course

The goal of this course is to present the theory of functions of bounded variation (in short: BV functions) in the context of variational problems and associated PDEs. The space of BV functions appears naturally when one considers variational problems with linear growth, i.e., where the minimised object depends on the gradient via a term

$$\int_\Omega f(x, Du)$$

where $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Borel function which satisfies

$$c_1|p| - c_2 \leqslant f(x,p) \leqslant c_3(1 + |p|)$$

for positive constants c_1, c_2 and c_3. Variational problems with linear growth are currently the subject of intense mathematical research and commonly arise in image processing and denoising, in materials science, and in phase transitions. Some important examples of such minimisation problems include the minimal surface equation, which corresponds to the minimisation of the *area functional*

$$\int_\Omega \sqrt{1 + |Du|^2}$$

and the least gradient problem, which is the minimisation of the total variation

$$\int_\Omega |Du|,$$

the Rudin-Osher-Fatemi functional, used for image denoising,

$$\int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega (u - f)^2 \, dx,$$

or the Mumford-Shah functional, used for image segmentation,

$$\int_{\Omega \backslash K} |\nabla u|^2 \, dx + \alpha \int_{\Omega \backslash K} (u - f)^2 \, dx + \mathcal{H}^{N-1}(K \cap \Omega).$$

These problems are usually considered in the presence of some boundary conditions or a penalisation term. Since the functionals of the above form are not lower semicontinuous in the space $W^{1,1}$, due to the fact that a bounded sequence in $W^{1,1}$ does not necessarily have a limit in $W^{1,1}$ even in the weak* topology, we need to consider a larger function space.

The goal of this course is to provide an introduction to the theory of functions of bounded variation and present some functional analytical tools which enable the mathematical treatment of linear-growth functionals, most notably the Anzellotti pairing theory. Then, we will discuss some examples of minimisations and degenerate partial differential equations including the Rudin-Osher-Fatemi model of image denoising, the least gradient problem, and the total variation flow.

The main references to this lecture are: for the theory of BV functions, Chapters 2-3 of the monograph by Ambrosio, Fusco and Pallara [**1**] and Chapter 5 of the book by Evans and Gariepy [**23**]; most of the necessary measure-theoretic notions appear in earlier chapters of these books. For the part concerning the area-minimising sets, and a different view on the basic properties of BV functions, see the book of Giusti [**28**]. The main reference concerning the Anzellotti pairings, the subdifferential of the total variation, and the total variation flow is the monograph by Andreu, Caselles and Mazón [**3**]; one can also find there the basic information concerning subdifferentials and gradient flows of convex functionals.

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Basic notation

In this Section, we briefly summarize the standard notations used throughout the lectures. The underlying assumption in most of them is that Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, i.e. that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary. We typically denote (possibly with indices) by u, v, w functions defined in Ω and by f, g, h functions defined on $\partial\Omega$. Below, we briefly present the notation for function spaces which we often use and some objects associated with them.

$C(X)$: space of continuous functions on a set $X \subset \mathbb{R}^N$;

$C_c(X)$: space of continuous functions with compact support in $X \subset \mathbb{R}^N$;

$C_c^\infty(X)$: space of smooth functions with compact support in $X \subset \mathbb{R}^N$;

$W^{k,p}(\Omega)$: Sobolev spaces on an open set $\Omega \subset \mathbb{R}^N$;

Weak derivative of a Sobolev function: ∇u;

$BV(\Omega)$: space of functions of bounded variation on an open set $\Omega \subset \mathbb{R}^N$;

Distributional derivative of a BV function: Du;

Total variation of a BV function: $|Du|(\Omega)$ or $\int_\Omega |Du|$;

Traces of Sobolev/BV functions on $\partial\Omega$: Tu, $u|_{\partial\Omega}$, or u if clear from the context;

ν^Ω: the outer unit normal to a Lipschitz boundary $\partial\Omega$;

$\mathcal{H}^k$: Hausdorff measure of dimension k;

$L^p(\partial\Omega)$: the L^p space on $\partial\Omega$ (for $\Omega \subset \mathbb{R}^N$ open) with respect to $\mathcal{H}^{N-1}$;

$L^p(\Omega, \mathbb{R}^N)$: space of integrable vector fields;

$X_p(\Omega)$: space of bounded vector fields with divergence in $L^p(\Omega)$;

$\mathcal{M}(X)$: space of finite Radon measures on a set $X \subset \mathbb{R}^N$;

$\mathcal{M}_+(X)$: space of positive finite Radon measures;

$\mathcal{M}(X, \mathbb{R}^N)$: space of finite vector-valued Radon measures;

S^{N-1}: the unit sphere in $\mathbb{R}^N$;

ω_N: the measure of the unit ball in $\mathbb{R}^N$;

Furthermore, we will use the following two sign functions

$$\operatorname{sign}_0(r) := \begin{cases} 1 & \text{if } r > 0; \\ 0 & \text{if } r = 0; \\ -1 & \text{if } r < 0 \end{cases} \qquad \operatorname{sign}(r) := \begin{cases} 1 & \text{if } r > 0; \\ [-1, 1] & \text{if } r = 0; \\ -1 & \text{if } r < 0. \end{cases}$$

CHAPTER 1

Functions of bounded variation

The space of functions of bounded variation shares many properties with the Sobolev spaces $W^{1,p}$ and, indeed, it is introduced by a generalisation of the distributional definition of Sobolev spaces. As such, these spaces share many properties; therefore, we start this lecture with a short recollection of definition and properties of Sobolev functions. For this purpose, assume that Ω is a sufficiently regular open subset of $\mathbb{R}^N$.

Definition 1.1. For $p \in [1, \infty]$, the Sobolev space $W^{1,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ whose distributional derivative ∇u (also called the weak derivative) lies in $L^p(\Omega; \mathbb{R}^N)$. In other words, $u \in W^{1,p}(\Omega)$ if and only if $u \in L^p(\Omega)$ and

$$\int_\Omega u \operatorname{div}(\varphi)\, dx = -\int_\Omega \varphi \cdot \nabla u\, dx \quad \forall \varphi \in C^\infty_{\mathrm{c}}(\Omega; \mathbb{R}^N).$$

The space $W^{1,p}(\Omega)$, endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = (\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^N)})^{1/p}$$

is a Banach space.

Equivalently, $u \in W^{1,p}(\Omega)$ if and only if $u \in L^p(\Omega)$ and there exist functions $u_{x_1}, ..., u_{x_N} \in L^p(\Omega)$ such that for all $i = 1, ..., N$

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi\, u_{x_i}\, dx \quad \forall \varphi \in C^\infty_{\mathrm{c}}(\Omega).$$

Here, $\nabla u = (u_{x_1}, ..., u_{x_N})$.

Among the properties of Sobolev functions, let us list these which are the most relevant to the present topic:

(a) Smooth functions with finite Sobolev norm form a dense subset of $W^{1,p}(\Omega)$;
(b) We have $W^{1,p}(\Omega) \hookrightarrow L^{Np/(N-p)}(\Omega)$;
(c) For bounded Ω, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \frac{Np}{N-p}$ and this embedding is compact;
(d) For $N > 1$, we have the Sobolev inequality

$$\|u\|_{L^{Np/(N-p)}(\mathbb{R}^N)} \leqslant C\bigg(\int_{\mathbb{R}^N} |\nabla u|^p\, dx\bigg)^{1/p}$$

for all $u \in W^{1,p}(\mathbb{R}^N)$;
(e) We have the Poincaré inequality

$$\|u - u_\Omega\|_{L^{Np/(N-p)}(\Omega)} \leqslant C\bigg(\int_{\Omega} |\nabla u|^p\, dx\bigg)^{1/p}$$

for some constant C depending only on the width of Ω. Here,

$$u_\Omega = \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u(x)\, dx$$

denotes the mean value of u in Ω;

(f) For bounded Ω, there is a bounded and linear trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ with the following property:

$$\int_\Omega u \operatorname{div}(\varphi)\, dx + \int_\Omega \varphi \cdot \nabla u\, dx = \int_{\partial\Omega} \varphi \cdot \nu^\Omega\, Tu\, d\mathcal{H}^{N-1}$$

for all $u \in W^{1,p}(\Omega)$ and $\varphi \in C^1(\mathbb{R}^N, \mathbb{R}^N)$;

(g) The image of the trace operator is the space $W^{1-\frac{1}{p},p}(\partial\Omega)$ (for $p > 1$), where

$$W^{1-\frac{1}{p},p}(\partial\Omega) = \left\{ f \in L^p(\partial\Omega) : \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{(n-1)+p(1-\frac{1}{p})}} dS(x,y) < \infty \right\},$$

or $L^1(\partial\Omega)$ (for $p = 1$). The extension operator in the reverse direction is linear for $p > 1$ and nonlinear for $p = 1$;

(h) Once we fix a direction in $\mathbb{R}^N$, Sobolev functions are absolutely continuous along almost every line in this direction.

We will see that the BV functions exhibit a lot of similar behaviour. The main difference concerns the last point - the property of absolute continuity along a.e. line implies that Sobolev functions cannot have jump-type discontinuities. On the other hand, the space of bounded variation functions includes characteristic functions of sufficiently regular sets, and allows for the study of functions which are discontinuous along a sufficiently regular set.

1.1. Definition and basic properties

First, let us recall the definition of vector-valued measures.

Memo 1 (Vector-valued measures). *Let $(X, \mathcal{F})$ be a measurable space. A set function $\mu : \mathcal{F} \to \mathbb{R}^N$ is a vector-valued measure, if $\mu(\varnothing) = 0$ and*

$$\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i)$$

for any sequence A_i or pairwise disjoint elements of $\mathcal{F}$. The variation $|\mu|$ of a vector-valued measure is given by

$$|\mu|(A) = \sup\left\{ \sum_{i=1}^\infty |\mu(A_i)| : A_i \in \mathcal{F},\ \bigcup_{i=1}^\infty A_i = A,\ A_i \text{ are pairwise disjoint} \right\}$$

for any $A \in \mathcal{F}$. This formula defines a positive measure on X. The space of vector-valued measures with finite total variation $|\mu|(X)$ is denoted $\mathcal{M}(X;\mathbb{R}^N)$, and equipped with the norm $\mu \mapsto |\mu|(X)$ it is a Banach space.

For $N = 1$, one usually says that μ is a signed measure. Then, one can uniquely decompose μ into a positive and negative part, i.e., $\mu = \mu^+ - \mu^-$, where

$$\mu^+ = \frac{1}{2}(|\mu| + \mu) \qquad \text{and} \qquad \mu^- = \frac{1}{2}(|\mu| - \mu)$$

are positive measures on X.

DEFINITION 1.2. The space $BV(\Omega)$ consists of functions $u \in L^1(\Omega)$ distributional derivative Du lies in $\mathcal{M}(\Omega;\mathbb{R}^N)$, i.e., it is a vector-valued Radon measure with finite total variation in Ω. In other words, $u \in L^1(\Omega)$ is a *function of bounded variation* (i.e., $u \in BV(\Omega)$) if and only if $u \in L^1(\Omega)$ and

$$\int_\Omega u \operatorname{div}(\varphi)\, dx = -\int_\Omega \varphi\, d[Du] \quad \forall \varphi \in C_c^\infty(\Omega;\mathbb{R}^N).$$

Endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$$

it is a Banach space.

Equivalently, $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and there exist Radon measures $\mu_1, ..., \mu_N$ with finite total mass in Ω such that for all $i = 1, ..., N$

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi\, d\mu_i \quad \forall \varphi \in C_c^\infty(\Omega).$$

Here, $Du = (\mu_1, ..., \mu_N)$.

MEMO 2 (Riesz representation theorem). *Let $L : C_c(\mathbb{R}^N;\mathbb{R}^N) \to \mathbb{R}$ be a linear functional which satisfies*

$$\sup\{L(f) : f \in C_c(\mathbb{R}^N;\mathbb{R}^N),\ |f| \leqslant 1,\ \operatorname{supp}(f) \subset K\} < \infty$$

for each compact set $K \subset \mathbb{R}^N$. Then, there exists a Radon measure μ on $\mathbb{R}^N$ and a μ-measurable function $\sigma : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$|\sigma(x)| = 1 \quad \textit{for } \mu - \textit{a.e. } x \in \mathbb{R}^N$$

and

$$L(f) = \int_{\mathbb{R}^N} f \cdot \sigma\, d\mu$$

for all $f \in C_c(\mathbb{R}^N;\mathbb{R}^N)$. We call μ the variation measure associated with L, and in each open set $V \subset \mathbb{R}^N$ it holds that

$$\mu(V) = \sup\{L(f) : f \in C_c(\mathbb{R}^N;\mathbb{R}^N),\ |f| \leqslant 1,\ \operatorname{supp}(f) \subset V\}.$$

THEOREM 1.3 (An equivalent definition). *Suppose that $u \in L^1(\Omega)$. If*

$$\sup\left\{\int_\Omega u \operatorname{div}(\varphi)\, dx : \varphi \in C_c^\infty(\Omega;\mathbb{R}^N),\ |\varphi(x)| \leqslant 1 \textit{ for } x \in \Omega\right\} < \infty, \tag{1.1}$$

then $u \in BV(\Omega)$. In the other direction, if $u \in BV(\Omega)$, then for any open set $U \subset \Omega$

$$|Du|(U) = \sup\left\{\int_U u \operatorname{div}(\varphi)\, dx : \varphi \in C_c^\infty(U;\mathbb{R}^N),\ |\varphi(x)| \leqslant 1 \textit{ for } x \in U\right\}.$$

In particular, the total variation of the measure Du is

$$|Du|(\Omega) = \sup\left\{\int_\Omega u\,\mathrm{div}(\varphi)\,dx : \varphi \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^N),\ |\varphi(x)| \leqslant 1 \text{ for } x \in \Omega\right\}.$$

Therefore, in the literature it is sometimes equivalently taken as a definition of BV functions that $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and $|Du|(\Omega) < \infty$, where $|Du|(\Omega)$ is given by the formula above.

Proof. Suppose that $u \in L^1(\Omega)$ satisfies condition (1.1). Define a linear functional $L : C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ by the formula

$$L(\varphi) := -\int_\Omega u\,\mathrm{div}(\varphi)\,dx$$

for any $\varphi \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^N)$. By condition (1.1), we have that

$$\sup\{L(\varphi) : \varphi \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^N),\ \|\varphi\|_\infty \leqslant 1\} < \infty$$

and consequently for all $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$

$$|L(\varphi)| \leqslant C\|\varphi\|_\infty, \tag{1.2}$$

where C is the left-hand side of (1.1).

We now extend the functional L to the space $C_{\mathrm{c}}(\Omega;\mathbb{R}^N)$. For each $\varphi \in C_{\mathrm{c}}(\Omega;\mathbb{R}^N)$, pick a sequence $\varphi_n \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^N)$ which converges uniformly to φ. Define

$$L(\varphi) := \lim_{n\to\infty} L(\varphi_n);$$

by estimate (1.2) this limit exists and does not depend on the choice of the approximating sequence. Therefore, L can be uniquely extended to a linear functional $L : C_{\mathrm{c}}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ which satisfies

$$\sup\{L(\varphi) : \varphi \in C_{\mathrm{c}}(\Omega;\mathbb{R}^N),\ \|\varphi\|_\infty \leqslant 1\} < \infty.$$

We conclude by the Riesz representation theorem: we get existence of a function σ with norm one and a (positive) measure σ such that

$$-\int_\Omega u\,\mathrm{div}(\varphi)\,dx = \int_\Omega \varphi\cdot\sigma\,d\mu$$

for all $\varphi \in C_{\mathrm{c}}(\Omega;\mathbb{R}^N)$. Therefore, once we denote $|Du| = \mu$ and $Du = \sigma|Du|$, we have that Du is the distributional derivative of u; its total variation $|Du|(\Omega)$ is finite by virtue of condition (1.1) and the explicit formula for μ.

For the second part, assume that $u \in BV(\Omega)$. Then, for any $\varphi \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^N)$ with $\|\varphi\|_\infty \leqslant 1$ we have

$$\left|\int_\Omega u\,\mathrm{div}(\varphi)\,dx\right| = \left|-\int_\Omega \varphi\,d[Du]\right| \leqslant \int_\Omega |Du|,$$

so condition (1.1) is satisfied. Thus, applying the Riesz representation theorem as above, the claim follows from the explicit formula for μ. □

This proof, with minimal modifications, works also for functions which are locally of bounded variation, i.e., the total variation of their distributional derivative is finite on bounded subsets.

EXERCISE 1.4. Show that this is a more general property, i.e., for any vector-valued measure $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$ we have

$$|\mu|(\Omega) = \sup \left\{ \varphi \, d\mu : \, \varphi \in C_c(\Omega; \mathbb{R}^N), \, |\varphi(x)| \leqslant 1 \text{ for } x \in \Omega \right\}.$$

It is clear from the definition that $W^{1,1}(\Omega) \subset BV(\Omega)$, since the distributional derivative of Sobolev functions can be viewed as a Radon measure which is absolutely continuous with respect to $\mathcal{L}^N$; let us see that this inclusion is strict.

DEFINITION 1.5. An $\mathcal{L}^N$ measurable subset E of $\mathbb{R}^N$ has *finite perimeter* in Ω if $\chi_E \in BV(\Omega)$. The perimeter of E in Ω is $P(E, \Omega) = |D\chi_E|(\Omega)$.

We denote $P(E, \mathbb{R}^N)$ by $\mathrm{Per}(E)$. The following example shows that sufficiently regular subsets of $\mathbb{R}^N$ are sets of finite perimeter; since a characteristic function of a set with positive Lebesgue measure cannot lie in $W^{1,1}(\Omega)$ (because it is not absolutely continuous along a.e. line in a given direction), this implies that the inclusion $W^{1,1}(\Omega) \subset BV(\Omega)$ is strict.

EXAMPLE 1.6. Assume that E is an open smooth subset of $\mathbb{R}^N$ such that $\mathcal{H}^{N-1}(\partial E \cap \Omega) < \infty$. Then, by the classical Gauss-Green formula,

$$\int_E \mathrm{div}(\varphi) \, dx = \int_{\partial E} \varphi \cdot \nu \, d\mathcal{H}^{N-1}$$

for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^N)$, where ν denotes the outer unit normal to ∂E. Thus, if we consider $\varphi \in C_c^\infty(\Omega; \mathbb{R}^N)$ with $\|\varphi\|_\infty \leqslant 1$,

$$\int_E \mathrm{div}(\varphi) \, dx = \int_{\partial E} \varphi \cdot \nu \, d\mathcal{H}^{N-1} \leqslant \mathcal{H}^{N-1}(\partial E \cap \Omega) < \infty,$$

so E is a set of finite perimeter. Since E is smooth, we may find $\varphi \in C_c^\infty(\Omega; \mathbb{R}^N)$ such that $\varphi \cdot \nu \equiv 1$ on any compact subset of $\partial E \cap \Omega$; by taking supremum over such φ we get that

$$P(E, \Omega) = \mathcal{H}^{N-1}(\partial E \cap \Omega)$$

and ν_E agrees with ν $\mathcal{H}^{N-1}-$a.e. on $\partial E \cap \Omega$.

A well-known "defect" of the Sobolev space $W^{1,1}$ is that its unit ball is not weakly closed. More explicitly, there exist bounded sequences in $W^{1,1}(\Omega)$ which converge in $L^1(\Omega)$ to a function which does not lie in $W^{1,1}(\Omega)$; see the following example.

EXAMPLE 1.7. Let $\Omega = (-1, 1)$ and take the function $u : \Omega \to \mathbb{R}$ given by

$$u_\varepsilon(x) = \begin{cases} -1 & \text{if } x < -\varepsilon; \\ \frac{x}{\varepsilon} & \text{if } -\varepsilon \leqslant x \leqslant \varepsilon; \\ 1 & \text{if } x > \varepsilon. \end{cases}$$

Then, for all $\varepsilon \in (0, 1)$ we have that $u_\varepsilon \in W^{1,1}((-1, 1))$ and $\|u_\varepsilon\|_{W^{1,1}(\Omega)} \leqslant 3$. Clearly, $u_\varepsilon \to u$ in $L^1((-1, 1))$, where

$$u(x) = \begin{cases} -1 & \text{if } x < 0; \\ 1 & \text{if } x > 0. \end{cases}$$

However, $u \notin W^{1,1}((-1, 1))$, as it is not absolutely continuous.

The BV space does not have this defect, and the reason is that in the space of vector-valued measures every bounded sequence has a convergent subsequence. Looking at it from another perspective, the total variation is lower semicontinuous with respect to convergence in $L^1(\Omega)$. More concretely, we have the following result.

THEOREM 1.8. *Suppose that $u_n \in BV(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$. Then,*

$$|Du|(\Omega) \leqslant \liminf_{i\to\infty} |Du_n|(\Omega). \tag{1.3}$$

PROOF. Let $\varphi \in C^1_c(\Omega;\mathbb{R}^N)$. Assume that $\|\varphi\|_\infty \leqslant 1$. Then,

$$\int_\Omega u_n \operatorname{div}(\varphi)\,dx = -\int_\Omega \varphi\, d[Du_n] \leqslant \int_\Omega |\varphi|\,|Du_n| \leqslant \int_\Omega |Du_n|,$$

hence after taking the limit as $n \to \infty$ we get

$$\int_\Omega u \operatorname{div}(\varphi)\,dx = \lim_{n\to\infty}\int_\Omega u_n \operatorname{div}(\varphi)\,dx \leqslant \liminf_{n\to\infty} |Du_n|(\Omega).$$

We conclude by taking a supremum of the left-hand side with respect to φ . □

COROLLARY 1.9. *The same result holds if we replace the convergence in the L^1 norm with weak convergence in $L^p(\Omega)$ for any $p \in [1,\infty)$.*

It is easy to see that we may have a strict inequality in (1.3); consider the following one-dimensional example.

EXAMPLE 1.10. Let $\Omega = (-1,1)$ and take the function $u : \Omega \to \mathbb{R}$ given by

$$u_\varepsilon(x) = \begin{cases} 0 & \text{if } x < -\varepsilon; \\ 1 & \text{if} -\varepsilon \leqslant x \leqslant \varepsilon; \\ 0 & \text{if } x > \varepsilon. \end{cases}$$

Then, for all $\varepsilon \in (0,1)$ we have that $u_\varepsilon \in BV((-1,1))$ and $Du_\varepsilon = \delta_{-\varepsilon} - \delta_\varepsilon$. Clearly, $u_\varepsilon \to u$ in $L^1((-1,1))$, where $u \equiv 0$. Hence,

$$|Du_\varepsilon|((-1,1)) = 2 > 0 = |Du|((-1,1))$$

for all $\varepsilon > 0$, so the sequence u_ε is such that there is a strict inequality in (1.3).

For many applications, the requirement that a sequence converges in the BV norm is too strong; for instance, it is clear from the definition that one cannot approximate a general BV function by smooth functions in the norm topology; for this purpose, we revisit the previous Example.

MEMO 3. *Given a measure space (X,Σ), i.e., a set and a collection of measurable sets, and two measures μ,ν defined on (X,Σ), we say μ and ν are mutually singular (denoted by $\mu \perp \nu$) if for every $A \in \Sigma$ there exist disjoint sets $E, F \in \Sigma$ such that*

$$\mu(A) = \mu(A \cap E) \quad \textit{and} \quad \nu(A) = \nu(A \cap F)$$

for all $A \in \Sigma$. For instance: measures with disjoint supports are mutually singular (e.g. Dirac deltas at different points); the measures $\mathcal{L}^1$, δ_0 and the derivative of the Cantor function are mutually singular on $\mathbb{R}$; and the Lebesgue measure $\mathcal{L}^N$ is mutually singular with any measure supported on an $n-1$-dimensional object.

EXERCISE 1.11. Show that for mutually singular measures μ and ν, we have $|\mu + \nu| = |\mu| + |\nu|$ as measures.

Example 1.12. Let $\Omega = (-1,1)$ and take

$$u(x) = \begin{cases} -1 & \text{if } x < 0; \\ 1 & \text{if } x > 0. \end{cases}$$

Then, $u \in BV((-1,1))$ and

$$Du = 2\delta_0.$$

However, for any approximating sequence $u_n \to u$ with $u_n \in W^{1,1}((-1,1))$, we have that $Du_n = \nabla u_n \, dx$, so $Du_n \ll \mathcal{L}^1$; since the measures $\mathcal{L}^1$ and δ_0 are mutually singular, we have that

$$|D(u - u_n)|((-1,1)) = |Du|((-1,1)) + |Du_n|((-1,1)) \geqslant 2,$$

so the sequence u_n does not converge in the BV norm to u.

Therefore, we will often rely on weaker modes of convergence, namely strict convergence and weak* convergence.

Definition 1.13. Let $u_n, u \in BV(\Omega)$. We say that u_n *strictly converges* to u in $BV(\Omega)$ if the following conditions hold:

(i) $u_n \to u$ in $L^1(\Omega)$;
(ii) $|Du_n|(\Omega) \to |Du|(\Omega)$ as $n \to \infty$.

Definition 1.14. Let $u_n, u \in BV(\Omega)$. We say that u_n *weakly* converges* to u in $BV(\Omega)$ if the following conditions hold:

(i) $u_n \to u$ in $L^1(\Omega)$;
(ii) $Du_n \rightharpoonup Du$ weakly* as measures as $n \to \infty$.

We have the following characterization of weak* convergence in BV (which, essentially, comes from the fact that $BV(\Omega)$ is the dual of a separable space; for more information we refer to [**1**]).

Theorem 1.15. *Let $u_n, u \in BV(\Omega)$. Then, u_n weakly* converges to u in $BV(\Omega)$ if and only if $u_n \to u$ in $L^1(\Omega)$ and $\{u_n\}$ is a bounded sequence in $BV(\Omega)$.*

Proof. One implication is very simple: if Du_n is a weakly* convergent sequence of measures, it is bounded in $\mathcal{M}(\Omega;\mathbb{R}^N)$; thus, any weakly* convergent sequence u_n in $BV(\Omega)$ is bounded in $BV(\Omega)$. In the other direction, assume that $\{u_n\}$ is a bounded sequence in $BV(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$. Then, $\{Du_n\}$ is a bounded sequence in $\mathcal{M}(\Omega;\mathbb{R}^N)$, so it has a weakly* convergent subsequence. Thus, we only need to show that any limit point of Du_n in the weak* topology coincides with Du. By definition of the weak derivative

$$\int_\Omega u_n \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega \varphi \, dD_i(u_n) \qquad \text{for all } \varphi \in C_c^\infty(\Omega)$$

for all $i = 1, ..., N$, where without restriction Du_n is the convergent subsequence and $\mu = w^*\text{-}\lim_{k\to\infty} Du_n$. Passing to the limit $k \to \infty$ we get

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega \varphi \, d\mu_i \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Thus, μ satisfies the definition of the distributional gradient of u. □

Therefore, it is clear that strict convergence implies weak* convergence; the converse is not always true. Due to compactness properties of the BV space (see later in Theorem 1.24) it is easy to obtain a weakly* converging subsequence of a bounded family of functions; on the other hand, the strict topology appears naturally when we try to approximate general BV functions by smooth functions. Namely, we have the following result.

THEOREM 1.16. *Assume that $u \in BV(\Omega)$. There exists a sequence of functions $u_n \in C^\infty(\Omega) \cap BV(\Omega)$ such that*

(i) $u_n \to u$ *in* $L^1(\Omega)$;
(ii) $|Du_n|(\Omega) \to |Du|(\Omega)$ *as* $n \to \infty$.

Moreover,

(iii) *if* $u \in BV(\Omega) \cap L^q(\Omega)$ *for some* $q < \infty$, *we can additionally require that* $u_n \in L^q(\Omega)$ *and* $u_n \to u$ *in* $L^q(\Omega)$;
(iv) *if* $u \in BV(\Omega) \cap L^\infty(\Omega)$, *we can additionally require that* $\|u_n\|_\infty \leqslant \|u\|_\infty$ *and* $u_n \rightharpoonup u$ *weakly* in* $L^\infty(\Omega)$.

Moreover, if $u \in W^{1,1}(\Omega)$, *we also have that* $\nabla u_n \to \nabla u$ *in* $L^1(\Omega; \mathbb{R}^N)$.

PROOF. We proceed similarly to the proof of the corresponding result for Sobolev functions. Take a sequence of open sets Ω_j with the following property: $\Omega_j \Subset \Omega$ and every point $x \in \Omega$ lies in at most four sets Ω_j. Take a partition of unity φ_j relative to this covering, i.e. $\varphi_j \in C^\infty_c(\Omega)$, $\varphi_j \geqslant 0$, $\mathrm{supp}(\varphi_j) \subset \Omega_j$ and $\sum_{j=1}^\infty \varphi_j \equiv 1$ in Ω.

We give the proof for $q < \infty$. Let ρ_ε be a family of standard mollifiers and take $\delta \in (0,1)$. We may require that Ω_1 is large enough so that $|Du|(\Omega\backslash\Omega_1) < \delta$. Then, for every $j \in \mathbb{N}$ there exists $\varepsilon_j > 0$ such that $\mathrm{supp}(\rho_{\varepsilon_j} * (\varphi_j u)) \subset \Omega_j$ and the following conditions hold:

$$\int_\Omega |\rho_{\varepsilon_j} * (\varphi_j u) - \varphi_j u|^q \, dx < (2^{-j}\delta)^q$$

and

$$\int_\Omega |\rho_{\varepsilon_j} * (u\nabla\varphi_j) - u\nabla\varphi_j| \, dx < 2^{-j}\delta.$$

If $u \in W^{1,1}(\Omega)$, then instead we have

$$\int_\Omega |\rho_{\varepsilon_j} * \nabla(\varphi_j u) - \nabla(\varphi_j u)| \, dx < 2^{-j}\delta.$$

Then, we set $u_\delta = \sum_{j=1}^\infty \rho_{\varepsilon_j} * (u\varphi_j)$. The function u_δ is smooth because each of the terms is smooth and the sum is locally finite. Our choice of the sequence ε_j yields that

$$\left(\int_\Omega |u_\delta - u|^q \, dx \right)^{1/q} \leqslant \sum_{j=1}^\infty \left(\int_\Omega |\rho_{\varepsilon_j} * (\varphi_j u) - \varphi_j u|^q \, dx \right)^{1/q} < \delta;$$

similarly,

$$\int_\Omega |Du_\delta| = \int_\Omega |\nabla u_\delta|\,dx < \sum_{j=1}^{\infty}\int_\Omega \varphi_j|Du| + \delta = \int_\Omega |Du| + \delta.$$

Since $\delta \in (0,1)$ was arbitrary, we conclude the proof of points (i)-(iii). Finally, the claim for functions in $W^{1,1}(\Omega)$ follows from

$$\int_\Omega |\nabla u_\delta - \nabla u|\,dx \leqslant \sum_{j=1}^{\infty}\int_\Omega |\rho_{\varepsilon_j} * \nabla(\varphi_j u) - \nabla(\varphi_j u)|\,dx < \delta.$$

and letting $\delta \to 0$. □

Exercise 1.17. Prove point (iv).

Exercise 1.18. For u_n as in the statement of the Theorem, show that the measures $\nabla u_n\,dx$ converge weakly to the measure Du.

One of the most important properties of functions of bounded variation is the coarea formula, which related the total variation of a BV function with perimeters of its superlevel sets.

Memo 4 (Co-area formula for Lipschitz functions). *Suppose that $\Omega \subset \mathbb{R}^N$ is open and $u : \Omega \to \mathbb{R}$ is Lipschitz. Then, for any $g \in L^1(\Omega)$ (or nonnegative), we have*

$$\int_\Omega g\,|\nabla u|\,dx = \int_{-\infty}^{\infty}\left(\int_{u^{-1}(t)} g\,d\mathcal{H}^{N-1}\right)dt.$$

In particular, taking $g \equiv 1$ we obtain

$$\int_\Omega |\nabla u|\,dx = \int_{-\infty}^{\infty}\mathcal{H}^{N-1}(u^{-1}(t))\,dt.$$

This second equality can be generalised to the following statement.

Theorem 1.19 (**Coarea formula**). *For $u \in L^1(\Omega)$, denote $E_t := \{x \in \Omega : u(x) > t\}$. If $u \in BV(\Omega)$, then E_t has finite perimeter for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ and*

$$|Du|(\Omega) = \int_{-\infty}^{\infty} P(E_t,\Omega)\,dt.$$

Conversely, if $u \in L^1(\Omega)$ and

$$\int_{-\infty}^{\infty} P(E_t,\Omega)\,dt < \infty, \tag{1.4}$$

then $u \in BV(\Omega)$.

An even stronger claim is true: for every Borel set $B \subset \Omega$, we have

$$|Du|(B) = \int_{-\infty}^{\infty}|D\chi_{E_t}|(B)\,dt \quad\text{and}\quad Du(B) = \int_{-\infty}^{\infty} D\chi_{E_t}(B)\,dt.$$

Proof. **Step 1.** We start by proving the second part. Assume that $u \in L^1(\Omega)$ satisfies condition (1.4). First, we show that for all $\varphi \in C_c^\infty(\Omega;\mathbb{R}^N)$ with $\|\varphi\|_\infty \leqslant 1$

we have

$$\int_\Omega u \operatorname{div}(\varphi)\, dx = \int_{-\infty}^{\infty} \left(\int_{E_t} \operatorname{div}(\varphi)\, dx \right) dt. \tag{1.5}$$

To see this, consider the following two cases. First, let $u \geqslant 0$. Then, for $\mathcal{L}^N$-a.e. $x \in \Omega$ we have

$$u(x) = \int_0^\infty \chi_{E_t}(x)\, dt.$$

Therefore,

$$\begin{aligned}\int_\Omega u \operatorname{div}(\varphi)\, dx &= \int_\Omega \left(\int_0^\infty \chi_{E_t}(x)\, dt \right) \operatorname{div}(\varphi)(x)\, dx \\ &= \int_0^\infty \left(\int_\Omega \chi_{E_t}(x) \operatorname{div}(\varphi)(x)\, dx \right) dt = \int_0^\infty \left(\int_{E_t} \operatorname{div}(\varphi)\, dx \right) dt.\end{aligned}$$

Similarly, for $u \leqslant 0$ observe that for $\mathcal{L}^N$-a.e. $x \in \Omega$ we have

$$u(x) = \int_{-\infty}^0 (\chi_{E_t}(x) - 1)\, dt$$

and hence

$$\begin{aligned}\int_\Omega u \operatorname{div}(\varphi)\, dx &= \int_\Omega \left(\int_{-\infty}^0 (\chi_{E_t}(x) - 1)\, dt \right) \operatorname{div}(\varphi)(x)\, dx \\ &= \int_{-\infty}^0 \left(\int_\Omega (\chi_{E_t}(x) - 1) \operatorname{div}(\varphi)(x)\, dx \right) dt = \int_{-\infty}^0 \left(\int_{E_t} \operatorname{div}(\varphi)\, dx \right) dt.\end{aligned}$$

The general case follows by decomposing u into a positive and negative part, i.e., $u = u^+ - u^-$; thus, formula (1.5) is proved. Consequently,

$$\int_\Omega u \operatorname{div}(\varphi)\, dx = \int_{-\infty}^{\infty} \left(\int_{E_t} \operatorname{div}(\varphi)\, dx \right) dt \leqslant \int_{-\infty}^{\infty} |D\chi_{E_t}|(\Omega)\, dt.$$

By taking supremum over φ, we get

$$|Du|(\Omega) \leqslant \int_{-\infty}^{\infty} |D\chi_{E_t}|(\Omega)\, dt,$$

which concludes the proof of the second part.

Memo 5 (Fatou lemma). *Fix a measure space (X, μ). Let $f_n : X \to [0, +\infty]$ be a sequence of μ-measurable functions (not necessarily integrable). Then*

$$\int_X \left(\liminf_{n\to\infty} f_n \right) d\mu \leqslant \liminf_{n\to\infty} \int_X f_n\, d\mu.$$

Step 2. We now prove the first part. For smooth functions, by Sard's theorem the preimage of almost every level set is a smooth manifold, and therefore the perimeter of E_t coincides with the Hausdorff measure of ∂E_t, so the claim follows from the co-area formula for Lipschitz functions (Memo 4).

We need to show that this implies the co-area formula for any BV function. Take $u \in BV(\Omega)$ and let $u_n \in C^\infty(\Omega)$ be the approximation sequence given by the Meyers-Serrin theorem (Theorem 1.16). Then, $u_n \to u$ in $L^1(\Omega)$, and if we denote

$$E_t^n = \{x \in \Omega : \, u_n(x) > t\}$$

we get that

$$\int_{-\infty}^{\infty} |\chi_{E_t^n}(x) - \chi_{E_t}(x)|\, dt = \int_{\min(u(x),u_n(x))}^{\max(u(x),u_n(x))} dt = |u_n(x) - u(x)|,$$

and consequently

$$\int_\Omega |u_n(x) - u(x)|\, dx = \int_{-\infty}^{\infty} \left(\int_\Omega |\chi_{E_t^n}(x) - \chi_{E_t}(x)|\, dx \right) dt.$$

Since $u_n \to u$ in $L^1(\Omega)$, by the above equation we may find a subsequence which satisfies $\chi_{E_t^n} \to \chi_{E_t}$ in $L^1(\Omega)$ for a.e. $t \in \mathbb{R}$. Then, lower semicontinuity of the total variation implies

$$|D\chi_{E_t}|(\Omega) \leqslant \liminf_{n\to\infty} |D\chi_{E_t^n}|(\Omega).$$

Applying the Fatou lemma (Memo 5) gives

$$\int_{-\infty}^{\infty} |D\chi_{E_t}|(\Omega)\, dt \leqslant \liminf_{n\to\infty} \int_{-\infty}^{\infty} |D\chi_{E_t^n}|(\Omega) = \liminf_{n\to\infty} |Du_n|(\Omega) = |Du|(\Omega),$$

since the co-area formula holds for smooth functions and u_n converges strictly to u. $\square$

1.2. Embedding theorems and compactness

The next several results concern bounds on the $L^{N/(N-1)}$ norm of a function of bounded variation in terms of its total variation. We present results both of Sobolev and Poincaré type.

Theorem 1.20 (**Sobolev inequality**). *Let $N > 1$. There exists a constant $C > 0$ depending only on the dimension such that*

$$\|u\|_{L^{N/N-1}(\mathbb{R}^N)} \leqslant C|Du|(\mathbb{R}^N)$$

for all $u \in BV(\mathbb{R}^N)$.

Proof. The result follows by approximating with Sobolev functions and a corresponding inequality for Sobolev functions. To be precise, let $u_n \in C_c^\infty(\mathbb{R}^N)$ be the sequence given by the Meyers-Serrin theorem (Theorem 1.16), i.e., $u_n \to u$ in $L^1(\mathbb{R}^N)$ and $\mathcal{L}^N$-a.e., and $|Du_n|(\mathbb{R}^N) \to |Du|(\mathbb{R}^N)$. Then, by the Fatou lemma (Memo 5),

$$\|u\|_{L^{N/(N-1)}(\mathbb{R}^N)} \leqslant \liminf_{n\to\infty} \|u_n\|_{L^{N/(N-1)}(\mathbb{R}^N)},$$

and since for the approximating sequence we have the Gagliardo-Sobolev-Nirenberg inequality, i.e.,

$$\|u_n\|_{L^{N/(N-1)}(\mathbb{R}^N)} \leqslant C\|\nabla u_n\|_{L^1(\mathbb{R}^N)},$$

we conclude that

$$\|u\|_{L^{N/(N-1)}(\mathbb{R}^N)} \leqslant \liminf_{n\to\infty} C\|\nabla u_n\|_{L^1(\mathbb{R}^N)} = \lim_{n\to\infty} C\|\nabla u_n\|_{L^1(\mathbb{R}^N)} = C|Du|(\mathbb{R}^N)$$

and in particular the left-hand side is finite. □

A second result of this type is the Poincaré inequality. The proof is very similar, but we give it for completeness. In what follows, we consider Ω to be a bounded Lipschitz domain. The main reason is that Lipschitz domains are extension domains for the Sobolev space $W^{1,1}$, and so we will be able to deduce the $L^{N/(N-1)}$ bound using approximations by Sobolev functions.

THEOREM 1.21 (**Poincaré inequality**). *Fix $N > 1$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. If Ω is connected, then for all $u \in BV(\Omega)$ we have*

$$\|u - u_\Omega\|_{L^{N/(N-1)}(\Omega)} \leqslant C|Du|(\Omega)$$

for some constant C depending only on the width of Ω. Here,

$$u_\Omega = \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u(x)\,dx$$

denotes the mean value of u in Ω.

PROOF. Let $u_n \in C_c^\infty(\Omega)$ be the sequence given by the Meyers-Serrin theorem (Theorem 1.16), i.e., $u_n \to u$ in $L^1(\Omega)$ and $\mathcal{L}^N$-a.e., and $|Du_n|(\Omega) \to |Du|(\Omega)$. Clearly, the condition that $u_n \to u$ in $L^1(\Omega)$ implies that $(u_n)_\Omega \to u_\Omega$. Thus, the Fatou lemma (Memo 5) yields

$$\|u - u_\Omega\|_{L^{N/(N-1)}(\Omega)} \leqslant \liminf_{n\to\infty} \|u_n - (u_n)_\Omega\|_{L^{N/(N-1)}(\Omega)},$$

and since for the approximating sequence we have the Poincaré inequality, i.e.,

$$\|u_n - (u_n)_\Omega\|_{L^{N/(N-1)}(\Omega)} \leqslant C\|\nabla u_n\|_{L^1(\Omega)},$$

we conclude that

$$\|u - u_\Omega\|_{L^{N/(N-1)}(\Omega)} \leqslant \liminf_{n\to\infty} C\|\nabla u_n\|_{L^1(\Omega)} = \lim_{n\to\infty} C\|\nabla u_n\|_{L^1(\Omega)} = C|Du|(\Omega)$$

and in particular the left-hand side is finite. □

Let us turn to some interesting geometric implications of the two embedding theorems above. First, as a consequence of the Sobolev inequality, we get a simple proof of the isoperimetric inequality in a very general setting.

THEOREM 1.22. *Let $N > 1$. For any set E of finite perimeter in $\mathbb{R}^N$ we have*

$$\mathcal{L}^N(E) \leqslant C[\mathrm{Per}(E)]^{\frac{N}{N-1}}$$

for some dimensional constant C.

PROOF. Take $u = \chi_E$ in the Sobolev inequality (Theorem 1.20). Then,

$$\left(\int_{\mathbb{R}^N} (\chi_E)^{N/(N-1)}\,dx\right)^{(N-1)/N} \leqslant C \cdot \mathrm{Per}(E).$$

Since χ_E takes only values zero and one, we have $(\chi_E)^{N/(N-1)} = \chi_E$, and taking both sides to power $\frac{N}{N-1}$ yields the result. □

EXERCISE 1.23. Use the Poincaré inequality (Theorem 1.21) in a similar fashion to prove the following *relative isoperimetric inequality*: take any ball $B(x, r) \subset \mathbb{R}^N$. For any set E of finite perimeter in $B(x, r)$, show that

$$\min\left\{\mathcal{L}^N(B(x,r) \cap E), \mathcal{L}^N(B(x,r)\backslash E)\right\} \leqslant C[P(E, B(x,r))]^{\frac{N}{N-1}}$$

for some dimensional constant C. Hint: take $f = \chi_{B(x,r)\cap E}$.

THEOREM 1.24 (**Embedding Theorem**). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then, the embedding $BV(\Omega) \hookrightarrow L^{N/(N-1)}(\Omega)$ is continuous and the embedding $BV(\Omega) \hookrightarrow L^p(\Omega)$ is compact for all $1 \leqslant p < \frac{N}{N-1}$.*

PROOF. Continuity of the embedding follows from the Poincaré inequality (Theorem 1.21); observe that

$$\begin{aligned}\|u\|_{L^{N/(N-1)}(\Omega)} &\leqslant \|u_\Omega\|_{L^{N/(N-1)}(\Omega)} + \|u - u_\Omega\|_{L^{N/(N-1)}(\Omega)} \\ &\leqslant |u_\Omega| \cdot \mathcal{L}^N(\Omega)^{(N-1)/N} + C|Du|(\Omega) \leqslant C\|u\|_{BV(\Omega)}.\end{aligned}$$

Consequently, we also get the embeddings for all $p < \frac{N}{N-1}$.

Concerning compactness of the embeddings for $p < \frac{N}{N-1}$: by the Meyers-Serrin approximation theorem, for any sequence f_n bounded in $BV(\Omega)$, one can find a sequence g_n bounded in $W^{1,1}(\Omega)$ such that $\|f_n - g_n\|_{L^p(\Omega)} < \frac{1}{n}$. By the Rellich-Kondrachov theorem for Sobolev functions, the sequence g_n (and therefore also f_n) converges to some f in $L^p(\Omega)$, which in this case does not need to lie in any Sobolev space, but by the lower semicontinuity of the total variation lies in $BV(\Omega)$. □

The main application of the above result is that from a bounded family in $BV(\Omega)$ we can extract a subsequence which converges in $L^1(\Omega)$.

COROLLARY 1.25. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. For every sequence $u_n \in BV(\Omega)$ with $\|u_n\|_{BV(\Omega)} \leqslant M$, there exists a subsequence which converges to some $u \in BV(\Omega)$ $\mathcal{L}^N$−a.e. and in $L^p(\Omega)$ for all $p \in [1, \frac{N}{N-1})$. In particular, $u_n \rightharpoonup u$ weakly* in $BV(\Omega)$.*

Finally, let us comment on the one-dimensional case. Then, the isoperimetric inequality cannot be formulated in a similar manner, and it is clear that the Lebesgue measure of a set cannot be estimated from above by its perimeter.

EXAMPLE 1.26. Consider the sequence of sets $E_n = [-n, n] \subset \mathbb{R}$. Then, for all $n \in \mathbb{N}$ we have $\mathrm{Per}(E_n) = |D\chi_{E_n}|(\mathbb{R}) = 2$, but $\mathcal{L}^1(E_n) \to \infty$.

However, the Sobolev and Poincaré inequality themselves are valid for $N = 1$ if we understand the exponent $\frac{N}{N-1}$ as $+\infty$, but we need to proceed a bit differently; we leave the proof as an exercise (a similar result holds on bounded domains).

EXERCISE 1.27. Show that for all $u \in C_c^\infty(\mathbb{R})$ we have

$$|u(b) - u(a)| \leqslant \int_a^b |u'|\, dx,$$

and thus $\|u\|_\infty \leqslant C\|u\|_{W^{1,1}(\mathbb{R})}$. Conclude using the Meyers-Serrin approximation theorem that for all $u \in BV(\mathbb{R})$ we have $\|u\|_\infty \leqslant C\|u\|_{BV(\mathbb{R})}$.

1.3. Traces of BV functions

We now turn our attention to boundary values of functions of bounded variation. Similarly to the case of Sobolev spaces, one can define a trace operator from $BV(\Omega)$ to $L^1(\partial\Omega, \mathcal{H}^{N-1})$, which for continuous functions agrees with the restriction to $\partial\Omega$.

Memo 6. *Let $A \subset \mathbb{R}^N$ and fix $s \in [0, \infty)$. Then, for any $\delta > 0$, we define*

$$\mathcal{H}^s_\delta(A) = \inf\left\{ \sum_{j=1}^\infty \alpha(s)\left(\frac{\operatorname{diam}(C_j)}{2}\right)^s : A \subset \bigcup_{j=1}^\infty C_j,\ \operatorname{diam}(C_j) \leqslant \delta \right\},$$

where $\alpha(s) = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$. For s integer, it is the volume of the unit ball. We call

$$\mathcal{H}^s(A) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A)$$

the s-dimensional Hausdorff measure on $\mathbb{R}^N$. It is a Borel regular measure.

Keeping in mind that for an open bounded set Ω with Lipschitz boundary the outer unit normal ν^Ω exists $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$ (as a consequence of the Rademacher theorem, which states that Lipschitz functions are differentiable almost everywhere), we have the following result.

Theorem 1.28. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. There exists a bounded linear mapping*

$$T : BV(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{N-1})$$

such that

$$\int_\Omega u \operatorname{div}(\varphi)\, dx + \int_\Omega \varphi \cdot d[Du] = \int_{\partial\Omega} \varphi \cdot \nu^\Omega\, Tu\, d\mathcal{H}^{N-1}$$

for all $u \in BV(\Omega)$ and $\varphi \in C^1(\mathbb{R}^N, \mathbb{R}^N)$. Furthermore, T is continuous between $BV(\Omega)$ endowed with the topology induced by strict convergence and $L^1(\partial\Omega, \mathcal{H}^{N-1})$.

Proof (omitted). Adaptation of the analogous result for $W^{1,p}$ functions, with some minor complications due to using strict approximation in place of approximation in norm. See for instance [**1, 23**]. □

The function Tu is called the *trace* of u on $\partial\Omega$. To simplify the notation, we write $L^p(\partial\Omega)$ in place of $L^p(\partial\Omega, \mathcal{H}^{N-1})$ for all $p \in [1, \infty]$. We will also use $u|_{\partial\Omega}$ and u^Ω for Tu, and when it is clear from the context, we omit the letter T and simply denote it by u. Similarly, one can consider traces of functions defined in $\mathbb{R}^N\backslash\overline{\Omega}$ in place of functions in Ω. We still denote it by T when it is clear from the context, and use $u^{\mathbb{R}^N\backslash\overline{\Omega}}$ when necessary.

Corollary 1.29. *For any $u \in BV(\Omega)$ and $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ we have*

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |u(y) - Tu(x)|\, dy = 0. \tag{1.6}$$

In particular, the trace defined in the previous Theorem agrees with restriction to the boundary for continuous functions, i.e., for any $u \in C(\overline{\Omega}) \cap BV(\Omega)$ we have

$$Tu = u|_{\partial\Omega}.$$

Proof (omitted). Applying the Vitali covering theorem and the Lebesgue-Besicovitch differentiation theorem (Memo 8 below) to some fine estimates from the previous proof. See for instance [**1, 23**]. □

Also, let us notice that property (1.6) shows that the pointwise trace can be defined only using information about the behaviour of u near $\partial\Omega$, and is itself suitable as a definition of the trace.

Exercise 1.30. Using a similar argument as in the proof above, prove that the approximation given by the Meyers-Serrin theorem (Theorem 1.16) has the same trace as the original function.

The trace operator is not continuous with respect to weak* convergence in $BV(\Omega)$; in other words, if we only assume that a sequence u_n converges weakly* to u in $BV(\Omega)$, it does not follow that the traces also converge. To this end, consider the following example.

Example 1.31. Let $\Omega = (-1,1)$ and let $u_n = \chi_{[-1+\frac{1}{n},1-\frac{1}{n}]}$. Clearly, $u_n \to u$ weakly* in $BV(\Omega)$, where $u \equiv 1$. However, $u_n(-1) = u_n(1) = 0$ for all $n \in \mathbb{N}$, but $u(-1) = u(1) = 1$.

It was proved by Gagliardo in [**24**] that the trace operator is actually onto $L^1(\partial\Omega)$; given a boundary datum in $L^1(\partial\Omega)$, one can find a function in $W^{1,1}(\Omega)$ with desired trace. Moreover, one can require some additional properties of the extension; to be exact, we have the following result.

Lemma 1.32. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then, for any given function $f \in L^1(\partial\Omega)$ and $\varepsilon > 0$ there exists a function $u \in W^{1,1}(\Omega)$ satisfying:*

$$u|_{\partial\Omega} = f;$$

$$\int_\Omega |\nabla u|\, dx \leqslant \int_{\partial\Omega} |f|\, d\mathcal{H}^{N-1} + \varepsilon;$$

$$\|u\|_{L^1(\Omega)} \leqslant \varepsilon.$$

In the case when the boundary datum is continuous, one can require that the extension is also continuous.

Proof (simplified). Since $\partial\Omega$ is Lipschitz and we only need to extend the boundary datum in a neighbourhood of $\partial\Omega$, using an argument based on a partition of unity and a straightening of the boundary we can reduce the proof to the case when $\partial\Omega = \mathbb{R}^{N-1}$, f has compact support in $\mathbb{R}^{N-1}$, and u is a function defined in

$$\mathbb{R}^N_+ := \{(y_1, ..., y_N) : y_1 > 0\}.$$

We first pick a sequence of smooth functions $f_j \in C_c(\mathbb{R}^{N-1})$ which converges to f in $L^1(\mathbb{R}^{N-1})$ as $j \to \infty$. We can assume that $f_0 \equiv 0$ and

$$\sum_{j=0}^\infty \|f_j - f_{j+1}\|_{L^1(\mathbb{R}^{N-1})} < \infty.$$

Since f_j have compact support, for every $j \in \mathbb{N} \cup \{0\}$ we have that

$$g_j := \sum_{l=2}^{N} \int_{\mathbb{R}^N_+} \left(\left| \frac{\partial}{\partial y_l} f_j \right| + \left| \frac{\partial}{\partial y_l} f_{j+1} \right| \right) dx < \infty. \tag{1.7}$$

Take a decreasing sequence t_j converging to zero; we will fix the exact values of t_j at the end of the proof. Denote the y_1 variable by t and set

$$u(t, y') = \begin{cases} 0 & \text{if } t > t_0; \\ \frac{t - t_{j+1}}{t_j - t_{j+1}} f_j(y') + \frac{t_j - t}{t_j - t_{j+1}} f_{j+1}(y') & \text{if } t \in [t_{j+1}, t_j] \end{cases}$$

for $t > 0$ and $y' \in \mathbb{R}^{N-1}$. By the mean integral formula for the trace (Corollary 1.29) the trace of u is correct; we only need to prove the desired bound.

To this end, observe that for $t \in [t_{j+1}, t_j]$ we have the following pointwise bounds:

$$\left| \frac{\partial}{\partial t} u(t, y') \right| \leqslant |f_j(y') - f_{j+1}(y')| (t_j - t_{j+1})^{-1};$$

and for all $l = 2, ..., N$ we have

$$\left| \frac{\partial}{\partial y_l} u(t, y') \right| \leqslant \left| \frac{\partial}{\partial y_l} f_j(y') \right| + \left| \frac{\partial}{\partial y_l} f_{j+1}(y') \right|.$$

We will show that the desired estimates follow. Observe that

$$|\nabla u| \leqslant \left| \frac{\partial}{\partial t} u \right| + \sum_{l=2}^{N} \left| \frac{\partial}{\partial y_l} u \right| \leqslant |f_j - f_{j+1}| (t_j - t_{j+1})^{-1}$$
$$+ \sum_{l=2}^{N} \left(\left| \frac{\partial}{\partial y_l} f_j \right| + \left| \frac{\partial}{\partial y_l} f_{j+1} \right| \right),$$

and integrating this inequality over $\mathbb{R}^N_+$ we get

$$\int_{\mathbb{R}^N_+} |\nabla u| \, dx \leqslant \sum_{j=0}^{\infty} \| f_j - f_{j+1} \|_{L^1(\mathbb{R}^{N-1})} + \sum_{j=0}^{\infty} (t_j - t_{j+1}) \, g_j, \tag{1.8}$$

where g_j is given by (1.7). Choosing the sequence t_j in such a way that $t_0 < \varepsilon$ and

$$t_j - t_{j+1} \leqslant \frac{\|f\|_{L^1(\mathbb{R}^{N-1})}}{1 + g_j} 2^{-j-2} \varepsilon,$$

using (1.8) we obtain that the estimate involving the gradient. It is easy to see that the bound on the support holds, which concludes the proof. □

EXERCISE 1.33. Make precise the first part of the argument involving the partition of unity.

EXERCISE 1.34. Make precise the part of the argument showing that the trace is correct.

The following result concerns the total variation of a function constructed from a function in $BV(\Omega)$ and a function in $BV(\mathbb{R}^N \backslash \Omega)$. It turns out that it is given by is the total variations of the original functions plus a boundary term.

THEOREM 1.35. *Let $u_1 \in BV(\Omega)$ and $u_2 \in BV(\mathbb{R}^N \backslash \overline{\Omega})$. We define*

$$v(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega; \\ u_2(x) & \text{if } x \in \mathbb{R}^N \backslash \overline{\Omega}. \end{cases}$$

Then, $v \in BV(\mathbb{R}^N)$ and

$$|Dv|(\mathbb{R}^N) = |Du_1|(\Omega) + |Du_2|(\mathbb{R}^N \backslash \overline{\Omega}) + \int_{\partial\Omega} |Tu_1 - Tu_2| \, d\mathcal{H}^{N-1}.$$

PROOF. Take a test function $\varphi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$ with $|\varphi| \leqslant 1$. Then, applying the trace theorem (technically, $\mathbb{R}^N \backslash \overline{\Omega}$ does not satisfy the assumptions as it is not a bounded domain, but we may restrict to the case of a bounded domain as φ has compact support), we see that

(1.9)
$$\begin{aligned} \int_{\mathbb{R}^N} v \operatorname{div}(\varphi) \, dx &= \int_\Omega u_1 \operatorname{div}(\varphi) \, dx + \int_{\mathbb{R}^N \backslash \overline{\Omega}} u_2 \operatorname{div}(\varphi) \, dx \\ &= -\int_\Omega \varphi \, d[Du_1] - \int_{\mathbb{R}^N \backslash \overline{\Omega}} \varphi \, d[Du_2] + \int_{\partial\Omega} (Tu_1 - Tu_2) \, \varphi \cdot \nu^\Omega \, d\mathcal{H}^{N-1} \\ &\leqslant \int_\Omega |Du_1| + \int_{\mathbb{R}^N \backslash \overline{\Omega}} |Du_2| + \int_{\partial\Omega} |Tu_1 - Tu_2| \, d\mathcal{H}^{N-1}, \end{aligned}$$

which proves the inequality in one direction, and in particular we conclude that $v \in BV(\mathbb{R}^N)$.

To obtain an equality, observe that by testing the definition of the distributional derivative with functions whose support lies entirely in the open set Ω or $\mathbb{R}^N \backslash \overline{\Omega}$,

$$Dv = \begin{cases} Du_1 & \text{in } \Omega; \\ Du_2 & \text{in } \mathbb{R}^N \backslash \overline{\Omega}. \end{cases} \tag{1.10}$$

Then, applying the first part of (1.9), we have

$$\begin{aligned} -\int_{\mathbb{R}^N} \varphi \, d[Dv] &= \int_{\mathbb{R}^N} v \operatorname{div}(\varphi) \, dx \\ &= -\int_\Omega \varphi \, d[Du_1] - \int_{\mathbb{R}^N \backslash \overline{\Omega}} \varphi \, d[Du_2] + \int_{\partial\Omega} (Tu_1 - Tu_2) \, \varphi \cdot \nu^\Omega \, d\mathcal{H}^{N-1}, \end{aligned}$$

so equation (1.10) implies that

$$-\int_{\partial\Omega} \varphi \, d[Dv] = \int_{\partial\Omega} (Tu_1 - Tu_2) \, \varphi \cdot \nu^\Omega \, d\mathcal{H}^{N-1}.$$

Thus, $|Dv|(\partial\Omega) = \int_{\partial\Omega} |Tu_1 - Tu_2| \, d\mathcal{H}^{N-1}$, which concludes the proof. ☐

1.4. Fine properties of BV functions

Finally, we turn our attention to pointwise properties of functions of bounded variation. We recall the notions of the reduced boundary and measure-theoretic boundary of a set of finite perimeter and use them to describe pointwise behavior of BV functions at the discontinuity points. As a preparation, let us recall the Radon-Nikodym theorem.

Memo 7. *Let μ and ν be two Radon measures on $\mathbb{R}^N$ and assume that $\nu \ll \mu$, i.e., for every Borel set $A \subset \mathbb{R}^N$ the condition $\mu(A) = 0$ implies $\nu(A) = 0$. For every $x \in \mathbb{R}^N$, denote*

$$\frac{d\nu}{d\mu}(x) := \begin{cases} \lim\limits_{r\to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r)) > 0 \text{ for all } r > 0; \\ +\infty & \text{if } \mu(B(x,r)) = 0 \text{ for some } r > 0. \end{cases}$$

Then, $\frac{d\nu}{d\mu}(x)$ is well-defined μ-a.e., and for any Borel set $A \subset \mathbb{R}^N$

$$\nu(A) = \int_A \frac{d\nu}{d\mu}\, d\mu,$$

i.e., $\frac{d\nu}{d\mu}$ is the density of ν with respect to μ. In particular, every Radon measure μ is absolutely continuous with respect to its total variation $|\mu|$, so the Radon-Nikodym derivative $\frac{d\mu}{d|\mu|}$ is well-defined $|\mu|$-a.e.

With this definition in mind, from now on we will denote by $\frac{Du}{|Du|}$ the Radon-Nikodym derivative of the distributional gradient Du with respect to its total variation $|Du|$; and by ν_E we denote the measure-theoretical outer normal to a set E of finite perimeter, i.e., $\nu_E = -\frac{Du}{|Du|}$.

Definition 1.36. Let E be a set of finite perimeter in $\mathbb{R}^N$. We say that $x \in \partial^* E$, the *reduced boundary* of E, if the following conditions hold:

(i) $|D\chi_E|(B(x,r)) > 0$ for all $r > 0$;
(ii)
$$\lim_{r\to 0^+} \fint_{B(x,r)} \nu_E \, d|D\chi_E| = \nu_E(x);$$
(iii) $|\nu_E(x)| = 1$.

By definition, the reduced boundary of a set of finite perimeter is where the perimeter measure $|D\chi_E|$ is concentrated; by the Lebesgue-Besikovitch differentiation theorem (Memo 8 below), we have

$$|D\chi_E|(\mathbb{R}^N \backslash \partial^* E) = 0.$$

Memo 8 (Lebesgue-Besikovitch differentiation theorem). *Let μ be a Radon measure on $\mathbb{R}^N$ and let $f \in L^1_{\text{loc}}(\mathbb{R}^N, \mu)$. Then,*

$$\lim_{r\to 0^+} \fint_{B(x,r)} f \, d\mu = f(x)$$

for μ-a.e. $x \in \mathbb{R}^N$.

An important related notion is that of the measure-theoretic boundary of E, which has a bit weaker properties, but is easier to deal with in specific applications.

Definition 1.37. Given a measurable set $E \subset \mathbb{R}^N$, we denote

$$E^{(1)} := \left\{ x \in \mathbb{R}^N \;:\; \lim_{r\to 0} \frac{\mathcal{L}^N(B(x,r) \cap E)}{\mathcal{L}^N(B(x,r))} = 1 \right\}$$

to be the set of points of density one. Similarly,

$$E^{(0)} := \left\{ x \in \mathbb{R}^N \ : \ \lim_{r \to 0} \frac{\mathcal{L}^N(B(x,r) \cap E)}{\mathcal{L}^N(B(x,r))} = 0 \right\}$$

is the set of points of density zero. Finally, we call

$$\partial_m E = \mathbb{R}^N \backslash (E^{(0)} \cup E^{(1)})$$

the measure-theoretic boundary of E.

EXERCISE 1.38. Let $E = [0,1]^2 \subset \mathbb{R}^2$. Find the sets $\partial^* E$ and $\partial_m E$.

THEOREM 1.39. *Suppose that E is a set of finite perimeter. Then, we have the inclusions*

$$\partial^* E \subset \partial_m E \subset \partial E.$$

PROOF. The inclusion $\partial_m E \subset \partial E$ is obvious; let us focus on the inclusion $\partial^* E \subset \partial_m E$. First, observe that replacing E with $E^{(1)}$ does not change $\partial^* E$ or $\partial_m E$, as these objects remain the same after modifications of E on a set of measure zero; for the rest of the proof, we will work with $E = E^{(1)}$.

Fix $x \in \partial^* E$. By the trace theorem (Theorem 1.28) applied to the case $\Omega = B(x,r)$ and $u = \chi_{E \cap B(x,r)}$, for all $\varphi \in C^1_c(\mathbb{R}^N; \mathbb{R}^N)$ we have

$$\int_{E \cap B(x,r)} \operatorname{div}(\varphi)\, dy = \int_{B(x,r)} \varphi \cdot \nu_E \, d|D\chi_E| + \int_{E \cap \partial B(x,r)} \varphi \cdot \nu^\Omega d\mathcal{H}^{N-1} \tag{1.11}$$

for $\mathcal{L}^1$-a.e. $r > 0$, i.e., those for which $|D\chi_E|(B(x,r)) = 0$. The minus sign in the second term disappears because ν_E is the outer normal. Thus, considering $|\varphi| \leqslant 1$ and using the representation of the total variation as a supremum, we get

$$|D\chi_{E \cap B(x,r)}|(\mathbb{R}^N) \leqslant \| D\chi_E|(B(x,r)) + \mathcal{H}^{N-1}(E \cap \partial B(x,r)). \tag{1.12}$$

Choose a test function $\varphi \in C^\infty_c(\mathbb{R}^N; \mathbb{R}^N)$ such that $\varphi \equiv \nu_E(x)$ on $B(x,r)$; this is clearly possible since $\nu_E(x)$ is a constant. Then, equation (1.11) becomes

$$0 = \int_{B(x,r)} \nu_E(x) \cdot \nu_E \, d|D\chi_E| + \int_{E \cap \partial B(x,r)} \nu_E(x) \cdot \nu^\Omega d\mathcal{H}^{N-1}$$

We rewrite this as

$$\int_{B(x,r)} \nu_E(x) \cdot \nu_E \, d|D\chi_E| = - \int_{E \cap \partial B(x,r)} \nu_E(x) \cdot \nu^\Omega d\mathcal{H}^{N-1},$$

and observe that the left hand side is arbitrarily close to $|D\chi_E|(B(x,r))$ as $r \to 0$ by the definition of reduced boundary; on the other hand, the right-hand side can be estimated from above by the $N-1$-dimensional Hausdorff measure of $E \cap \partial B(x,r)$. Thus, for sufficiently small $r > 0$ we have

$$\frac{1}{2}|D\chi_E|(B(x,r)) \leqslant H^{N-1}(E \cap \partial B(x,r)).$$

The above and estimate (1.12) give that

$$|D\chi_{E \cap B(x,r)}|(\mathbb{R}^N) \leqslant 3\mathcal{H}^{N-1}(E \cap \partial B(x,r)). \tag{1.13}$$

for sufficiently small $r > 0$.

Now, denote $g(r) = \mathcal{L}^N(B(x,r) \cap E)$. By the Fubini theorem in spherical coordinates, we have that

$$g(r) = \int_0^r \mathcal{H}^{N-1}(\partial B(x,s) \cap E)\, ds$$

and consequently $g'(r) = \mathcal{H}^{N-1}(\partial B(x,r) \cap E)$ for a.e. $r > 0$. Thus, by the isoperimetric inequality (Theorem 1.22) and estimate (1.13) we obtain

$$\begin{aligned} g(r)^{1-\frac{1}{N}} = (\mathcal{L}^N(B(x,r) \cap E))^{1-\frac{1}{N}} &\leqslant C|D\chi_{B(x,r)\cap E)}|(\mathbb{R}^N) \\ &\leqslant C\mathcal{H}^{N-1}(\partial B(x,r) \cap E) = Cg'(r). \end{aligned}$$

This differential inequality for g implies that

$$\frac{1}{C} \leqslant g'(r)g(r)^{\frac{1}{N}-1} = n(g^{\frac{1}{N}}(r))',$$

so $g^{\frac{1}{N}}(r) \geqslant \frac{r}{C\cdot N}$ and consequently $g(r) \geqslant cr^N$ for some $c > 0$ and sufficiently small $r > 0$. Thus,

$$\liminf_{r\to 0} \frac{\mathcal{L}^N(B(x,r) \cap E)}{r^N} > c > 0.$$

An analogous argument applied to $\mathbb{R}^N \backslash \Omega$ yields

$$\liminf_{r\to 0} \frac{\mathcal{L}^N(B(x,r) \backslash E)}{r^N} > c > 0,$$

and consequently $x \in \partial_m E$. The constant c depends only on N. □

The main feature of the reduced boundary of a set of finite perimeter is a well-defined approximate tangent hyperplane. For each $x \in \partial^* E$, we define the hyperplane

$$H(x) := \{y \in \mathbb{R}^N : \ \nu_E(x) \cdot (y - x) = 0\}$$

and the half-spaces

$$H^+(x) := \{y \in \mathbb{R}^N : \ \nu_E(x) \cdot (y - x) \geqslant 0\}$$

and

$$H^-(x) := \{y \in \mathbb{R}^N : \ \nu_E(x) \cdot (y - x) \leqslant 0\}.$$

The following result describes the local behaviour of E around a point in $\partial^* E$.

Theorem 1.40. *Assume $x \in \partial^* E$. Then,*

$$\lim_{r\to 0^+} \frac{\mathcal{L}^N(B(x,r) \cap E \cap H^+(x))}{r^N} = 0,$$

similarly

$$\lim_{r\to 0^+} \frac{\mathcal{L}^N((B(x,r) \backslash E) \cap H^-(x))}{r^N} = 0,$$

and

$$\lim_{r\to 0^+} \frac{|D\chi_E|(B(x,r))}{\omega_{N-1} r^{N-1}} = 1.$$

In fact, even a stronger claim is true, called the *blow-up of the reduced boundary*: for each $x \in \partial^* E$, we have

$$\chi_{\{y \in \mathbb{R}^N : x + r(y-x) \in E\}} \to \chi_{H^-(x)} \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N)$$

as $r \to 0$.

To conclude the discussion on the set $\partial^* E$, we recall (without proof) the main result underlining the importance of $\partial^* E$, which in particular shows that this set is nonempty for any set of finite perimeter.

THEOREM 1.41 (Structure theorem for sets of finite perimeter). *Let E be a set of finite perimeter. Then, we have*

$$|D\chi_E| = \mathcal{H}^{N-1}|_{\partial^* E}.$$

Furthermore,

$$\partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N,$$

where $|D\chi_E|(N) = 0$, K_k is a compact subset of a C^1 hypersurface S_k, and $\nu_E|_{S_k}$ is normal to S_k.

As an immediate consequence of this result, we have that

$$\mathcal{H}^{N-1}(\partial^* E) = P(E, \mathbb{R}^N).$$

Moreover, for any $x \in \partial_m E$ there exists a subsequence $r_n \to 0$ such that $\frac{\mathcal{L}^N(B(x,r) \cap E)}{r^N}$ converges to $\alpha \in (0,1)$; applying the relative isoperimetric inequality to this subsequence, we get

$$0 < \limsup_{r \to 0} \frac{\min(\alpha, 1-\alpha)\, r^N}{r^N} \leqslant C \cdot \limsup_{r \to 0} \frac{|D\chi_E|(B(x,r))}{r^{N-1}},$$

and since $|D\chi_E|(\mathbb{R}^N \backslash \partial^* E) = 0$, using a standard covering argument we get that $\mathcal{H}^{N-1}(\partial_m E \backslash \partial^* E) = 0$. As a consequence, $\partial_m E$ has finite $\mathcal{H}^{N-1}$ measure and

$$\mathcal{H}^{N-1}(\partial_m E) = \mathcal{H}^{N-1}(\partial^* E) = P(E, \mathbb{R}^N).$$

The sets $\partial^* E$ and $\partial_m E$ are used to describe a regular part of the boundary of a set of finite perimeter; the topological boundary, on the other hand, can in general be quite irregular.

EXAMPLE 1.42. Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and consider the following sequence of open balls $B(x_k, r_k)$. Enumerate the points of $(\mathbb{Q} \times \mathbb{Q}) \cap \Omega$ as $\{x_k\}$. Fix $\varepsilon > 0$, take $r_1 \leqslant \frac{\varepsilon}{2}$ small enough so that $B(x_1, r_1) \subset \Omega$, and denote $F_1 = B_1(x_1, r_1)$. Then, for any $k \geqslant 2$, we denote by $x_{n(k)}$ the first point among x_k for which $x_{n(k)} \notin F_{k-1}$, fix $r_k \leqslant \frac{\varepsilon}{2^k}$ small enough so that $B(x_{n(k)}, r_k) \subset \Omega \backslash F_{k-1}$, and set

$$F_k := \bigcup_{i=1}^{k} B(x_{n(i)}, r_i).$$

Then, F_k is an open set of finite perimeter, which satisfies

$$|F_k| = \sum_{i=1}^{k} \pi r_k^2 \leqslant \sum_{i=1}^{k} \pi \varepsilon^2 2^{-2i} \leqslant \pi \varepsilon^2$$

and

$$P(F_k, \Omega) = \sum_{i=1}^{k} 2\pi r_k \leqslant \sum_{i=1}^{k} 2\pi\varepsilon 2^{-i} \leqslant 2\pi\varepsilon.$$

Thus, if we denote

$$F_\infty := \bigcup_{i=1}^{\infty} B(x_{n(i)}, r_i),$$

we see that by $\chi_{F_k} \to \chi_{F_\infty}$ in $L^1(\Omega)$ and the above bounds we have $\chi_{F_\infty} \in BV(\Omega)$. Yet, F_∞ is dense in Ω, so $\overline{F_\infty} = \overline{\Omega}$, and consequently ∂F_∞ has positive Lebesgue measure (almost equal to the measure of Ω).

EXERCISE 1.43. Identify $\partial^* F$ and $\partial_m F$ in the above example.

We turn our attention to a more precise description of the discontinuity set of functions of bounded variation. We first recall the notion of the approximate discontinuity set of a locally integrable function.

DEFINITION 1.44. For a function $u \in L^1_{\mathrm{loc}}(\Omega)$, we denote by $u^\wedge(x)$ and $u^\vee(x)$ respectively its lower and upper approximate limits, i.e.:

$$u^\wedge(x) = \sup\left\{t \in \mathbb{R} : \lim_{r\to 0^+} \frac{\mathcal{L}^N(\{u \geqslant t\} \cap B(x,r))}{\mathcal{L}^N(B(x,r))} = 1\right\},$$

$$u^\vee(x) = \inf\left\{t \in \mathbb{R} : \lim_{r\to 0^+} \frac{\mathcal{L}^N(\{u \leqslant t\} \cap B(x,r))}{\mathcal{L}^N(B(x,r))} = 1\right\}.$$

We say that $x \in S_u$, the *approximate discontinuity set* of u, if $u^\wedge(x)$ and $u^\vee(x)$ do not coincide. For any $x \in \Omega\backslash S_u$, the real number $u^\wedge(x) = u^\vee(x)$, is called the *approximate limit* of u at x and is denoted by $\tilde{u}(x)$. Note that for $x \in \Omega\backslash S_u$, $\tilde{u}(x)$ is the unique real number satisfying

$$\lim_{r\to 0^+} \fint_{B(x.r)} |u(y) - \tilde{u}(x)|\, dy = 0.$$

Now, we recall the definition of the jump set J_u of a BV function.

DEFINITION 1.45. Let $u \in BV(\Omega)$. We say that $x \in J_u$, the *jump set* of u, if there exists a unit vector ν (called the normal vector) and real numbers $a \neq b$ such that

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\{\langle y-x,\nu\rangle>0\}} |u(y) - a|\, dy = 0,$$

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\{\langle y-x,\nu\rangle<0\}} |u(y) - b|\, dy = 0.$$

The triple (a, b, ν) is uniquely determined up to permutation of a, b and the sign of ν and is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

Clearly, for a set E of finite perimeter, the approximate discontinuity set S_{χ_E} agrees with $\partial_m E$, and by the structure theorem for sets of finite perimeter the jump set J_{χ_E} agrees with the reduced boundary $\partial^* E$ up to a set of zero $\mathcal{H}^{N-1}-$measure zero. In general, the relationship between the two sets is a bit more complicated; we give without proof the following result (which essentially follows from the structure theorem for sets of finite perimeter).

Theorem 1.46 (Federer-Vol'pert theorem). *Let $u \in BV(\Omega)$. Then, S_u is countably $\mathcal{H}^{N-1}$-rectifiable, i.e., it is contained in a countable union of Lipschitz (even C^1) hypersurfaces up to a set of zero $\mathcal{H}^{N-1}$-measure; J_u is a Borel subset of S_u; and*

$$\mathcal{H}^{N-1}(S_u \backslash J_u) = 0.$$

On a final note, let us discuss how we can use the definition of the jump set to analyse in more detail the properties of the distributional derivative Du. To this end, we first recall the Lebesgue decomposition theorem.

Memo 9 (Lebesgue decomposition theorem). *Let μ and ν be Radon measures on $\mathbb{R}^N$. Then, one can uniquely decompose*

$$\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{s}},$$

where ν_{ac}, ν_{s} are Radon measures on $\mathbb{R}^N$ which satisfy $\nu_{\mathrm{ac}} \ll \mu$ and $\nu_{\mathrm{s}} \perp \mu$.

Definition 1.47. For $u \in BV(\Omega)$, we call

$$D^a u = \nabla u \, \mathcal{L}^N$$

where ∇u is the Radon-Nikodym derivative of Du with respect to the Lebesgue measure $\mathcal{L}^N$, the *absolutely continuous part of the derivative*; we call the measure

$$D^j u := D^s u|_{J_u}$$

the *jump part of derivative*; and we call

$$D^c u := D^s u|_{\Omega \backslash S_u}$$

the *Cantor part of derivative.*

Observe that the definition of the jump set implies that

$$D^j u = (u^+ - u^-)\, \nu_u \, \mathcal{H}^{N-1}|_{J_u}$$

as measures. Applying twice the Lebesgue decomposition theorem, we see that the following decomposition of Du holds:

$$Du = D^a u + D^j u + D^c u.$$

We stress that this decomposition of the measure Du does not necessarily hold at the level of function; the following example appears in [**1**].

Example 1.48. Let $\Omega = B(0,1) \subset \mathbb{R}^2$ and take $S = (-1, 0] \times 0$. Define the function $u : \Omega \to \mathbb{R}$ using polar coordinates by the formula $u(r,\theta) = \sqrt{r}\sin(\frac{\theta}{2})$. Then, it is clear that $u \in BV(\Omega)$ with $J_u = S_u = S\backslash\{0,0\}$, and that the Cantor part of the derivative is equal to zero. Then, if one could decompose u as $u = u_a + u_j$, where $u \in W^{1,1}(\Omega)$ and u_j has only jump-type derivative, we would have

$$\nabla(u - u_a) = \nabla u_j = 0,$$

so by the Poincaré inequality $u - u_a$ is a constant, since $u - u_a \in W^{1,1}(\Omega\backslash S)$ and $\Omega\backslash S$ is connected. But then $u = u_a + (u - u_a) \in W^{1,1}(\Omega)$, a contradiction.

Definition 1.49. We say that $u \in BV(\Omega)$ is a *special function of bounded variation*, and we write $u \in SBV(\Omega)$, if the Cantor part of its derivative $D^c u$ is zero. In other words, for all $u \in SBV(\Omega)$ we have

$$Du = D^a u + D^j u = \nabla u \, \mathcal{L}^N + (u^+ - u^-)\, \nu_u \, \mathcal{H}^{N-1} \llcorner J_u.$$

The space $SBV(\Omega)$ is a closed subspace of $BV(\Omega)$.

Further reading

For more information, we refer to [**1**], [**23**], [**28**], and [**43**].

CHAPTER 2

First look at variational problems

In this lecture, we present some basic features a variational problems involving functionals of linear growth. It turns out that many techniques known from the p-growth case, when the problems are formulated in the Sobolev space $W^{1,p}$ with $p > 1$, fail in this case. As a model problem, consider the Dirichlet problem for the p-Laplacian in a smooth domain $\Omega \subset \mathbb{R}^N$, i.e.,

$$\begin{cases} -\mathrm{div}\Big(|\nabla u|^{p-2} \nabla u \Big) = 0 & \text{in } \Omega; \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where $f \in W^{1-\frac{1}{p},p}(\partial\Omega)$ (the trace space of Sobolev functions), which is the Euler-Lagrange equation of the minimisation problem

$$\min \bigg\{ \frac{1}{p} \int |\nabla u|^p : \quad u \in W^{1,p}(\Omega), \quad u|_{\partial\Omega} = f \bigg\}. \tag{2.1}$$

Using the direct method of calculus of variations, it is easy to prove existence of minimisers to this problem. Clearly, the functional on $W^{1,p}(\Omega)$ given by

$$F(u) = \begin{cases} \dfrac{1}{p} \displaystyle\int |\nabla u|^p & \text{if } u|_{\partial\Omega} = f \\ +\infty & \text{otherwise} \end{cases}$$

is bounded from below and proper (i.e., not identically equal to $+\infty$). Thus, we can find a minimising sequence u_n for the minimisation problem (2.1). Now, observe that the functional F is finite and coercive on

$$W^{1,p}_f(\Omega) := \bigg\{ u \in W^{1,p}(\Omega) : \quad u|_{\partial\Omega} = f \bigg\},$$

i.e., boundedness of $F(u_n)$ implies boundedness of $\|u_n\|_{W^{1,p}(\Omega)}$: indeed, if $v \in W^{1,p}(\Omega)$ denotes any extension of f, then by the Poincaré inequality for any $u \in W^{1,p}_f(\Omega)$ we have

$$\begin{aligned} \|u\|_{L^p(\Omega)} \leqslant \|u - v\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} &\leqslant \|\nabla(u - v)\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \\ &\leqslant \|\nabla u\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \\ &= (pF(u))^{1/p} + \|\nabla v\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}, \end{aligned}$$

which yields the claim since v is a fixed function. Therefore, the minimising sequence u_n is bounded in $W^{1,p}(\Omega)$ and as such has a weakly convergent subsequence. Since $W^{1,p}_f(\Omega)$ is weakly closed, we obtain that the limit function u lies in $W^{1,p}_f(\Omega)$, and hence the functional F is sequentially weakly lower semicontinuous. Therefore,

$$F(u) \leqslant \liminf_{n\to\infty} F(u_n) \to \inf F,$$

so u is a solution to problem (2.1). It is unique since F is a strictly convex functional.

If we now consider the *least gradient problem*, i.e., set $p = 1$ in the above calculation, the argument falls apart. The corresponding functional is bounded from below and proper, and it is finite and coercive on

$$BV_f(\Omega) := \Big\{ u \in BV(\Omega) : \quad u|_{\partial\Omega} = f \Big\},$$

so the minimising sequence u_n exists and is bounded in $BV(\Omega)$, but this is not enough to conclude the proof. Indeed, the subsequence we would obtain is only weakly* convergent, and the space $BV_f(\Omega)$ is not closed with respect to weak* convergence (as we saw in Example 1.31). Thus, the functional F is not sequentially weakly* lower semicontinuous, and we cannot conclude that a minimiser exists; indeed, at the end of the lecture we will give an example of nonexistence of minimisers for this problem. Furthermore, we will also see that uniqueness of solutions may also fail.

In general, we are interested in minimisation problems which involve functionals of linear growth, i.e., ones for which the term involving the gradient is of the form $\int_\Omega f(Du)$ with

$$m|p| - c \leqslant f(p) \leqslant M(1 + |p|)$$

for all $p \in \mathbb{R}^N$. The particular cases we consider in these lectures are the ROF functional, the associated gradient flow, and the least gradient problem. For simplicity, from now on in the whole lecture series we assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary.

2.1. First example: ROF functional

The Rudin-Osher-Fatemi functional $E : L^2(\Omega) \to [0, +\infty]$, which is the basis of total variation denoising, is defined by the formula

$$E(u) = \begin{cases} \displaystyle\int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega (u - f)^2 \, dx & \text{if } u \in BV(\Omega) \cap L^2(\Omega); \\ +\infty & \text{if } u \in L^2(\Omega) \backslash BV(\Omega), \end{cases}$$

where $\lambda > 0$ is a bias parameter which measures how close the denoised image u is to the original image f. This model first appeared in [**40**] and was designed to sharpen existing edges in a given picture; we will say more about motivations and relationship to other problems in the next lectures, and for now we focus only on its analytical properties.

Proposition 2.1. *The functional E is lower semicontinuous with respect to convergence in $L^2(\Omega)$.*

Proof. Suppose that $u_n \to u$ in $L^2(\Omega)$. Without loss of generality, we may assume that

$$\liminf_{n \to \infty} E(u_n) \leqslant M < \infty,$$

which implies

$$\liminf_{n \to \infty} \int_\Omega |Du_n| \leqslant M < \infty,$$

and consequently by the lower semicontinuity of the total variation $u \in BV(\Omega)$ (since u_n also converges to u in $L^1(\Omega)$). Thus,

$$\liminf_{n\to\infty}\left(\int_\Omega |Du_n| + \frac{\lambda}{2}\int_\Omega (u_n - f)^2\,dx\right)$$
$$= \liminf_{n\to\infty}\int_\Omega |Du_n| + \lim_{n\to\infty}\frac{\lambda}{2}\int_\Omega (u_n - f)^2\,dx \geqslant \int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega (u-f)^2\,dx,$$

which concludes the proof. □

Recall that for any given normed space X if a convex function $E : X \to (-\infty, +\infty]$ is lower semicontinuous with respect to norm convergence, then it is lower semicontinuous with respect to weak convergence; thus, E is also lower semicontinuous with respect to weak convergence in $L^2(\Omega)$.

THEOREM 2.2. *The functional E has a unique minimiser in $L^2(\Omega)$.*

PROOF. We use the direct method of calculus of variations. Obviously, the functional E is bounded from below and proper. Thus, there exists a minimising sequence u_n for the minimisation of E; since a minimising sequence satisfies

$$\liminf_{n\to\infty} E(u_n) \leqslant M < \infty,$$

we also have

$$\liminf_{n\to\infty}\int_\Omega |u_n - f|^2 \leqslant M < \infty,$$

and therefore u_n is bounded in $L^2(\Omega)$ (up to taking a subsequence), so u_n converges on a subsequence in the weak topology of $L^2(\Omega)$ to some $u \in L^2(\Omega)$. Then, since E is lower semicontinuous with respect to weak convergence in $L^2(\Omega)$, we have

$$\inf E = \lim_{n\to\infty} E(u_n) \geqslant E(u),$$

hence u is a minimiser of E. Since E is strictly convex, the minimiser is unique. □

EXERCISE 2.3. Verify whether the ROF functional with L^1 fidelity term also has similar properties, i.e., if $E_{L^1} : L^1(\Omega) \to [0, +\infty]$ given by

$$E_{L^1}(u) = \begin{cases} \displaystyle\int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega |u-f|\,dx & \text{if } u \in BV(\Omega); \\ +\infty & \text{if } u \in L^1(\Omega)\backslash BV(\Omega) \end{cases}$$

is lower semicontinuous with respect to L^1-convergence and has a unique minimiser.

2.2. Relaxation of the functional for the least gradient problem

The next part of this lecture is dedicated to an example of a functional which is not lower semicontinuous. The lack of lower semicontinuity here is related to the Dirichlet boundary data; we briefly mention other types of failure of lower semicontinuity in Theorem 2.9.

The least gradient problem concerns the minimisation of total variation of a function for given Dirichlet boundary data $f \in L^1(\partial\Omega)$, i.e.,

$$\min \left\{ \int_\Omega |Du| : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \right\}. \tag{2.2}$$

Let us consider the energy functional $J : L^1(\Omega) \to [0, \infty]$ associated to the least gradient problem (2.2), i.e., defined by the formula

$$J(u) = \begin{cases} \displaystyle\int_\Omega |Du|, & \text{if } u \in BV(\Omega) \text{ and } u = f \text{ on } \partial\Omega; \\ +\infty & \text{otherwise.} \end{cases}$$

Following the classical reasoning appearing for instance in [**2**] or [**26**], we will see that the functional $\mathcal{J} : L^1(\Omega) \to (-\infty, +\infty]$ defined by

$$\mathcal{J}(u) = \begin{cases} \displaystyle\int_\Omega |Du| + \int_{\partial\Omega} |u - f| \, d\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega); \\ +\infty & \text{if } u \in L^1(\Omega) \backslash BV(\Omega), \end{cases}$$

is the relaxation of the functional J.

MEMO 10 (Relaxation). *Given a functional $F : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$, we call its sequentially lower semicontinuous envelope $\mathcal{F} : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$, i.e.,*

$$\mathcal{F}(u) = \inf \left\{ \liminf_{n\to\infty} F(u_n) : \ u_n \to u \text{ in } L^1(\Omega) \right\}$$

the relaxation of F.

This definition is strongly related to the notion of Γ-convergence: relaxation of a functional arises once one considers a Γ-limit of a constant sequence.

MEMO 11 (Γ-convergence). *Assume that X is a topological space such that each point has a countable local basis of neighbourhoods (e.g., X is metric). Then, we say that a sequence of functionals $F_n : X \to \mathbb{R} \cup \{+\infty\}$ Γ-converges to a functional $F : X \to \mathbb{R} \cup \{+\infty\}$, if the following two conditions are satisfied:*

1. *For any sequence $x_n \in X$ such that $x_n \to x$, we have*
$$F(x) \leqslant \liminf_{n\to\infty} F_n(x_n);$$
2. *For any $x \in X$ there exists a sequence $x_n \to x$ such that*
$$F(x) \geqslant \limsup_{n\to\infty} F_n(x_n).$$

The main property of Γ-convergence related to calculus of variations is that if x_n are minimisers of F_n, then every cluster point of the sequence $\{x_n\}$ is a minimiser of F. Furthermore, Γ-limits are automatically lower semicontinuous, and the Γ-limit of a constant sequence $F_n := F$ is its relaxation $\mathcal{F}$.

To compute the relaxation of J, we first prove the following result.

PROPOSITION 2.4. *The functional $\mathcal{J}$ is lower semicontinuous on $L^1(\Omega)$.*

PROOF. Let $\psi \in W^{1,1}(\mathbb{R}^N \backslash \overline{\Omega})$ be a function with compact support and trace f on $\partial\Omega$. Denote by $u_\psi \in BV(\mathbb{R}^N)$ the function defined by

$$u_\psi(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ \psi(x) & \text{if } x \in \mathbb{R}^N \backslash \overline{\Omega}. \end{cases}$$

By Theorem 1.35,

$$\int_{\mathbb{R}^N} |Du_\psi| = \int_\Omega |Du| + \int_{\partial\Omega} |u - f| \, d\mathcal{H}^{N-1} + \int_{\mathbb{R}^N \backslash \Omega} |\nabla\psi(x)| \, dx.$$

We rewrite the above as follows:

$$\mathcal{J}(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u - f| \, d\mathcal{H}^{N-1} = \int_{\mathbb{R}^N} |Du_\psi| - \int_{\mathbb{R}^N \backslash \Omega} |\nabla\psi(x)| \, dx.$$

Now, suppose that $u_n \to u$ in $L^1(\Omega)$. In particular, also $(u_n)_\psi \to u_\psi$ in $L^1(\mathbb{R}^N)$. Then, by the lower semicontinuity of the total variation,

$$\begin{aligned} \liminf_{n\to\infty} \mathcal{J}(u_n) &= \liminf_{n\to\infty} \int_{\mathbb{R}^N} |D(u_n)_\psi| - \int_{\mathbb{R}^N \backslash \Omega} |\nabla\psi(x)| \, dx \\ &\geqslant \int_{\mathbb{R}^N} |Du_\psi| - \int_{\mathbb{R}^N \backslash \Omega} |\nabla\psi(x)| \, dx = \mathcal{J}(u), \end{aligned}$$

so the functional $\mathcal{J}$ is lower semicontinuous on $L^1(\Omega)$. □

PROPOSITION 2.5. *Given $u \in BV(\Omega)$, there exists a sequence $u_n \in W^{1,1}(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$, $u_n = f$ on $\partial\Omega$ and*

$$\mathcal{J}(u) = \lim_{n\to\infty} J(u_n).$$

PROOF. We set $g = f - u$ on $\partial\Omega$. Let w_n be the sequence given by the Meyers-Serrin theorem (Theorem 1.16), applied for u, and let v_n be the sequence given by the Gagliardo extension theorem (Lemma 1.32), applied for g. We have $w_n \to u$ in $L^1(\Omega)$, $v_n \to 0$ in $L^1(\Omega)$ and $v_n = g$ on $\partial\Omega$. Moreover, we rewrite the estimate in Lemma 1.32 as

$$\int_\Omega |Dv_n| \leqslant \int_{\partial\Omega} |u - f| \, d\mathcal{H}^{N-1} + \frac{1}{n}.$$

Set $u_n = v_n + w_n$. Then, $u_n \in W^{1,1}(\Omega)$, $u_n \to u$ in $L^1(\Omega)$ and $u_n = f$ on $\partial\Omega$. We estimate

$$\mathcal{J}(u_n) = \int_\Omega |Du_n| \leqslant \int_\Omega |Dv_n| + \int_\Omega |Dw_n| \leqslant \int_{\partial\Omega} |u - f| \, d\mathcal{H}^{N-1} + \frac{1}{n} + \int_\Omega |Dw_n|.$$

Now, we take the upper limit in the above series of inequalities. By the lower semicontinuity of $\mathcal{J}$ given in Proposition 2.4, we get

$$\begin{aligned} \mathcal{J}(u) \leqslant \liminf_{n\to\infty} \mathcal{J}(u_n) \leqslant \limsup_{n\to\infty} \mathcal{J}(u_n) &\leqslant \limsup_{n\to\infty} \int_\Omega |Dw_n| + \int_{\partial\Omega} |u - f| \, d\mathcal{H}^{N-1} \\ &= \lim_{n\to\infty} \int_\Omega |Dw_n| + \int_{\partial\Omega} |u - f| \, d\mathcal{H}^{N-1} = \mathcal{J}(u). \end{aligned}$$

Hence, all the inequalities above are in fact equalities and u_n satisfies all the desired properties. □

Finally, notice that Propositions 2.4 and 2.5 immediately imply the following Theorem.

THEOREM 2.6. *Then, the relaxation of the functional J is the functional $\mathcal{J}$, i.e.,*

$$\mathcal{J}(u) = \inf \left\{ \liminf_{n\to\infty} J(u_n) \ : \ u_n \to u \text{ in } L^1(\Omega), \ u_n = f \text{ on } \partial\Omega \right\}.$$

Therefore, the functional $\mathcal{J}$ is the 'correct' functional when we want to study the least gradient problem. We will come back to this in the last lecture. Once we identified the relaxed functional, we can apply the direct method of calculus of variations to conclude that the functional $\mathcal{J}$ has a minimiser.

PROPOSITION 2.7. *The functional $\mathcal{J}$ has a minimiser for any $f \in L^1(\Omega)$.*

PROOF. Clearly, the functional $\mathcal{J}$ is bounded from below and proper (i.e., not identically equal to $+\infty$) on $BV(\Omega)$. Moreover, $\mathcal{J}$ is coercive, i.e., boundedness of a family $\mathcal{J}(u_n)$ implies boundedness of $\|u_n\|_{BV(\Omega)}$: indeed, if we extend u by zero to a function in $BV(\mathbb{R}^N)$, then by the Sobolev inequality (Theorem 1.20; see that the support of u is bounded) and Theorem 1.35 we have

$$\begin{aligned}
\|u\|_{L^1(\Omega)} = \|u\|_{L^1(\mathbb{R}^N)} &\leqslant C \int_{\mathbb{R}^N} |Du| = C \int_\Omega |Du| + C \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \\
&\leqslant C \int_\Omega |Du| + C \int_{\partial\Omega} |u - f| \, d\mathcal{H}^{N-1} + C \int_{\partial\Omega} |f| \, d\mathcal{H}^{N-1} \\
&\leqslant C\mathcal{J}(u) + C \int_{\partial\Omega} |f| \, d\mathcal{H}^{N-1}.
\end{aligned}$$

Thus, there is a minimising sequence u_n for the least gradient problem and it is bounded in $BV(\Omega)$. By Theorem 1.24 it admits a convergent subsequence in $L^1(\Omega)$. Since $\mathcal{J}$ is lower semicontinuous in $L^1(\Omega)$ by virtue of Proposition 2.4,

$$\mathcal{J}(u) \leqslant \liminf_{n\to\infty} \mathcal{J}(u_n) \to \inf \mathcal{J},$$

so u is a minimiser of $\mathcal{J}$. □

However, in this case the minimiser may fail to be unique as $\mathcal{J}$ is only convex and not strictly convex; consider the following simple example.

EXERCISE 2.8. Let $\Omega = (0,1)$ and take boundary data given by $f(0) = 0$ and $f(1) = 1$. Then, show that any nondecreasing function $u \in BV((0,1))$ with $u(0), u(1) \in [0,1]$ is a minimiser of $\mathcal{J}$.

For general functionals of linear growth, identification of the relaxed functional is one of the main priorities in their study. The results are much more difficult to show and we restrict ourselves to the following statement (without proof; the assumptions are not optimal, see [**4**] or [**1**]).

THEOREM 2.9. *Suppose that $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is smooth and that for all $p \in \mathbb{R}^N$ the following limit exists:*

$$f^\infty(x,p) := \lim_{t\to\infty} \frac{f(x,tp)}{t}.$$

Assume additionally that there exists positive constants such that the following conditions are satisfied:

(a) f is nonnegative and convex in the second variable;
(b) f has linear growth, i.e.,

$$c_1|p| - c_2 \leqslant f(x,p) \leqslant c_3(1+|p|);$$

(c) f is locally uniformly continuous with respect to x, or more precisely

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x,p) - f(x_0,p)| \leqslant \varepsilon c_4(1+|p|);$$

(d) The rate of convergence f to f^∞ along each ray is of order $O(t^{-m})$, i.e.,

$$\text{For all } t > 1 \text{ we have } |f(x,tp)/t - f^\infty(x,p)| \leqslant c_5 t^{-m}.$$

Then, if we define a functional $E : L^1(\Omega) \to [0,+\infty]$ by the formula

$$E(u) = \begin{cases} \displaystyle\int_\Omega f(x, \nabla u(x))\, dx & \text{if } u \in W^{1,1}(\Omega); \\ +\infty & \text{otherwise,} \end{cases}$$

its relaxation is the functional $\overline{E} : L^1(\Omega) \to [0,+\infty]$ given by

$$\overline{E}(u) = \begin{cases} \displaystyle\int_\Omega f(x, \nabla u(x))\, dx + \int_\Omega f^\infty\left(x, \frac{dD^s u}{d|D^s u|}\right) d|D^s u| & \text{if } u \in BV(\Omega); \\ +\infty & \text{otherwise,} \end{cases}$$

where $D^s u$ is the singular part of Du with respect to the Lebesgue measure (i.e., $D^s u = D^j u + D^c u$) and $\frac{dD^s u}{d|D^s u|}$ is the Radon-Nikodym derivative.

Example 2.10. Using this theorem, we immediately see that the relaxation of

$$E_1(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|\, dx & \text{if } u \in W^{1,1}(\Omega); \\ +\infty & \text{otherwise} \end{cases}$$

is the total variation, i.e.,

$$\overline{E_1}(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|\, dx + \int_\Omega |D^s u| & \text{if } u \in BV(\Omega); \\ +\infty & \text{otherwise,} \end{cases}$$

and the relaxation of the area functional

$$E_2(u) = \begin{cases} \displaystyle\int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx & \text{if } u \in W^{1,1}(\Omega); \\ +\infty & \text{otherwise} \end{cases}$$

is the functional

$$\overline{E_2}(u) = \begin{cases} \displaystyle\int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx + \int_\Omega |D^s u| & \text{if } u \in BV(\Omega); \\ +\infty & \text{otherwise.} \end{cases}$$

2.3. Why we need to consider relaxations

The variational problem of minimising the integral of the gradient of a function was first considered by Miranda in [**36**] in connection to the study of area-minimising sets. Because this functional has linear growth, the natural energy space for minimizers is the space of functions of bounded variation. The first rigorous definition of solutions was proposed by Miranda in [**36**].

DEFINITION 2.11. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Given $u \in BV(\Omega)$, we say that u is a *function of least gradient* in Ω, if for all $g \in BV(\Omega)$ with compact support $K \subset \Omega$ we have

$$\int_K |Du| \leqslant \int_K |D(u+g)|.$$

Equivalently, one may assume that g is a BV function with zero trace on $\partial\Omega$.

This definition is a local version of the minimisation of $\mathcal{J}$; the relationship is described in the following result.

PROPOSITION 2.12. *For $v \in BV(\Omega)$ satisfying $v|_{\partial\Omega} = f \in L^1(\partial\Omega)$, the following conditions are equivalent:*

(i) $\mathcal{J}(v) \leqslant \mathcal{J}(u)$ for all $u \in BV(\Omega)$.
(ii) v is a function of least gradient.

PROOF. (i) implies (ii): If we consider competitors which satisfy $u|_{\partial\Omega} = f$, we get

$$\int_\Omega |Dv| = \mathcal{J}(v) \leqslant \mathcal{J}(u) = \int_\Omega |Du|,$$

which proves the first implication.

(ii) implies (i): Given $u \in BV(\Omega)$, we have to see that $\mathcal{J}(v) \leqslant \mathcal{J}(u)$. Fix $\varepsilon > 0$ and apply the Gagliardo extension theorem (Lemma 1.32). We find $w \in W^{1,1}(\Omega)$ satisfying

$$w|_{\partial\Omega} = u|_{\partial\Omega} - f; \tag{2.3}$$

$$\int_\Omega |Dw| \leqslant \int_{\partial\Omega} |u-f|\, d\mathcal{H}^{N-1} + \varepsilon; \tag{2.4}$$

and

$$w(x) = 0 \quad \text{if} \quad \operatorname{dist}(x,\partial\Omega) > \varepsilon.$$

Now, consider the function $u - w$. By (2.3), its trace on $\partial\Omega$ is f. So we may use (iii) to deduce that

$$\begin{aligned}\int_\Omega |Dv| \leqslant \int_\Omega |D(u-w)| &\leqslant \int_\Omega |Du| + \int_\Omega |Dw| \\ &\leqslant \int_\Omega |Du| + \int_{\partial\Omega} |u-f|\, d\mathcal{H}^{N-1} + \varepsilon\end{aligned}$$

due to (2.4). Thus,

$$\mathcal{J}(v) = \int_\Omega |Dv| \leqslant \mathcal{J}(u) + \varepsilon.$$

Since ε was arbitrary, it follows that $\mathcal{J}(v) \leqslant \mathcal{J}(u)$ holds. □

Now, we prove Miranda's theorem on stability of functions of least gradient functions (see [**36**]). It was first used to study the local properties of regular points of area-minimising sets.

THEOREM 2.13. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Suppose that $u_n \in BV(\Omega)$ is a sequence of functions of least gradient in Ω, uniformly bounded in $L^\infty(\Omega)$, and let $u_n \to u$ in $L^1(\Omega)$. Then, $u \in BV(\Omega)$ and it is a function of least gradient in Ω.*

In place of the assumption that u_n is uniformly bounded in $L^\infty(\Omega)$, we may just assume that the limit function lies in $BV(\Omega)$; this assumption is only used to show that the sequence u_n is bounded in $BV(\Omega)$.

PROOF. By the lower semicontinuity of the total variation, it is enough to show the following estimate

$$\sup_{n \in \mathbb{N}} \int_\Omega |Du_n| < \infty. \tag{2.5}$$

To see this, denote

$$\Omega_t = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > t\}$$

and recall that for sufficiently small $t > 0$ the set Ω_t has Lipschitz boundary. Clearly, $\overline{\Omega_t}$ is relatively compact in Ω. Then, for almost all $t > 0$ we can pick t in such a way that for all $n \in \mathbb{N}$

$$\int_{\partial\Omega_t} |Du_n| = 0. \tag{2.6}$$

Now, for any $n \in \mathbb{N}$, we let

$$g_n(x) := \begin{cases} 0 & \text{if } x \in \Omega_t; \\ u_n & \text{if } x \in \Omega \backslash \Omega_t. \end{cases}$$

Since $\partial\Omega_t$ is Lipschitz, by Theorem 1.35 and equation (2.6) we have

$$\int_{\overline{\Omega_t}} |Dg_n| = \int_{\partial\Omega_t} |u_n| \, d\mathcal{H}^{N-1} \quad \forall n \in \mathbb{N}.$$

Since u_n is a function of least gradient in Ω, we get

$$\int_{\Omega_t} |Du_n| = \int_{\overline{\Omega_t}} |Du_n| \leqslant \int_{\overline{\Omega_t}} |Dg_n| = \int_{\partial\Omega_t} |u_n| \, d\mathcal{H}^{N-1}.$$

By the assumption that u_n is uniformly bounded in $L^\infty(\Omega)$, the right hand side is uniformly bounded for all $n \in \mathbb{N}$ and $t > 0$, so (2.5) holds and by the lower semicontinuity of the total variation we have $u \in BV(\Omega)$.

Finally, let us see that u is a function of least gradient in Ω. Suppose that $g \in BV(\Omega)$ has compact support $K \subset \Omega$. We need to show that

$$\int_K |Du| \leqslant \int_K |D(u+g)|. \tag{2.7}$$

Let A be an open set with Lipschitz boundary, relatively compact in Ω. Assume additionally that $K \subset A$, it satisfies (2.6),

$$\int_{\partial A} |Du| = 0$$

and

$$\lim_{n\to\infty}\int_{\partial A}|u-u_n|\,d\mathcal{H}^{N-1}=0$$

(we can find such a set thanks to the co-area formula for Lipschitz functions applied to $u=\operatorname{dist}(x,\partial\Omega)$, see Memo 4). For $n\in\mathbb{N}$, let

$$f_n:=\begin{cases} u+g & \text{in } A;\\ u_n & \text{in } \Omega\backslash A.\end{cases}$$

Then, by (2.6) and (2.7), applying Theorem 1.35 and the fact that $g=0$ in $\Omega\backslash K$, we have

$$\int_{\overline{A}}|Df_n|=\int_A|D(u+g)|+\int_{\partial A}|u-u_n|\,d\mathcal{H}^{N-1}\quad\forall n\in\mathbb{N}.$$

Hence, since u_n is a function of least gradient in Ω, we have

$$\int_{\overline{A}}|Du_n|\leqslant\int_A|D(u+g)|+\int_{\partial A}|u-u_n|\,d\mathcal{H}^{N-1}\quad\forall n\in\mathbb{N},$$

so

$$\int_{A}|Du_n|\leqslant\int_A|D(u+g)|+\int_{\partial A}|u-u_n|\,d\mathcal{H}^{N-1}\quad\forall n\in\mathbb{N}.$$

Thus, by the lower semicontinuity of the total variation with respect to the convergence in L^1, we obtain that

$$\int_A|Du|\leqslant\int_A|D(u+g)|. \tag{2.8}$$

Finally, (2.7) is consequence of the inclusion $K\subset A$, property (2.8) and the fact that $g=0$ in $\Omega\backslash K$. □

EXERCISE 2.14. Let $\Omega=(0,1)$. Show that if $u\in L^1((0,1))$ is an unbounded increasing function, then the truncations $u_n=T_n(u)$ are functions of least gradient which converge to u in $L^1(\Omega)$, but the limit does not lie in $BV(\Omega)$.

The main motivation behind the above result (as suggested by the title of Miranda's paper [**36**]) is that if we consider a set which is a limit of a sequence of area-minimising sets, then it is itself area-minimising. We start with the following classical definition.

DEFINITION 2.15. Suppose that $E\subset\mathbb{R}^N$ is a set of finite perimeter in an open set Ω. We say that E is *area-minimising* in Ω, whenever

$$P(E,\Omega)=\inf\{P(F,\Omega):\ E\Delta F\Subset\Omega\}.$$

Clearly, whenever E is a set of finite perimeter in an open set Ω and χ_E is a function of least gradient in Ω, then E is area-minimising in Ω.

EXERCISE 2.16. Use the co-area formula to prove that the converse also holds, i.e., E is area-minimising in Ω if and only if χ_E is a function of least gradient in Ω.

The most important property of functions of least gradient is their connection to area-minimising sets. The result in the general setting was proved in [**8**] and states that boundaries of superlevel sets of a function of least gradient are area-minimising.

THEOREM 2.17. *Suppose that $u \in BV(\Omega)$ is a function of least gradient in Ω. Then, for all $t \in \mathbb{R}$, the functions $\chi_{\{u>t\}}$ and $\chi_{\{u \geqslant t\}}$ are also functions of least gradient in Ω.*

PROOF. By the coarea formula (Theorem 1.19), for almost all $t \in \mathbb{R}$ we have $\chi_{\{u>t\}} \in BV(\Omega)$ and

$$\int_\Omega |Du| = \int_{-\infty}^{+\infty} \left(\int_\Omega |D\chi_{\{u>t\}}| \right) dt. \tag{2.9}$$

For $t \in \mathbb{R}$, let

$$u_1 := \max\{u - t, 0\}, \quad u_2 := \min\{u, t\}.$$

Clearly, we have $u_1, u_2 \in BV(\Omega)$. Moreover, $u = u_1 + u_2$ and by equation (2.9)

$$\int_\Omega |Du| = \int_\Omega |Du_1| + \int_\Omega |Du_2|.$$

Then, given any $g \in BV(\Omega)$, we have

$$\int_\Omega |Du_1| + \int_\Omega |Du_2| = \int_\Omega |Du| \leqslant \int_\Omega |D(u+g)| \leqslant \int_\Omega |D(u_1+g)| + \int_\Omega |Du_2|$$

and

$$\int_\Omega |Du_1| + \int_\Omega |Du_2| = \int_\Omega |Du| \leqslant \int_\Omega |D(u+g)| \leqslant \int_\Omega |D(u_2+g)| + \int_\Omega |Du_1|,$$

which shows that u_1 and u_2 are functions of least gradient in Ω. Therefore, the functions

$$u_{\varepsilon,t} := \frac{1}{\varepsilon} \min\{\varepsilon, (u-t)^+\}$$

are functions of least gradient in Ω for every $\varepsilon > 0$ and $t \in \mathbb{R}$. Now, since

$$\lim_{\varepsilon \to 0^+} \int_\Omega |u_{\varepsilon,t} - \chi_{\{u>t\}}| \, dx = 0,$$

by Theorem 2.13 it follows that $\chi_{\{u>t\}} \in BV(\Omega)$ and that $\chi_{\{u>t\}}$ is a function of least gradient in Ω. If additionally $\mathcal{L}^N(\{x \in \Omega : u(x) = t\}) = 0$, then the two functions coincide and the proof for $\chi_{\{u \geqslant t\}}$ is also concluded.

Now, consider the case when $\mathcal{L}^N(\{x \in \Omega : u(x) = t\}) > 0$. Then, there exists a sequence $t_n \nearrow t$ such that $\mathcal{L}^N(\{x \in \Omega : u(x) = t_n\}) = 0$ and

$$\lim_{n \to \infty} \int_\Omega |\chi_{\{u \geqslant t_n\}} - \chi_{\{u \geqslant t\}}| \, dx = 0,$$

whence by the previous result and Theorem 2.13 we have that $\chi_{\{u \geqslant t\}} \in BV(\Omega)$ and that $\chi_{\{u \geqslant t\}}$ is a function of least gradient in Ω. □

In other words, whenever $u \in BV(\Omega)$ is a function of least gradient, then the sets $\{u > t\}$ and $\{u \geqslant t\}$ are area-minimising in Ω, and by the regularity theory for area-minimising sets to conclude that the singular set of their boundaries is of dimension $N - 8$ (and in dimensions up to seven it is empty).

Memo 12. *For a set $E \subset \mathbb{R}^N$, we say that $x \in \partial E$ is a regular point of ∂E, if there exists $r > 0$ such that $\partial E \cap B(x,r)$ is a C^2 hypersurface. We denote the set of all regular points of ∂E by* $\mathrm{reg}(\partial E)$. *We also say that $x \in \partial E$ is a singular point of ∂E, if $x \notin \mathrm{reg}(\partial E)$, and denote the set of all singular points of ∂E by* $\mathrm{sing}(\partial E)$.

The size of the singular set for area-minimising hypersurfaces is a classical problem in geometric measure theory. We have the following result, fully presented in [**28**]*: suppose that E is area-minimising in an open set $\Omega \subset \mathbb{R}^N$ and $E = E^{(1)}$. Then:*

(a) If $N \leqslant 7$, we have $\mathrm{sing}(\partial E) \cap \Omega = \varnothing$;
(b) If $N = 8$, the set $\mathrm{sing}(\partial E) \cap \Omega$ consists only of isolated points;
(c) If $N > 8$, we have $\dim_{\mathcal{H}}(\mathrm{sing}(\partial E) \cap \Omega) \leqslant N - 8$.

The estimate in point (a) follows from the fact that for $N \leqslant 7$ there are no minimal cones in $\mathbb{R}^N$ other than halfspaces. As a particular case, in two dimensions ∂E consists of a locally finite union of pairwise disjoint line segments. This result is optimal in the sense that in dimension eight the Simons cone

$$S = \{x \in \mathbb{R}^8 : \, x_1^2 + x_2^2 + x_3^2 + x_4^2 > x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

is minimal in $\mathbb{R}^8$. The analysis in points (b) and (c) follows from the estimates on the Hausdorff dimension of the singular set of the minimal cones.

This result implies that if u is a function of least gradient in two dimensions, then (up to a choice of representative) for all $t \in \mathbb{R}$ each connected component of $\partial\{u > t\}$ is a line segment. To be more precise, we may write

$$\partial\{u > t\} = \bigcup_{i=1}^{\infty} \ell_{t,i}$$

where each $\ell_{t,i}$ is a line segment or the empty set. This union is locally finite and the sets $\ell_{t,i}$ are pairwise disjoint in Ω. If Ω is convex, they are pairwise disjoint in $\overline{\Omega}$. As a consequence of this and the pointwise formula for the trace (Corollary 1.29), whenever one of the line segments $\ell_{t,i}$ intersects Ω, it does so either at a discontinuity point of the boundary datum f or at a point in $f^{-1}(t)$.

We will use these facts to give explicit solutions to the least gradient problem in order to highlight some possible irregular behaviour of solutions. The first two examples concern an explicit construction of solutions (which is a general technique for strictly convex domains and continuous boundary data, see [**42**] for the least gradient problem or [**33**] for the area functional).

Example 2.18. Let $\Omega = B(0,1) \subset \mathbb{R}^2$ and consider the boundary datum $f(x,y) = x$. Then, the unique solution to the least gradient problem $u(x,y) = x$.

Example 2.19. Let $\Omega = B(0,1) \subset \mathbb{R}^2$ and consider the boundary datum $f(\theta) = \cos(2\theta)$. Then, the unique solution to the least gradient problem is given by

$$u(x,y) = \begin{cases} 2x^2 - 1 & \text{if } |x| > \frac{\sqrt{2}}{2}, \; |y| < \frac{\sqrt{2}}{2}; \\ 0 & \text{if } |x| < \frac{\sqrt{2}}{2}, \; |y| < \frac{\sqrt{2}}{2}; \\ 1 - 2y^2 & \text{if } |x| < \frac{\sqrt{2}}{2}, \; |y| > \frac{\sqrt{2}}{2}. \end{cases}$$

The second example shows that in general solutions may be nonunique.

EXAMPLE 2.20. Let $\Omega = B(0,1) \subset \mathbb{R}^2$ and consider the boundary datum $f : \partial\Omega \to \mathbb{R}$ given by the formula

$$f(x,y) = \begin{cases} 1 & \text{if } |x| > \frac{1}{\sqrt{2}}; \\ -1 & \text{if } |y| > \frac{1}{\sqrt{2}}. \end{cases}$$

Then, the functions u_λ given by

$$u_\lambda(x,y) = \begin{cases} 1 & \text{if } |x| > \frac{\sqrt{2}}{2},\ |y| < \frac{\sqrt{2}}{2}; \\ \lambda & \text{if } |x| < \frac{\sqrt{2}}{2},\ |y| < \frac{\sqrt{2}}{2}; \\ -1 & \text{if } |x| < \frac{\sqrt{2}}{2},\ |y| > \frac{\sqrt{2}}{2} \end{cases}$$

with $\lambda \in [-1,1]$ are functions of least gradient.

The final example shows that when the domain fails to be strictly convex (or, in higher dimensions, when the mean curvature of the boundary is not positive), then the boundary condition may fail to be satisfied in the trace sense even for smooth boundary data. Therefore, in general we are forced to consider relaxations.

EXAMPLE 2.21. Let $\Omega = [0,1]^2 \subset \mathbb{R}^2$ and consider a boundary datum $f \in C^\infty(\partial\Omega)$ such that $f = 0$ on three sides of the square $\{0\} \times [0,1]$, $\{1\} \times [0,1]$ and $[0,1] \times \{0\}$. On the remaining side $[0,1] \times \{0\}$, we allow for any $f \in C_c^\infty((0,1))$. Then, there is no solution to the least gradient problem in the trace sense (clearly, the solution to the relaxed problem exists and it is everywhere equal to zero).

Further reading

The Rudin-Osher-Fatemi model first appeared in [**40**]; a mathematical overview of the problem can be found in [**3**]. Relaxation theorems for general linear growth functionals first appeared in [**4**]; the current statement was taken from [**1**]. Finally, the Miranda theorem first appeared in [**36**] and the Bombieri-de Giorgi-Giusti theorem in [**8**]; in both cases, the proof is a bit different as the original formulations were for functions which only locally have the least gradient property. An up-to-date overview of topics concerning the least gradient problem can be found in [**31**].

CHAPTER 3

Anzellotti pairing theory

In this lecture we give a brief description of the pairing between measures and bounded variation functions given by Anzellotti in [**4**] and its main properties. Let $1 \leqslant p \leqslant N$; following Anzellotti, for an open bounded set with Lipschitz boundary $\Omega \subset \mathbb{R}^N$ we denote

$$X_p(\Omega) := \{\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N) \ : \ \operatorname{div}(\mathbf{z}) \in L^p(\Omega)\}.$$

Our goal is to define a weak normal trace of a vector field in $X_p(\Omega)$ and a pairing between such a vector field and a function in $BV(\Omega) \cap L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, which will act as a replacement of the pointwise product $\mathbf{z} \cdot \nabla u$ and enable a generalisation of the Gauss-Green formula. We assume that $N \geqslant 2$; the case $N = 1$ is considered separately due to the fact that the divergence is just the derivative and vector fields with integrable divergence are Sobolev functions. Throughout the whole lecture, we assume that Ω is an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary, and we denote by $q \in [\frac{N}{N-1}, \infty]$ the conjugate exponent of $p \in [1, N]$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

3.1. The generalised pairing $(\mathbf{z}, Du)$

Our main assumption for this lecture is the following joint condition on the function u and vector field $\mathbf{z}$. Let $1 \leqslant p \leqslant N$; from now on, we assume that

$$u \in BV(\Omega) \cap L^q(\Omega) \quad \text{and} \quad \mathbf{z} \in X_p(\Omega). \tag{3.1}$$

The main settings to which we apply this construction is when $p = N$ and $q = \frac{N}{N-1}$, i.e., the exponent given by the Sobolev embedding, and $p = q = 2$, which we will use for the study of the total variation flow in the last lecture.

Definition 3.1. For every test function $\varphi \in C_c^\infty(\Omega)$, we set

$$\langle (\mathbf{z}, Du), \varphi \rangle := -\int_\Omega u \operatorname{div}(\varphi \mathbf{z})\, dx = -\int_\Omega u\, \varphi \operatorname{div}(\mathbf{z})\, dx - \int_\Omega u\, \mathbf{z} \cdot \nabla \varphi\, dx.$$

We call $(\mathbf{z}, Du)$ the *Anzellotti pairing.*

Note that under condition (3.1) all the integrals are well-defined and finite. The newly defined object $(\mathbf{z}, Du)$ a priori is a distribution; in the next Proposition, we prove that it is actually a Radon measure.

Proposition 3.2. *Assume that $u \in BV(\Omega) \cap L^q(\Omega)$ and $\mathbf{z} \in X_p(\Omega)$. Then, the distribution $(\mathbf{z}, Du)$ is a Radon measure in Ω. Moreover,*

$$\left| \int_B (\mathbf{z}, Du) \right| \leqslant \|\mathbf{z}\|_\infty \int_B |Du|$$

for any Borel set $B \subseteq \Omega$, i.e., it is absolutely continuous with respect to $|Du|$ with density bounded in $L^\infty(\Omega)$ by $\|\mathbf{z}\|_\infty$.

PROOF. For now, assume additionally that $u \in C^\infty(\Omega)$. We note that $\varphi\mathbf{z} \in X_p(\Omega)$ for all $\varphi \in C_c^\infty(\Omega)$. Therefore, using the distributional definition of the divergence we get

$$|\langle(\mathbf{z}, Du), \varphi\rangle| = \left|- \int_\Omega u \operatorname{div}(\varphi\mathbf{z})\, dx\right| = \left|\int_\Omega \nabla u \cdot (\varphi\mathbf{z})\, dx\right| = \left|\int_\Omega \varphi(\mathbf{z} \cdot \nabla u)\, dx\right|$$
$$\leqslant \|\varphi\|_\infty \left|\int_\Omega \mathbf{z} \cdot \nabla u\, dx\right| \leqslant \|\varphi\|_\infty \|\mathbf{z}\|_\infty \int_\Omega |\nabla u|\, dx.$$

In the general case, assuming that $u \in BV(\Omega)$ satisfies the assumption (3.1), take the sequence $u_j \in W^{1,1}(\Omega) \cap C^\infty(\Omega)$ given by the Meyers-Serrin theorem (Theorem 1.16). Then, for any $\varphi \in C_c^\infty(\Omega)$ we get

$$\lim_{j\to\infty} \langle(\mathbf{z}, Du_j), \varphi\rangle = \lim_{j\to\infty} - \int_\Omega u_j \operatorname{div}(\varphi\mathbf{z})\, dx = - \int_\Omega u \operatorname{div}(\varphi\mathbf{z})\, dx = \langle(\mathbf{z}, Du), \varphi\rangle. \tag{3.2}$$

As a consequence,

$$|\langle(\mathbf{z}, Du), \varphi\rangle| = \lim_{j\to\infty} |\langle(\mathbf{z}, Du_j), \varphi\rangle| \leqslant \lim_{j\to\infty} \|\varphi\|_\infty \|\mathbf{z}\|_\infty \int_\Omega |\nabla u_j|\, dx$$
$$= \|\varphi\|_\infty \|\mathbf{z}\|_\infty \int_\Omega |Du|.$$

Thus, $(\mathbf{z}, Du)$ is a continuous functional on the space of smooth functions (equipped with the supremum norm). Since smooth functions are dense in continuous functions in the supremum norm, $(\mathbf{z}, Du)$ defines a continuous functional on the space $C(\Omega)$. By the Riesz representation theorem (Memo 2), we get that $(\mathbf{z}, Du)$ is a Radon measure and

$$\left|\int_B (\mathbf{z}, Du)\right| \leqslant \|\mathbf{z}\|_\infty \int_B |Du|,$$

which concludes the proof. $\square$

As a consequence of the above result, by the Radon-Nikodym theorem (Memo 7) there exists a $|Du|$-measurable function

$$\theta(\mathbf{z}, Du, \cdot) : \Omega \to \mathbb{R}$$

such that

$$\int_B (\mathbf{z}, Du) = \int_B \theta(\mathbf{z}, Du, x)\, |Du| \qquad \text{for all Borel sets} \quad B \subset \Omega$$

and

$$\|\theta(\mathbf{z}, Du, \cdot)\|_{L^\infty(\Omega, |Du|)} \leqslant \|\mathbf{z}\|_\infty.$$

EXERCISE 3.3. Show that for $u \in W^{1,1}(\Omega)$, we have

$$(\mathbf{z}, Du) = \mathbf{z} \cdot \nabla u\, dx$$

as measures in Ω.

3.2. Weak integration by parts formula

Our main goal in this lecture is to prove a weak integration by parts formula, i.e., the weak Gauss-Green formula given in Theorem 3.9. We first prove existence of the weak normal trace of a vector field with integrable divergence in Theorem 3.5, from which follows the Gauss-Green formula for Sobolev functions, and then use an approximation as in the Meyers-Serrin theorem (Theorem 1.16) to conclude the proof in the general case. The heart of the proof lies in Proposition 3.4 and Theorem 3.5.

We now prove that there exists a function $[\mathbf{z}, \nu^{\Omega}]$ which has an interpretation of a weak normal trace of the vector field $\mathbf{z} \in X_1(\Omega)$ on $\partial\Omega$. To simplify the notation, we denote

$$BV_{\infty}(\Omega) = BV(\Omega) \cap L^{\infty}(\Omega).$$

The proof follows in two steps: in Proposition 3.4 we introduce an auxiliary pairing

$$\langle \mathbf{z}, u \rangle_{\partial\Omega} : X_1(\Omega) \times BV_{\infty}(\Omega) \to \mathbb{R}$$

and then in Theorem 3.5 we provide its integral representation, from which we deduce existence of a function in $L^{\infty}(\partial\Omega)$ which has an interpretation of a weak normal trace of the vector field $\mathbf{z}$.

PROPOSITION 3.4. *There exists a bilinear map* $\langle \mathbf{z}, u \rangle_{\partial\Omega} : X_1(\Omega) \times BV_{\infty}(\Omega) \to \mathbb{R}$ *such that*

$$\langle \mathbf{z}, u \rangle_{\partial\Omega} = \int_{\partial\Omega} u\, \mathbf{z} \cdot \nu^{\Omega} \, d\mathcal{H}^{N-1} \qquad \text{if } \mathbf{z} \in C^1(\overline{\Omega}; \mathbb{R}^N),$$

where ν^{Ω} *denotes the outer unit normal to* Ω*, and*

$$|\langle \mathbf{z}, u \rangle_{\partial\Omega}| \leqslant \|\mathbf{z}\|_{\infty} \cdot \|u\|_{L^1(\partial\Omega)}.$$

PROOF. For all $\mathbf{z} \in X_1(\Omega)$ and $u \in BV_{\infty}(\Omega) \cap W^{1,1}(\Omega)$ we set

$$\langle \mathbf{z}, u \rangle_{\partial\Omega} = \int_{\Omega} u \operatorname{div}(\mathbf{z}) \, dx + \int_{\Omega} \mathbf{z} \cdot \nabla u \, dx. \tag{3.3}$$

Clearly, this map is bilinear.

In the general case, due to the fact that Du may be only a measure, the formula above is not well-defined; we will extend it by approximating general $u \in BV_{\infty}(\Omega)$ using smooth functions. To this end, we notice that if $u, v \in BV_{\infty}(\Omega) \cap W^{1,1}(\Omega)$ have the same trace, then

$$\langle \mathbf{z}, u \rangle_{\partial\Omega} = \langle \mathbf{z}, v \rangle_{\partial\Omega}. \tag{3.4}$$

To prove this, consider an approximation g_j of the function $u-v$ by smooth functions given by the Meyers-Serrin theorem (Theorem 1.16). Since $u-v$ has trace zero, with a minor modification of the proof we can require that $u - v$ has compact support in Ω. Then,

$$\begin{aligned} \langle \mathbf{z}, u - v \rangle_{\partial\Omega} &= \int_{\Omega} (u - v) \operatorname{div}(\mathbf{z}) \, dx + \int_{\Omega} \mathbf{z} \cdot \nabla (u - v) \, dx \\ &= \lim_{j \to \infty} \left(\int_{\Omega} g_j \operatorname{div}(\mathbf{z}) \, dx + \int_{\Omega} \mathbf{z} \cdot \nabla g_j \, dx \right) = 0, \end{aligned}$$

where the last equality follows from the distributional definition of the divergence; this concludes the proof of property (3.4). Since by the Gagliardo extension theorem (Lemma 1.32) for every $u \in BV(\Omega)$ there exists a function in $W^{1,1}(\Omega)$ with the same trace, for arbitrary $u \in BV_\infty(\Omega)$ we define $\langle \mathbf{z}, u\rangle_{\partial\Omega}$ by

$$\langle \mathbf{z}, u\rangle_{\partial\Omega} = \langle \mathbf{z}, v\rangle_{\partial\Omega},$$

where v is any function in $BV_\infty(\Omega) \cap W^{1,1}(\Omega)$ with the same trace as u. In view of equation (3.4), this uniquely defines $\langle \mathbf{z}, u\rangle_{\partial\Omega}$ for any $u \in BV_\infty(\Omega)$.

Now, we have to prove the second property. Let us take a sequence $u_j \in BV_\infty(\Omega) \cap C^\infty(\Omega)$ which converges to u as in the Meyers-Serrin theorem (Theorem 1.16). Then, we get that

$$\begin{aligned} |\langle \mathbf{z}, u\rangle_{\partial\Omega}| = |\langle \mathbf{z}, u_j\rangle_{\partial\Omega}| &= \left| \int_\Omega u_j \operatorname{div}(\mathbf{z})\, dx + \int_\Omega \mathbf{z} \cdot \nabla u_j \, dx \right| \\ &\leqslant \left| \int_\Omega u_j \operatorname{div}(\mathbf{z})\, dx \right| + \|\mathbf{z}\|_\infty \int_\Omega |\nabla u_j|\, dx. \end{aligned}$$

We pass to the limit with $j \to \infty$ and obtain

$$|\langle \mathbf{z}, u\rangle_{\partial\Omega}| \leqslant \left| \int_\Omega u \operatorname{div}(\mathbf{z})\, dx \right| + \|\mathbf{z}\|_\infty \int_\Omega |Du|. \tag{3.5}$$

Fix $\varepsilon > 0$. Observe that by property (3.4), we may take u to be the function given by the variant of the Gagliardo extension theorem proved in Proposition 1.32; therefore,

$$\int_\Omega |\nabla u|\, dx \leqslant (1+\varepsilon)\|u\|_{L^1(\partial\Omega)},$$

and u is supported in $\Omega \backslash \Omega_\varepsilon$, where

$$\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

We insert it in the estimate (3.5) and obtain

$$|\langle \mathbf{z}, u\rangle_{\partial\Omega}| \leqslant \|u\|_\infty \left| \int_{\Omega\backslash\Omega_\varepsilon} \operatorname{div}(\mathbf{z})\, dx \right| + (1+\varepsilon)\|\mathbf{z}\|_\infty \|u\|_{L^1(\partial\Omega)}$$

Since ε was arbitrary, we pass to the limit $\varepsilon \to 0$ and obtain

$$|\langle \mathbf{z}, u\rangle_{\partial\Omega}| \leqslant \|\mathbf{z}\|_\infty \|u\|_{L^1(\partial\Omega)},$$

which concludes the proof. □

Now, we provide an integral representation of the bilinear map $\langle \mathbf{z}, u\rangle_{\partial\Omega}$, from which follows that for every vector field $\mathbf{z} \in X_1(\Omega)$ there exists a function in $L^\infty(\partial\Omega)$ which has an interpretation of its normal trace.

THEOREM 3.5. *There exists a linear operator $\gamma : X_1(\Omega) \to L^\infty(\partial\Omega)$ such that*

$$\|\gamma(\mathbf{z})\|_{L^\infty(\partial\Omega)} \leqslant \|\mathbf{z}\|_\infty,$$

and we have the following integral representation: for all $u \in BV_\infty(\Omega)$

$$\langle \mathbf{z}, u\rangle_{\partial\Omega} = \int_{\partial\Omega} u\, \gamma(\mathbf{z})\, d\mathcal{H}^{N-1}, \tag{3.6}$$

and

$$\gamma(\mathbf{z})(x) = \mathbf{z} \cdot \nu^\Omega \qquad \textit{if } \mathbf{z} \in C^1(\overline{\Omega}; \mathbb{R}^N).$$

The function $\gamma(\mathbf{z})$ is a weakly defined normal trace of $\mathbf{z}$ on $\partial\Omega$; for this reason, we will denote it by $[\mathbf{z}, \nu^\Omega]$.

PROOF. Given $\mathbf{z} \in X_1(\Omega)$, consider the linear functional $G : L^\infty(\partial\Omega) \to \mathbb{R}$ defined by the formula

$$G(f) = \langle \mathbf{z}, u \rangle_{\partial\Omega},$$

where $u \in BV_\infty(\Omega)$ is such that $u = f$ on $\partial\Omega$. By Proposition 3.4, we have

$$|G(f)| = |\langle \mathbf{z}, u \rangle_{\partial\Omega}| \leqslant \|\mathbf{z}_1\|_\infty \|f\|_{L^1(\partial\Omega)}.$$

Since G is a continuous functional on (a dense subset of) $L^1(\partial\Omega)$, there exists a function $\gamma(\mathbf{z}) \in L^\infty(\partial\Omega)$ with norm at most equal to $\|\mathbf{z}\|_\infty$ such that

$$G(f) = \int_{\partial\Omega} f\, \gamma(\mathbf{z})\, d\mathcal{H}^{N-1},$$

which concludes the proof. □

REMARK 3.6. In [**5**], Anzellotti proved the following pointwise characterization of the weak normal trace. Given $r, \rho > 0$, denote

$$C_{r,\rho}(x, \alpha) := \{\xi \in \mathbb{R}^N : \ |(\xi - x) \cdot \alpha| < r, \ |(\xi - x) - [(\xi - x) \cdot \alpha]\alpha| < \rho\}$$

for $x \in \partial\Omega$ and $\alpha \in S^{N-1}$. Assume that $\mathbf{z} \in X_1(\Omega)$. Then,

$$[\mathbf{z}, \nu^\Omega](x) = \lim_{\rho \to 0^+} \lim_{r \to 0^+} \frac{1}{2r\omega_{N-1}\rho^{N-1}} \int_{C_{r,\rho}(x, \nu^\Omega(x))} \mathbf{z}(y) \cdot \nu^\Omega(x)\, dy$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$.

COROLLARY 3.7. *For all $\mathbf{z} \in X_p(\Omega)$ and $u \in W^{1,1}(\Omega) \cap L^q(\Omega)$ we have*

$$\int_\Omega u \operatorname{div}(\mathbf{z})\, dx + \int_\Omega \mathbf{z} \cdot \nabla u\, dx = \int_{\partial\Omega} u\, [\mathbf{z}, \nu^\Omega]\, d\mathcal{H}^{N-1}.$$

PROOF. Since $\mathbf{z} \in X_p(\Omega)$, we also have that $\mathbf{z} \in X_1(\Omega)$ and in particular the weak normal trace $[\mathbf{z}, \nu^\Omega]$ is well-defined. Take a sequence u_j which approximates u as in the Meyers-Serrin theorem (Theorem 1.16). By considering truncations, we can assume that u_n is bounded; then, the functions may be no longer smooth or satisfy the trace constraint, but we have $u_j \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and the sequence still satisfies

$$u_j \to u \quad \text{in } L^q(\Omega)$$

and

$$\nabla u_j \to \nabla u \quad \text{in } L^1(\Omega; \mathbb{R}^N),$$

which follows from the assumption that $u \in W^{1,1}(\Omega)$. Then, by the definition of the bilinear form $\langle \mathbf{z}, u \rangle_{\partial\Omega}$ given in equation (3.3) and the integral representation (3.6), we have that

$$\int_\Omega u_j \operatorname{div}(\mathbf{z})\, dx + \int_\Omega \mathbf{z} \cdot \nabla u_j\, dx = \int_{\partial\Omega} u_j\, [\mathbf{z}, \nu^\Omega]\, d\mathcal{H}^{N-1}.$$

Passing to the limit $j \to \infty$, we obtain the desired result: convergence on the left-hand side follows from our assumption on the sequence u_j and on the right-hand side from the fact that the trace operator is continuous with respect to convergence in $W^{1,1}(\Omega)$. □

Before we prove the Gauss-Green formula, we need one additional technical result concerning the pairing $(\mathbf{z}, Du)$.

LEMMA 3.8. *Assume that $u \in BV(\Omega) \cap L^q(\Omega)$ and $\mathbf{z} \in X_p(\Omega)$. Let $u_j \in C^\infty(\Omega) \cap BV(\Omega)$ converge to $u \in BV(\Omega)$ as in the Meyers-Serrin theorem (Theorem 1.16). Then, we have*

$$\int_\Omega (\mathbf{z}, Du_j) \to \int_\Omega (\mathbf{z}, Du).$$

PROOF. Fix $\varepsilon > 0$ and choose an open set $A \Subset \Omega$ such that

$$\int_{\Omega\setminus A} |Du| < \varepsilon.$$

Let $g \in C_c^\infty(\Omega)$ be such that $0 \leqslant g \leqslant 1$ in Ω and $g \equiv 1$ in A. We write $1 = g + (1-g)$ and estimate

$$\begin{aligned}(3.7)\quad \left| \int_\Omega (\mathbf{z}, Du_j) - \int_\Omega (\mathbf{z}, Du) \right| \leqslant & \left| \langle (\mathbf{z}, Du_j), g\rangle - \langle (\mathbf{z}, Du), g\rangle \right| \\ & + \int_\Omega |(\mathbf{z}, Du_j)|(1-g) + \int_\Omega |(\mathbf{z}, Du)|(1-g).\end{aligned}$$

We already proved in equation (3.2) that for any $g \in C_c^\infty(\Omega)$ we have $\langle (\mathbf{z}, Du_j), g\rangle \to \langle (\mathbf{z}, Du), g\rangle$. Moreover, we have

$$\int_\Omega (1-g)|(\mathbf{z}, Du)| \leqslant \int_{\Omega\setminus A} |(\mathbf{z}, Du)| \leqslant \|\mathbf{z}\|_\infty \int_{\Omega\setminus A} |Du| < \varepsilon \|\mathbf{z}\|_\infty$$

and similarly

$$\limsup_{j\to\infty} \int_\Omega (1-g)|(\mathbf{z}, Du_j)| \leqslant \limsup_{j\to\infty} \|\mathbf{z}\|_\infty \int_{\Omega\setminus A} |Du_j| \leqslant \varepsilon \|\mathbf{z}\|_\infty,$$

so the right-hand side of (3.7) goes to zero as $j \to \infty$. □

We conclude by proving the Gauss-Green formula, which relates the measure $(\mathbf{z}, Du)$ with the weak normal trace $[\mathbf{z}, \nu^\Omega]$.

THEOREM 3.9 (Gauss-Green formula). *For all functions $u \in BV(\Omega) \cap L^q(\Omega)$ and vector fields $\mathbf{z} \in X_p(\Omega)$ we have*

$$\int_\Omega u \operatorname{div}(\mathbf{z})\, dx + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u\, [\mathbf{z}, \nu^\Omega]\, d\mathcal{H}^{N-1}.$$

PROOF. Take an approximation $u_j \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ of u given by the Meyers-Serrin theorem (Theorem 1.16). Then, by Corollary 3.7 we have

$$\int_\Omega u_j \operatorname{div}(\mathbf{z})\, dx + \int_\Omega \mathbf{z} \cdot \nabla u_j\, dx = \int_{\partial\Omega} u_j\, [\mathbf{z}, \nu^\Omega]\, d\mathcal{H}^{N-1}.$$

We now pass to the limit separately in each term. Since $u_j \to u$ in $L^q(\Omega)$ and $\operatorname{div}(\mathbf{z}) \in L^p(\Omega)$, we have

$$\lim_{j\to\infty} \int_\Omega u_j \operatorname{div}(\mathbf{z})\, dx = \int_\Omega u \operatorname{div}(\mathbf{z})\, dx.$$

By Lemma 3.8, we have

$$\lim_{j\to\infty}\int_\Omega \mathbf{z}\cdot\nabla u_j\,dx = \int_\Omega (\mathbf{z}, Du).$$

Finally, since $u_j|_{\partial\Omega} = u|_{\partial\Omega}$, we have

$$\lim_{j\to\infty}\int_{\partial\Omega} [\mathbf{z}, \nu^\Omega]\, u_j\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} [\mathbf{z}, \nu^\Omega]\, u\, d\mathcal{H}^{N-1},$$

which concludes the proof. □

EXERCISE 3.10. One can prove most of the results in this Section under a slightly weaker assumption on the vector fields and a slightly stronger assumption on the functions, namely in place of condition (3.1) assume that

$$u \in BV(\Omega)\cap C_b(\Omega) \quad \text{and} \quad \mathbf{z}\in X_\mu(\Omega),$$

where

$$X_\mu(\Omega) := \Big\{\mathbf{z}\in L^\infty(\Omega;\mathbb{R}^N) \,:\, \operatorname{div}(\mathbf{z})\in\mathcal{M}(\Omega)\Big\}.$$

This corresponds to assumption (c) from the classical paper [**4**] due to Anzellotti. Then, we can define $(\mathbf{z}, Du)$ as in Definition 3.1, and all the subsequent results remain true with only minor modifications of the proofs. Work out the necessary details.

EXERCISE 3.11. Make a similar construction for $N = 1$, when the divergence is just the derivative and vector fields with integrable divergence are Sobolev functions.

EXERCISE 3.12. Make a similar construction for $\Omega = \mathbb{R}^N$, leading to a Gauss-Green formula in the following form: for all functions $u \in BV(\Omega)$ and vector fields $\mathbf{z}\in X_p(\Omega)$ satisfying a suitable compatibility condition, we have

$$\int_\Omega u\operatorname{div}(\mathbf{z})\,dx + \int_\Omega (\mathbf{z}, Du) = 0.$$

3.3. Co-area formula for the pairing

We now move to the last topic concerning Anzellotti pairings, which is a generalisation of the co-area formula. For $u\in BV(\Omega)$, we denote by $\frac{Du}{|Du|}$ the Radon-Nikodym derivative of Du with respect to $|Du|$. Denote $E_t = \{u > t\}$. As a consequence of the co-area formula (Theorem 1.19), for almost all $t\in\mathbb{R}$

$$\frac{D\chi_{E_t}}{|D\chi_{E_t}|} = \frac{Du}{|Du|} \quad |D\chi_{E_t}| - \text{a.e. in } \Omega.$$

A natural question is whether the Anzellotti pairing defined above satisfies an analogue of the co-area formula, and how to formulate it. Our main goal in this section is to first prove that

$$\langle(\mathbf{z}, Du), \varphi\rangle = \int_{-\infty}^{\infty}\langle(\mathbf{z}, D\chi_{E_t}), \varphi\rangle\,dt$$

holds for all functions $\varphi\in C_c^\infty(\Omega)$, and from this deduce the co-area formula for the measure $(\mathbf{z}, Du)$ itself.

EXERCISE 3.13. Prove that for almost all $t \in \mathbb{R}$
$$\frac{D\chi_{E_t}}{|D\chi_{E_t}|} = \frac{Du}{|Du|} \quad |D\chi_{E_t}| - \text{a.e. in } \Omega.$$

First, let us state some technical results. By Proposition 3.2, the measure $(\mathbf{z}, Du)$ is absolutely continuous with respect to the measure $|Du|$. By the Radon-Nikodym theorem, there exists a measurable function $\theta(\mathbf{z}, Du, x)$ which is the density of the measure $(\mathbf{z}, Du)$ with respect to $|Du|$, i.e. for all Borel sets $B \subset \Omega$ we have
$$\int_B (\mathbf{z}, Du) = \int_B \theta(\mathbf{z}, Du, x)\, d|Du|.$$
Moreover, by the estimate in Proposition 3.2, we have that
$$|\theta(\mathbf{z}, Du, x)| \leqslant \|\mathbf{z}\|_\infty \quad |Du| - \text{a.e. in } \Omega.$$

Taking a sequence of mollifications of a vector field $\mathbf{z} \in X_p(\Omega)$, we can easily prove the following result.

LEMMA 3.14. *For every $\mathbf{z} \in X_p(\Omega)$, there exists a sequence $\mathbf{z}_n \in C^\infty(\Omega; \mathbb{R}^N) \cap X_p(\Omega)$ with the following properties:*

(a) $\|\mathbf{z}_n\|_\infty \leqslant \|\mathbf{z}\|_\infty$;
(b) $\mathbf{z}_n \rightharpoonup \mathbf{z}$ *weakly* in* $L^\infty(\Omega; \mathbb{R}^N)$ *and* $\mathbf{z}_n \to \mathbf{z}$ *in* $L^r_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ *for all* $r \in [1, \infty)$;
(c) $\mathbf{z}_n(x) \to \mathbf{z}(x)$ *at every Lebesgue point x of $\mathbf{z}$ and uniformly in sets of uniform continuity of $\mathbf{z}$;*
(d) $\mathrm{div}(\mathbf{z}_n) \to \mathrm{div}(\mathbf{z})$ *in* $L^p_{\mathrm{loc}}(\Omega)$.

As a consequence, we get the following pointwise representation result for the density function $\theta(\mathbf{z}, Du, x)$.

PROPOSITION 3.15. *Assume that $u \in BV(\Omega) \cap L^q(\Omega)$ and suppose that $\mathbf{z} \in X_p(\Omega) \cap C(\Omega; \mathbb{R}^N)$. Then, we have*

$$\theta(\mathbf{z}, Du, x) = \mathbf{z}(x) \cdot \frac{Du}{|Du|}(x) \qquad |Du| - \text{a.e. in } \Omega. \tag{3.8}$$

PROOF. By the definition of the Radon-Nikodym derivative $\frac{Du}{|Du|}$, condition (3.8) is equivalent to

$$\langle (\mathbf{z}, Du), \varphi \rangle = \int_\Omega \varphi\, \mathbf{z}\, d[Du] \qquad \text{for all } \varphi \in C^\infty_c(\Omega). \tag{3.9}$$

We first prove the claim for $\mathbf{z} \in C^1(\Omega; \mathbb{R}^N)$. Take a sequence $u_j \to u$ as given by the Meyers-Serrin theorem (Theorem 1.16). By the distributional definition of the divergence, for all $\varphi \in C^\infty_c(\Omega)$ we have
$$\langle (\mathbf{z}, Du_j), \varphi \rangle = -\int_\Omega u_j \,\mathrm{div}(\varphi \mathbf{z})\, dx = \int_\Omega \nabla u_j \cdot (\varphi \mathbf{z})\, dx = \int_\Omega \varphi (\mathbf{z} \cdot \nabla u_j)\, dx.$$
By the continuity of $\mathbf{z}$, passing to the limit $j \to \infty$ we get that equation (3.9) holds.

We now allow for general $\mathbf{z} \in C(\Omega; \mathbb{R}^N)$. Take a sequence of approximations $\mathbf{z}_n$ given by Lemma 3.14 and for any $\varphi \in C_c^\infty(\Omega)$ calculate

$$\langle(\mathbf{z}, Du), \varphi\rangle = \lim_{n\to\infty} \langle(\mathbf{z}_n, Du), \varphi\rangle = \lim_{n\to\infty} \int_\Omega \varphi\, \mathbf{z}_n\, d[Du] = \int_\Omega \varphi\, \mathbf{z}\, d[Du], \tag{3.10}$$

where the first equality is a consequence of Lemma 3.14 applied directly to the definition of the pairing, and the last equality follows from continuity of $\mathbf{z}$ and uniform convergence of $\mathbf{z}_n$ to $\mathbf{z}$ on the support of φ. □

Finally, we have the following coarea formula for the Anzellotti pairing.

THEOREM 3.16. *Assume that $u \in BV(\Omega) \cap L^q(\Omega)$ and $\mathbf{z} \in X_p(\Omega)$. Then:*

(i) For all $\varphi \in C_c(\Omega)$, the function $t \mapsto \langle(\mathbf{z}, D\chi_{E_t}), \varphi\rangle$ is $\mathcal{L}^1$-measurable and

$$\langle(\mathbf{z}, Du), \varphi\rangle = \int_{-\infty}^{+\infty} \langle(\mathbf{z}, D\chi_{E_t}), \varphi\rangle\, dt.$$

(ii) $\theta(\mathbf{z}, Du, x) = \theta(\mathbf{z}, D\chi_{E_t}, x)$ $|D\chi_{E_t}|$-a.e. in Ω for $\mathcal{L}^1$-almost all $t \in \mathbb{R}$.

(iii) For all Borel sets $B \subset \Omega$, the function $t \mapsto \int_B (\mathbf{z}, D\chi_{E_t})$ is $\mathcal{L}^1$-measurable and

$$\int_B (\mathbf{z}, Du) = \int_{-\infty}^{+\infty} \left(\int_B (\mathbf{z}, D\chi_{E_t}) \right) dt.$$

PROOF. (i) Take an approximating sequence $\mathbf{z}_n \in C^\infty(\Omega; \mathbb{R}^N) \cap X_p(\Omega)$ as given in Lemma 3.14. Then, using Proposition 3.15 and the co-area formula (Theorem 1.19), we get

$$\begin{aligned} \langle(\mathbf{z}_n, Du), \varphi\rangle &= \int_\Omega \mathbf{z}_n(x) \cdot \frac{Du}{|Du|}(x)\, \varphi(x)\, d|Du| \\ &= \int_{-\infty}^{\infty} \left(\int_\Omega \mathbf{z}_n(x) \cdot \frac{D\chi_{E_t}}{|D\chi_{E_t}|}(x)\, \varphi(x)\, d|D\chi_{E_t}| \right) dt \\ &= \int_{-\infty}^{\infty} \langle(\mathbf{z}_n, D\chi_{E_t}), \varphi\rangle\, dt. \end{aligned} \tag{3.11}$$

Since

$$|\langle(\mathbf{z}_n, D\chi_{E_t}), \varphi\rangle| \leqslant \|\mathbf{z}\|_\infty \|\varphi\|_\infty \int_\Omega |D\chi_{E_t}|$$

and by the co-area formula the map $t \mapsto \int_\Omega |D\chi_{E_t}|$ is an integrable function in t, we may apply the dominated convergence theorem to pass to the limit in (3.11). Using an argument as in (3.10) we conclude the proof of point (i).

(ii) For $a, b \in \mathbb{R}$ with $a < b$, denote by $T_{a,b}(u)$ the truncation of u at levels a, b, i.e.

$$T_{a,b}(u) = \begin{cases} b & u(x) \geqslant b; \\ u(x) & u(x) \in (a, b); \\ a & u(x) \leqslant a. \end{cases}$$

Then, by Theorem 1.19 we have $DT_{a,b}(u) \in BV(\Omega)$ and $\int_\Omega |DT_{a,b}(u)| \leqslant \int_\Omega |Du|$.

We first prove that for all $a, b \in \mathbb{R}$ we have

$$\theta(\mathbf{z}, Du, x) = \theta(\mathbf{z}, DT_{a,b}(u), x) \qquad |DT_{a,b}(u)| - \text{a.e. in } \Omega. \tag{3.12}$$

Suppose otherwise; then, there exists a Borel set $B \subset \Omega$ with positive $|DT_{a,b}(u)|$-measure such that $a \leqslant u(x) \leqslant b$ almost everywhere on B and

$$\theta(\mathbf{z}, Du, x) > \theta(\mathbf{z}, DT_{a,b}(u), x) \qquad |DT_{a,b}(u)| - \text{a.e. on } B;$$

the case when the opposite inequality holds is handled similarly. Hence,

$$\begin{aligned}\int_B (\mathbf{z}, Du) &= \int_B \theta(\mathbf{z}, Du, x)|Du| = \int_B \theta(\mathbf{z}, Du, x)|DT_{a,b}(u)| \\ &> \int_B \theta(\mathbf{z}, DT_{a,b}(u), x)|DT_{a,b}(u)| = \int_B (\mathbf{z}, DT_{a,b}(u)).\end{aligned} \tag{3.13}$$

Now, notice that

$$\begin{aligned}\left|\int_B (\mathbf{z}, Du) - \int_B (\mathbf{z}, DT_{a,b}(u))\right| &= \left|\int_B (\mathbf{z}, D(u - T_{a,b}(u)))\right| \\ &\leqslant \|\mathbf{z}\|_\infty \int_B |D(u - T_{a,b}(u))| = \|\mathbf{z}\|_\infty \int_{-\infty}^{\infty} \int_B |D\chi_{\{u - T_{a,b}(u) \geqslant t\}}|\, dt \\ &= \|\mathbf{z}\|_\infty \int_{-\infty}^{a} \int_B |D\chi_{E_t}|\, dt + \|\mathbf{z}\|_\infty \int_b^{\infty} \int_B |D\chi_{E_t}|\, dt = 0,\end{aligned}$$

since $a \leqslant u \leqslant b$ a.e. on B. This gives a contradiction with (3.13), so (3.12) holds.

Now, we use point (i) for the function $T_{a,b}(u)$: for any $\varphi \in C_c^\infty(\Omega)$

$$\langle (\mathbf{z}, DT_{a,b}(u)), \varphi \rangle = \int_a^b \langle (\mathbf{z}, D\chi_{E_t}), \varphi \rangle\, dt, \tag{3.14}$$

where expanding the right-hand side yields

$$\langle (\mathbf{z}, DT_{a,b}(u)), \varphi \rangle = \int_a^b \left(\int_\Omega \theta(\mathbf{z}, D\chi_{E_t}, x)\, \varphi(x)\, d|D\chi_{E_t}| \right) dt.$$

Since by equation (3.12) and the co-area formula (Theorem 1.19) we can write the left-hand side of (3.14) in the following way

$$\begin{aligned}\langle (\mathbf{z}, DT_{a,b}(u)), \varphi \rangle &= \int_\Omega \varphi\, d(\mathbf{z}, DT_{a,b}(u)) = \int_\Omega \theta(\mathbf{z}, DT_{a,b}(u), x)\, \varphi(x)\, d|DT_{a,b}(u)| \\ &= \int_\Omega \theta(\mathbf{z}, Du, x)\, \varphi(x)\, d|DT_{a,b}(u)| = \int_a^b \left(\int_\Omega \theta(\mathbf{z}, Du, x)\, \varphi(x)\, d|D\chi_{E_t}| \right) dt,\end{aligned}$$

we get that

$$\int_a^b \left(\int_\Omega \theta(\mathbf{z}, Du, x)\, \varphi(x)\, d|D\chi_{E_t}| \right) dt = \int_a^b \left(\int_\Omega \theta(\mathbf{z}, D\chi_{E_t}, x)\, \varphi(x)\, d|D\chi_{E_t}| \right) dt.$$

Since a and b were arbitrary, we get that for almost all $t \in \mathbb{R}$ we have

$$\int_\Omega \theta(\mathbf{z}, Du, x)\, \varphi(x)\, d|D\chi_{E_t}| = \int_\Omega \theta(\mathbf{z}, D\chi_{E_t}, x)\, \varphi(x)\, d|D\chi_{E_t}|,$$

and since φ was arbitrary, by a density argument we finish the proof of point (ii).

(iii) This point is an immediate consequence of (ii), since

$$\begin{aligned}\int_B (\mathbf{z}, Du) &= \int_B \theta(\mathbf{z}, Du, x)|Du| = \int_{-\infty}^{\infty} \left(\int_B \theta(\mathbf{z}, Du, x)|D\chi_{E_t}| \right) dt \\ &= \int_{-\infty}^{\infty} \left(\int_B \theta(\mathbf{z}, D\chi_{E_t}, x)|D\chi_{E_t}| \right) dt = \int_{-\infty}^{\infty} \left(\int_B (\mathbf{z}, D\chi_{E_t}) \right) dt.\end{aligned}$$

□

We now show that the Radon-Nikodym derivative θ is invariant under monotone Lipschitz transformations of the real line.

PROPOSITION 3.17. *Assume that $u \in BV(\Omega) \cap L^q(\Omega)$ and $\mathbf{z} \in X_p(\Omega)$. If $T : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous increasing function, then*

$$\theta(\mathbf{z}, D(T \circ u), x) = \theta(\mathbf{z}, Du, x) \qquad |Du| - \text{a.e. in } \Omega.$$

PROOF. Denote by F_s the superlevel sets of $T \circ u$, i.e., $F_s = \{(T \circ u) > s\}$. Then, observe that

$$E_t = \{x \in \Omega : \ u(x) > t\} = \{x \in \Omega : \ (T \circ u)(x) > T(t)\} = F_{T(t)},$$

so for almost all $t \in \mathbb{R}$ we have

$$D\chi_{E_t} = D\chi_{F_{T(t)}}$$

as measures. Hence, by Theorem 3.16(ii), for $\mathcal{L}^1$-almost all $t \in \mathbb{R}$

$$\theta(\mathbf{z}, Du, x) = \theta(\mathbf{z}, D\chi_{E_t}, x) = \theta(\mathbf{z}, D\chi_{F_{T(t)}}, x) = \theta(\mathbf{z}, D(T \circ u), x)$$

$|D\chi_{E_t}|$-a.e. in Ω. By the co-area formula (Theorem 1.19), this equality also holds $|Du|$-a.e., which concludes the proof. □

EXERCISE 3.18. Show that whenever the function u satisfies the chain rule $D(f \circ u) = f'(u)Du$ for all Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$, the result above extends to all nondecreasing Lipschitz functions $T : \mathbb{R} \to \mathbb{R}$ (with the desired property valid $|D(T \circ u)|$-a.e. in Ω).

Further reading

The original construction of the pairing $(\mathbf{z}, Du)$ and the weak normal trace $[\mathbf{z}, \nu^\Omega]$ is due to Anzellotti [**4**]. At the same time, a similar pairing has been introduced by Kohn and Temam in [**32**]. For further properties of Anzellotti pairings, see for instance [**14, 17**]. An overview of Anzellotti pairings, including their applications to PDEs, can be found in [**3**].

CHAPTER 4

Rudin-Osher-Fatemi model of image denoising

This lecture is devoted to the study of the Euler-Lagrange equation for the minimisation of the Rudin-Osher-Fatemi functional [**40**], introduced already in the second lecture, which is the basis of total variation denoising. We set $E : L^2(\Omega) \to [0, +\infty]$ by the formula

$$E(u) = \begin{cases} \displaystyle\int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega (u-f)^2\,dx & \text{if } u \in BV(\Omega) \cap L^2(\Omega); \\ +\infty & \text{if } u \in L^2(\Omega)\backslash BV(\Omega), \end{cases}$$

where $\lambda > 0$ is a bias parameter. This is a classical problem in image restoration: given a corrupted image $f \in L^2(\Omega)$, the goal is to remove the "noise" and recover the uncorrupted image $u \in L^2(\Omega)$. In other words, we aim to decompose f as

$$f = u + n,$$

where n is the additive noise with small L^2 norm. The function n includes both the "white noise" and the textured part, i.e., periodic structures with small amplitude.

The ROF model is closely related to another model from image processing, called the Chan-Vese model [**16**]. Let $\Omega \subset \mathbb{R}^2$ be sufficiently regular bounded domain and $\mu > 0$ be a fidelity parameter. Given an initial image $f : \Omega \to [0,1]$, we aim to find a set of finite perimeter $\Lambda \subset \Omega$ and two constants m_0 and m_1, which represent the light intensity in the foreground and background regions of an image, which minimise

$$P(\Lambda, \Omega) + \mu\bigg(\int_\Lambda (m_1 - f(x))^2\,dx + \int_{\Omega\backslash\Lambda} (m_0 - f(x))^2\,dx\bigg)$$

among all sets of finite perimeter and constants between 0 and 1. This corresponds to the segmentation of the image Ω into a light and dark area. It can be looked at as a simple case of the Mumford-Shah functional [**39**], in which one considers only piecewise constant functions with two values. It is difficult to study the CV functional directly, due to the lack of convexity, and instead one can do this through the ROF model; for a suitable choice of λ depending on m_0, m_1 and μ, level sets of the unique minimiser of the ROF functional are solutions to the constrained CV model with the parameters m_0, m_1 fixed [**12**].

Since the functional E is not regular enough, as it involves the total variation measure, we cannot assign to it an Euler-Lagrange in the classical sense (i.e., by considering variations of E in different directions). We will do so in a generalised sense using the notions of convex analysis we introduce below.

4.1. Subdifferentials of convex functions

Let $(E, \|\cdot\|)$ be a real Banach space. In this Section, by an *operator* in E we will understand a multivalued operator, i.e., a mapping

$$A : E \to 2^E,$$

where 2^E denotes the collection of all subsets of E. Equivalently, we can think of a multivalued operator as a subset of $E \times E$: if we denote by

$$G(A) := \{(x, y) \in E \times E : \ y \in Ax\}$$

the graph of an operator A, then the set $G(A)$ determines uniquely the operator A since

$$Ax = \{y \in E : \ (x, y) \in G(A)\}.$$

Furthermore, let us denote by

$$D(A) := \{x \in E \ : \ Ax \neq \varnothing\}$$

the *effective domain* of A and by

$$R(A) := \bigcup_{x \in E} \{Ax \ : \ x \in D(A)\}$$

its *range*. Clearly, to every multivalued operator we can assign its (also multivalued) inverse, i.e. the operator

$$A^{-1}x := \{y \in E \ : \ x \in Ay\}.$$

The most important example of a multivalued operator is the subdifferential of a convex function.

Definition 4.1. Let $\mathcal{F} : E \to (-\infty, +\infty]$ be proper (i.e. $\mathcal{F} \not\equiv +\infty$) and convex, i.e.

$$\mathcal{F}(tx + (1-t)y) \leqslant t\mathcal{F}(x) + (1-t)\mathcal{F}(y) \quad \forall\, x, y \in E \text{ and } t \in (0,1).$$

The *subdifferential* (or *subgradient*) $\partial\mathcal{F}$ of the functional $\mathcal{F}$ is defined as

$$\partial\mathcal{F}(x) = \left\{x^* \in E^* : \mathcal{F}(y) - \mathcal{F}(x) \geqslant \langle x^*, y - x\rangle \quad \forall\, y \in E\right\},$$

where E^* denotes the dual of E. Equivalently, if we identify a multivalued operator with its graph, it is a subset of $E \times E^*$ defined by

$$\partial\mathcal{F} = \left\{(x, x^*) \in E \times E^* : \mathcal{F}(y) - \mathcal{F}(x) \geqslant \langle x^*, y - x\rangle \quad \forall\, y \in E\right\}.$$

The geometric idea behind the subdifferential is that it describes the set of supporting hyperplanes which lie below the graph of a convex function. Let us first see several examples of subdifferentials; then, we will discuss how they arise in calculus of variations and in the study of evolution equations.

Example 4.2. Suppose that $E = \mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}$ is differentiable. Then, $\partial f(x) = \{\nabla f(x)\}$.

Example 4.3. Let $E = \mathbb{R}^N$ and take $f : \mathbb{R}^N \to \mathbb{R}$ given by $f(x) = |x|$. It is not differentiable at the origin, but we still can explicitly compute the subdifferential:

$$\partial f(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x \neq 0; \\ B(0,1) & \text{if } x = 0. \end{cases}$$

Example 4.4. Let Ω be an open bounded subset of $\mathbb{R}^N$ with smooth boundary. Let $\mathcal{F} : L^2(\Omega) \to [0, +\infty]$ be given by

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 \, dx & \text{if } u \in W_0^{1,2}(\Omega); \\ +\infty & \text{if } u \in L^2(\Omega) \backslash W_0^{1,2}(\Omega). \end{cases}$$

Then, $\partial \mathcal{F}(u) = -\Delta u$ and $D(\partial \mathcal{F}) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

The subdifferentials of convex functions in Banach spaces are important in the optimization theory due to the following fact: observe that

$$0 \in \partial \mathcal{F}(x) \iff \mathcal{F}(y) \geqslant \mathcal{F}(x) \quad \forall\, y \in E.$$

Therefore, we have that $0 \in \partial \mathcal{F}(x)$ is the *Euler-Lagrange equation* of the variational problem

$$\mathcal{F}(x) = \min_{y \in E} \mathcal{F}(y).$$

Example 4.5. In the notation of the previous example, we have that

$$\mathcal{F}(u) = \min_{w \in L^2(\Omega)} \mathcal{F}(w) \qquad \iff \qquad 0 = -\Delta u \text{ on } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

Note that in the case when E is a Hilbert space H, by the Riesz theorem we have that for a proper convex functional $\mathcal{F} : H \to (-\infty, +\infty]$ the subdifferential $\partial \mathcal{F}$ is the operator in H defined as

$$z \in \partial \mathcal{F}(x) \iff \mathcal{F}(y) - \mathcal{F}(x) \geqslant (z, y - x) \quad \forall\, y \in H.$$

Proposition 4.6. *Let $\mathcal{F}_1, \mathcal{F}_2 : H \to (-\infty, \infty]$ be two proper, convex, and lower semicontinuous functionals. If there exists $u_0 \in D(\mathcal{F}_1) \cap D(\mathcal{F}_2)$ such that $\mathcal{F}_1$ is continuous at u_0, then*

$$\partial(\mathcal{F}_1 + \mathcal{F}_2)(u) = \partial \mathcal{F}_1(u) + \partial \mathcal{F}_2(u) \quad \textit{for all } u \in H.$$

Definition 4.7. In the case E is a Hilbert space H equipped with a scalar product $(\cdot, \cdot)$ and a norm

$$\|x\|_H := \sqrt{(x, x)},$$

we will say that an operator A in H is *monotone* if

$$(x - \hat{x}, y - \hat{y}) \geqslant 0 \quad \text{for all } (x, y), (\hat{x}, \hat{y}) \in A.$$

We have the following result due to Minty [**37**].

Theorem 4.8 (Minty theorem). *Let H be a Hilbert space. An operator A in H is maximal monotone if and only if it is monotone and satisfies the range condition, i.e., $R(I + A) = H$.*

It is easy to see that $\partial\mathcal{F}$ is a monotone operator in H. Moreover, if $\mathcal{F}$ is lower semicontinuous, then $\partial\mathcal{F}$ is maximal monotone (see [**10**]); this property is crucial when it comes to the study of evolution equations.

EXERCISE 4.9. For a non-empty set $K \subset H$, its *indicator function* is defined as

$$I_K(x) := \begin{cases} 0 & \text{if } x \in K; \\ +\infty & \text{if } x \notin K. \end{cases}$$

Show that: I_K is convex if and only if K is convex; I_K is lower semicontinuous if and only if K is closed; and

$$z \in \partial I_K(x) \iff x \in K \text{ and } (y,z) \leqslant (x,z) \quad \forall\, y \in K.$$

4.2. Fenchel-Rockafellar duality theorem

In this Section, we briefly present some of the convex duality methods for calculus of variations, in particular the Fenchel-Rockafellar duality theorem. Our presentation follows the one in [**22**] (in particular Chapters III and V).

DEFINITION 4.10. Given a Banach space V and a convex function $F : V \to \mathbb{R} \cup \{+\infty\}$, we define its *Legendre-Fenchel transform* $F^* : V^* \to \mathbb{R} \cup \{+\infty\}$ by the formula

$$F^*(v^*) = \sup_{v \in V} \Big\{ \langle v, v^* \rangle - F(v) \Big\}.$$

EXERCISE 4.11. For a non-empty set $K \subset V$, if I_K is indicator function of K, show that

$$I_K^*(x^*) = \sup\{\langle x, x^* \rangle : \ x \in K\} =: p_K(x^*).$$

The function p_K is called the *support function* of K. Also, prove that if K is the unit ball of V, we have that

$$I_K^*(x^*) = \|x^*\| \quad \text{for all } \ x^* \in V^*.$$

We now state the Fenchel-Rockafellar duality theorem in the form suitable for calculus of variations and presented in [**22**]. Let X, Y be two Banach spaces and let $A : X \to Y$ be a continuous linear operator. Denote by $A^* : Y^* \to X^*$ its dual. Then, if the primal minimization problem is of the form

$$\text{(P)} \qquad \inf_{u \in X} \Big\{ E(Au) + G(u) \Big\},$$

then the dual problem is defined as the maximization problem

$$\text{(P}^*\text{)} \qquad \sup_{p^* \in Y^*} \Big\{ -E^*(-p^*) - G^*(A^* p^*) \Big\},$$

where E^* and G^* are the Legendre–Fenchel transformations (conjugate functions) of E and G respectively. Furthermore, the following result holds.

THEOREM 4.12 (Fenchel-Rockafellar duality theorem). *Assume that E and G are proper, convex and lower semicontinuous. If there exists $u_0 \in X$ such that $E(Au_0) < \infty$, $G(u_0) < \infty$ and E is continuous at Au_0, then*

$$\inf\text{(P)} = \sup\text{(P}^*\text{)}$$

and the dual problem (P*) *admits at least one solution. Moreover, the optimality condition of these two problems is given by*

$$A^*p^* \in \partial G(u), \quad -p^* \in \partial E(Au)),$$

where u is solution of (P) *and p^* is solution of* (P*). *Equivalently, we have*

$$E(Au) + E^*(-p^*) = \langle -p^*, Au \rangle$$

and

$$G(u) + G^*(A^*p^*) = \langle u, A^*p^* \rangle.$$

In the case when there is no solution to the primal problem, we have a similar result, but instead of optimality conditions we have the following $\varepsilon-$subdifferentiability property of minimizing sequences.

Proposition 4.13. *Assume that E and G are proper, convex and lower semicontinuous. If there exists $u_0 \in X$ such that $E(Au_0) < \infty$, $G(u_0) < \infty$ and E is continuous at Au_0, then*

$$\inf \text{(P)} = \sup \text{(P*)}$$

and the dual problem (P*) *admits at least one solution. Moreover, for any minimizing sequence u_n for* (P) *and a maximizer p^* of* (P*), *we have*

$$0 \leqslant E(Au_n) + E^*(-p^*) - \langle -p^*, Au_n \rangle \leqslant \varepsilon_n \tag{4.1}$$

and

$$0 \leqslant G(u_n) + G^*(A^*p^*) - \langle u_n, A^*p^* \rangle \leqslant \varepsilon_n \tag{4.2}$$

with $\varepsilon_n \to 0$.

4.3. Euler-Lagrange equation for the ROF functional

This section is devoted to finding the Euler-Lagrange equation corresponding to the minimisation of E, i.e., giving a precise meaning to the equation

$$0 \in \partial E(u)$$

and characterising the minimisers in this way. To this end, we use Proposition 4.6; we may decompose E as the sum of two functionals, i.e.,

$$E = \mathcal{F} + \mathcal{G},$$

where $\mathcal{F} : L^2(\Omega) \to [0, +\infty]$ is defined by

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_\Omega |Du| & \text{if } u \in BV(\Omega) \cap L^2(\Omega); \\ +\infty & \text{if } u \in L^2(\Omega) \backslash BV(\Omega). \end{cases}$$

and $\mathcal{G} : L^2(\Omega) \to [0, +\infty]$ is defined by

$$\mathcal{G}(u) := \frac{\lambda}{2} \int_\Omega |u - f|^2 \, dx.$$

Clearly, the functional $\mathcal{F}$ is proper, convex and lower semicontinuous with respect to convergence in $L^2(\Omega)$. Moreover, the functional $\mathcal{G}$ is finite everywhere on $L^2(\Omega)$, convex and continuous with respect to the norm in $L^2(\Omega)$; thus, applying Proposition 4.6, we get that for all $u \in L^2(\Omega)$

$$\partial E(u) = \partial \mathcal{F}(u) + \partial \mathcal{G}(u),$$

or equivalently, since $\mathcal{G}$ is Fréchet differentiable,

$$\partial E(u) = \partial \mathcal{F}(u) + \lambda(u - f). \tag{4.3}$$

Therefore, the problem of finding the subdifferential of E is reduced to studying the subdifferential of the total variation. To characterise the subdifferential of $\mathcal{F}$ in $L^2(\Omega)$, we will use convex duality in the setting presented in Memo 4.12; to this end, define the following multivalued operator.

Definition 4.14. We say that $(u, v) \in \mathcal{A}$ if and only if $u, v \in L^2(\Omega)$, $u \in BV(\Omega)$ and there exists a vector field $\mathbf{z} \in X_2(\Omega)$ such that the following conditions hold:

$$\|\mathbf{z}\|_\infty \leqslant 1;$$

$$(\mathbf{z}, Du) = |Du| \quad \text{as measures;}$$

$$-\mathrm{div}(\mathbf{z}) = v \quad \text{in } \Omega;$$

$$[\mathbf{z}, \nu^\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Lemma 4.15. *We have $\mathcal{A} \subset \partial \mathcal{F}$. In particular, $\mathcal{A}$ is a monotone operator.*

Proof. Let $(u, v) \in \mathcal{A}$ and $\mathbf{z} \in X_2(\Omega)$ satisfy the conditions in Definition 4.14. Given $w \in L^2(\Omega) \cap BV(\Omega)$, by the Gauss-Green formula (Theorem 3.9)

$$\begin{aligned}\int_\Omega (w-u)\, v\, dx = -\int_\Omega \mathrm{div}(\mathbf{z})(w-u)\, dx = \int_\Omega (\mathbf{z}, Dw) - \int_\Omega (\mathbf{z}, Du) \\ \leqslant \int_\Omega |Dw| - \int_\Omega |Du| = \mathcal{F}(w) - \mathcal{F}(u),\end{aligned}$$

which concludes the proof. □

We now prove the anticipated result that we can characterise the subdifferential of $\mathcal{F}$ using the auxiliary operator $\mathcal{A}$.

Theorem 4.16. *We have $\mathcal{A} = \partial \mathcal{F}$.*

Proof. **Step 1.** By Lemma 4.15, the operator $\mathcal{A}$ is monotone and contained in $\partial\mathcal{F}$. The operator $\partial\mathcal{F}$ is maximal monotone; hence, once we prove that $\mathcal{A}$ satisfies the range condition, i.e.

$$\forall\, g \in L^2(\Omega)\ \exists\, u \in D(\mathcal{A}) \text{ such that } g \in u + \mathcal{A}(u), \tag{4.4}$$

or equivalently that $(u, g - u) \in \mathcal{A}$, the Minty theorem implies that the operator $\mathcal{A}$ is maximal monotone and consequently that $\mathcal{A} = \partial\mathcal{F}$. Therefore, we need to prove existence of $u \in BV(\Omega)$ and $\mathbf{z} \in X_2(\Omega)$ such that the following conditions hold:

$$\|\mathbf{z}\|_\infty \leqslant 1; \tag{4.5}$$

$$(\mathbf{z}, Du) = |Du| \quad \text{as measures;} \tag{4.6}$$

$$-\mathrm{div}(\mathbf{z}) = g - u \quad \text{in } \Omega; \tag{4.7}$$

$$(4.8) \qquad [\mathbf{z}, \nu^\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

We will prove that the range condition (4.4) holds using the Fenchel-Rockafellar duality theorem; we need to present it in the framework described before Theorem 4.12.

Step 2. We first restrict our attention to $W^{1,1}(\Omega)$ and set

$$U = W^{1,1}(\Omega) \cap L^2(\Omega)$$

and

$$V = L^1(\partial\Omega) \times L^1(\Omega; \mathbb{R}^N).$$

We denote the points $v \in V$ in the following way: $v = (v_0, \overline{v})$, where $v_0 \in L^1(\partial\Omega)$ and $\overline{v} \in L^1(\Omega; \mathbb{R}^N)$. We will also need the explicit expression of the dual space to V, which is

$$V^* = L^\infty(\partial\Omega) \times L^\infty(\Omega; \mathbb{R}^N),$$

and we use a similar notation for points $v^* \in V^*$. The operator $A : U \to V$ is defined by the formula

$$Au = (u|_{\partial\Omega}, \nabla u).$$

Clearly, A is a linear and continuous operator.

Then, we set $E : V \to \mathbb{R}$ by the formula

$$E(v_0, \overline{v}) = E_0(v_0) + E_1(\overline{v}),$$

where

$$E_0(v_0) = 0$$

and

$$E_1(\overline{v}) = \int_\Omega |\overline{v}| \, dx.$$

Clearly, E is a proper, convex, and lower semicontinuous functional. We also set $G : W^{1,1}(\Omega) \cap L^2(\Omega) \to \mathbb{R}$ by

$$G(u) := \frac{1}{2} \int_\Omega u^2 \, dx - \int_\Omega ug \, dx$$

and see that it a proper, convex, and continuous functional. Observe that by the Young inequality

$$G(u) \geqslant \frac{1}{2} \int_\Omega u^2 \, dx - \varepsilon \int_\Omega u^2 \, dx - C(\varepsilon) \int_\Omega g^2 \, dx,$$

so if we choose $\varepsilon < \frac{1}{2}$, we get that G is bounded from below.

Step 3. We now compute the convex conjugates of E and G. The dual functional of G, i.e. $G^* : (W^{1,1}(\Omega) \cap L^2(\Omega))^* \to [0, +\infty]$ is given by

$$G^*(u^*) = \frac{1}{2} \int_\Omega (u^* + g)^2 \, dx.$$

Now, observe that we evaluate G^* at $A^* v^*$; we need to compute this value. By definition of the dual operator, we get

$$\int_\Omega u \, (A^* v^*) \, dx = \langle u, A^* v^* \rangle = \langle v^*, Au \rangle = \int_{\partial\Omega} v_0^* \, u \, d\mathcal{H}^{N-1} + \int_\Omega \overline{v}^* \cdot \nabla u \, dx.$$

If we now consider only functions $u \in W_0^{1,1}(\Omega) \cap L^2(\Omega)$, which are dense in $L^2(\Omega)$, we see that the boundary term disappears and get

$$A^* v^* = -\mathrm{div}(\overline{v}^*). \tag{4.9}$$

In particular, the divergence of $\overline{v}^*$ is square-integrable, so $\overline{v}^* \in X_2(\Omega)$. Therefore, for any $u \in W^{1,1}(\Omega) \cap L^2(\Omega)$ we may apply the Gauss-Green formula (Theorem 3.9) and get

$$\begin{aligned}\int_\Omega u\,(A^* v^*)\,dx = \langle u, A^* v^* \rangle = \langle v^*, Au \rangle &= \int_{\partial\Omega} v_0^*\, u\, d\mathcal{H}^{N-1} + \int_\Omega \overline{v}^* \cdot \nabla u\, dx \\ &= \int_{\partial\Omega} v_0^*\, u\, d\mathcal{H}^{N-1} - \int_\Omega u\, \mathrm{div}(\overline{v}^*)\, dx + \int_{\partial\Omega} u\, [\overline{v}^*, \nu^\Omega]\, d\mathcal{H}^{N-1} \\ &= -\int_\Omega u\, \mathrm{div}(\overline{v}^*)\, dx + \int_{\partial\Omega} u\, (v_0^* + [\overline{v}^*, \nu^\Omega])\, d\mathcal{H}^{N-1}.\end{aligned}$$

By (4.9), the integrals over Ω cancel out, so

$$\int_{\partial\Omega} u\, (v_0^* + [\overline{v}^*, \nu^\Omega])\, d\mathcal{H}^{N-1} = 0$$

for all $u \in W^{1,1}(\Omega) \cap L^2(\Omega)$. By a density argument, we conclude that

$$v_0^* = -[\overline{v}^*, \nu^\Omega] \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Therefore,

$$G^*(A^* v^*) = \frac{1}{2} \int_\Omega (-\mathrm{div}(\overline{v}^*) + g)^2\, dx < \infty.$$

We now turn to computing the convex conjugates of the functionals E_i (for $i = 0, 1$). It is clear that the functional $E_0^* : L^\infty(\partial\Omega) \to [0, \infty]$ is

$$E_0^*(v_0^*) = \begin{cases} 0 & \text{if } v_0^* = 0; \\ +\infty & \text{if } v_0^* \neq 0. \end{cases}$$

Furthermore, the functional $E_1^* : L^\infty(\Omega; \mathbb{R}^N) \to [0, \infty]$ is given by the formula

$$E_1^*(\overline{v}^*) = \begin{cases} 0 & \text{if } \|\overline{v}^*\|_\infty \leqslant 1; \\ +\infty & \text{otherwise,} \end{cases}$$

so we computed the convex conjugate of E coordinate-wise.

Step 4. We will infer that the range condition (4.4) holds in the following way. Consider the minimisation problem

$$\inf_{u \in U} \Big\{ E(Au) + G(u) \Big\} \tag{4.10}$$

with E and G defined as above. For $u_0 \equiv 0$ we have $E(Au_0) = G(u_0) = 0 < \infty$ and E is continuous at 0. Then, Theorem 4.12 implies that the dual problem given by

$$\sup_{v^* \in V^*} \Big\{ -E^*(-v^*) - G^*(A^* v^*) \Big\}$$

admits at least one solution and there is no duality gap, i.e. the infimum in the first problem is equal to the supremum in the second one. Since the value of E^*

is either 0 or $+\infty$, and the value of G^* is finite exactly on the domain of A^*, we conclude that any solution v^* to the dual problem satisfies

$$v_0^* = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega; \tag{4.11}$$

$$\|\overline{v}^*\|_\infty \leqslant 1; \tag{4.12}$$

and

$$\overline{v}^* \in X_2(\Omega).$$

Step 5. Now, consider the functional $\mathcal{G} : L^2(\Omega) \to (-\infty, +\infty]$ defined by

$$\mathcal{G}(v) := \mathcal{F}(v) + G(v),$$

i.e. an extension of the functional $E \circ A + G$, well-defined for functions in $W^{1,1}(\Omega) \cap L^2(\Omega)$, to the space $BV(\Omega) \cap L^2(\Omega)$ (and a further extension by $+\infty$ to the rest of $L^2(\Omega)$). By the properties of $\mathcal{F}$ and G, we get that $\mathcal{G}$ is bounded from below, convex and lower semicontinuous. It is also coercive, because whenever $\mathcal{G}(u) \leqslant M$, we have

$$\frac{1}{2} \int_\Omega u^2 \, dx + \mathcal{F}(u) \leqslant M + \int_\Omega ug \, dx,$$

and by positivity of $\mathcal{F}$ and the Young inequality for $\varepsilon < \frac{1}{2}$ we get

$$\left(\frac{1}{2} - \varepsilon \right) \int_\Omega u^2 \, dx \leqslant M + C(\varepsilon) \int_\Omega g^2 \, dx,$$

so the norm of u in $L^2(\Omega)$ is bounded. Therefore, the minimisation of $\mathcal{G}(v)$ in $L^2(\Omega)$ admits a solution u and by the Meyers-Serrin theorem (Theorem 1.16) we have

$$\min_{v \in L^2(\Omega)} \mathcal{G}(v) = \inf_{v \in U} \Big\{ E(Av) + G(v) \Big\}.$$

However, the solution u does not necessarily lie in $W^{1,1}(\Omega)$, which is the domain of the functional $E \circ A + G$. Therefore, we cannot use the extremality conditions given in Theorem 4.12, and we instead rely on the $\varepsilon-$subdifferentiability property of minimising sequences given in (4.1) and (4.2). From this, we will deduce that the vector field $\mathbf{z} = -\overline{v}^* \in X_2(\Omega)$ satisfies the conditions (4.5)-(4.8) required for the range condition (4.4). Observe that condition (4.5) is automatically satisfied due to (4.12) and the condition (4.8) holds by (4.11) and the constraint $v_0^* = [-\overline{v}^*, \nu^\Omega]$; we proceed to prove the other conditions.

Take a sequence $u_n \in W^{1,1}(\Omega) \cap L^2(\Omega)$ which approximates u as in the anisotropic Meyers-Serrin theorem (Theorem 1.16); in particular, it is a minimising sequence in (4.10). By the second subdifferentiability property (4.2), for every $w \in L^2(\Omega)$ we have

$$G(w) - G(u_n) \geqslant \langle (w - u_n), A^* v^* \rangle - \varepsilon_n,$$

and by passing to the limit $n \to \infty$ we get

$$G(w) - G(u) \geqslant \langle (w - u), A^* v^* \rangle.$$

Therefore,

$$\mathrm{div}(-\overline{v}^*) = A^* v^* \in \partial G(u) = \{u - g\},$$

so the divergence constraint (4.7) is satisfied once we choose $\mathbf{z} = -\overline{v}^*$.

By the first subdifferentiability property (4.1), we have

$$0 \leqslant \int_\Omega |\nabla u_n|\, dx + \int_{\partial\Omega} v_0^* u_n \, d\mathcal{H}^{N-1} + \int_\Omega \overline{v}^* \cdot \nabla u_n \, dx \leqslant \varepsilon_n.$$

Since the boundary terms disappears, the first subdifferentiability property (4.1) yields

$$0 \leqslant \int_\Omega |\nabla u_n|\, dx + \int_\Omega \overline{v}^* \cdot \nabla u_n \, dx \leqslant \varepsilon_n. \tag{4.13}$$

Since $u_n|_{\partial\Omega} = u|_{\partial\Omega}$, by the Gauss-Green formula (Theorem 3.9) we have

$$\begin{aligned}
\int_\Omega \overline{v}^* \cdot \nabla u_n \, dx &= -\int_\Omega u_n \operatorname{div}(\overline{v}^*)\, dx + \int_{\partial\Omega} u_n \, [\overline{v}^*, \nu^\Omega] \, d\mathcal{H}^{N-1} \\
&= -\int_\Omega u \operatorname{div}(\overline{v}^*)\, dx + \int_{\partial\Omega} u \, [\overline{v}^*, \nu^\Omega] \, d\mathcal{H}^{N-1} \\
&\qquad\qquad + \int_\Omega (u - u_n) \operatorname{div}(\overline{v}^*)\, dx \\
&= \int_\Omega (\overline{v}^*, Du) + \int_\Omega (u - u_n) \operatorname{div}(\overline{v}^*)\, dx.
\end{aligned}$$

Passing to the limit $n \to \infty$, we get

$$\lim_{n\to\infty} \int_\Omega \overline{v}^* \cdot \nabla u_n \, dx = \int_\Omega (\overline{v}^*, Du).$$

We now pass to the limit $n \to \infty$ in the inequality (4.13) and obtain

$$\int_\Omega |Du| + \int_\Omega (\overline{v}^*, Du) = 0.$$

Observe that the above expression is always nonnegative; since $\|\overline{v}^*\|_\infty \leqslant 1$, by Proposition 3.2 we have

$$\int_\Omega |Du| + \int_\Omega (\overline{v}^*, Du) \geqslant 0.$$

Therefore, this inequality needs to be an equality, so property (4.6) holds for the choice $\mathbf{z} = -\overline{v}^*$. Therefore, we proved that all the conditions (4.5)-(4.8) needed for the range condition (4.4) hold, so the operator $\mathcal{A}$ is maximal monotone. $\square$

Exercise 4.17. Prove that for any proper, convex and lower semicontinuous functional $\mathcal{F} : L^2(\Omega) \to (-\infty, +\infty]$ we have $\overline{D(\partial\mathcal{F})} = \overline{D(\mathcal{F})}$, and conclude that the domain of $\mathcal{A}$ is dense in $L^2(\Omega)$.

Exercise 4.18. Find the explicit form of the dual problem in Step 4.

Therefore, by Theorem 4.16 and formula (4.3), we get the following characterisation of the subdifferential of E.

Corollary 4.19. *For $v \in L^2(\Omega)$ and $u \in BV(\Omega) \cap L^2(\Omega)$, the following conditions are equivalent:*

(1) $v \in \partial E(u)$;

(2) *There exists* $\mathbf{z} \in X_2(\Omega)$ *such that*

$$\|\mathbf{z}\|_\infty \leqslant 1;$$

$$(\mathbf{z}, Du) = |Du| \quad \textit{as measures};$$

$$v = -\mathrm{div}(\mathbf{z}) + \lambda(u - f) \quad \textit{in } \Omega;$$

$$[\mathbf{z}, \nu^\Omega] = 0 \qquad \mathcal{H}^{N-1} - \textit{a.e. on } \partial\Omega.$$

In particular, taking $v = 0$ in the above result, we get the Euler-Lagrange equation for minimisers of E.

Corollary 4.20. *The following conditions are equivalent:*

(1) $u \in L^2(\Omega) \cap BV(\Omega)$ *is a minimiser of* E*;*
(2) *There exists* $\mathbf{z} \in X_2(\Omega)$ *such that*

$$\|\mathbf{z}\|_\infty \leqslant 1;$$

$$(\mathbf{z}, Du) = |Du| \quad \textit{as measures};$$

$$\mathrm{div}(\mathbf{z}) = \lambda(u - f) \quad \textit{in } \Omega;$$

$$[\mathbf{z}, \nu^\Omega] = 0 \qquad \mathcal{H}^{N-1} - \textit{a.e. on } \partial\Omega.$$

The Rudin-Osher-Fatemi model can be understood as a semidiscretisation in time of the total variation flow, which will be the main focus of the next lecture. Take an interval $[0, T]$ and consider the equation

$$\begin{cases} u_t(t,x) = \Delta_1 u & \text{in } \ (0,T) \times \Omega; \\ \frac{\partial u}{\partial \nu}(t) = 0 & \text{on } \ (0,T) \times \partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \ \Omega, \end{cases}$$

where $u_0 \in L^2(\Omega)$. Here, Δ_1 denotes the 1-Laplacian operator, i.e.

$$\Delta_1(u) = \mathrm{div}\bigg(\frac{Du}{|Du|}\bigg).$$

Divide the interval $[0, T]$ into k parts of length λ^{-1}, so that $k = \lambda T$. Let us iterate the ROF functional in the following way: set $\lambda \gg 0$ and denote $u^0 = u_0$. Then, for each $n \in \{0, ..., \lambda T - 1\}$ we iteratively solve the ROF problem with $f = u^0$ and denote its solution by u^{n+1}.Then, in the limit $\lambda \to \infty$ with $k = \lambda T$, we have

$$\frac{\partial u}{\partial t}\bigg(\cdot, \frac{n}{\lambda T}\bigg) \approx \frac{u^{n+1} - u^n}{\lambda^{-1}} = \lambda(u^{n+1} - u^n) \in \mathrm{div}\bigg(\frac{Du^{n+1}}{|Du^{n+1}|}\bigg),$$

where u is a piecewise constant (in time) function with value u^n on the interval $[\frac{n}{\lambda T}, \frac{n+1}{\lambda T}]$. Passing to the limit, we formally arrive at the differential inclusion

$$\frac{\partial u}{\partial t} \in \mathrm{div}\bigg(\frac{Du}{|Du|}\bigg).$$

The argument given here is of course purely heuristic, and the way to arrive to this conclusion in a precise way is the Crandall-Liggett generation theorem [**18**] (a more modern take can be found in [**3**]).

4.4. Regularity of solutions in one dimension

The question of regularity of minimisers to the Rudin-Osher-Fatemi functional, or to be more exact, whether regularity of the initial datum is preserved, was an object of intensive study in the last twenty years. It is still an active topic with many open questions; the only fully solved cases are when u is Hölder continuous, when it is a characteristic function of a convex smooth set, and the one-dimensional case. We present the argument in the last setting, mostly following [**25**], and only briefly discuss the other two.

THEOREM 4.21. *Suppose that $f \in BV((a,b))$ and let $u \in BV((a,b))$ be the unique minimiser of the functional E. Then, $|u'| \leqslant |f'|$ as measures, i.e.,*

$$|u'|(A) \leqslant |f'|(A) \quad \textit{for any Borel set } A \subseteq (a,b).$$

PROOF. Consider a regularisation of E of the following type: for $\varepsilon > 0$, set

$$E_\varepsilon(u) = \int_a^b \left(\frac{\lambda}{2}|u-f|^2 + \sqrt{(u')^2+\varepsilon^2} + \frac{\varepsilon^2}{2}|u'|^2 \right) dx,$$

i.e., we separate the total variation term from zero and add a second-order term. Then, for any $f \in L^2((a,b))$ this is a smooth and uniformly convex functional, which has a unique minimiser $u_\varepsilon \in W^{2,2}((a,b))$, so in particular $u \in C^1([a,b])$, and the minimiser satisfies the Euler-Lagrange equation

$$\lambda(u-f) = \left(\frac{u_\varepsilon'}{\sqrt{(u')^2+\varepsilon^2}} \right)' + \varepsilon^2 u_\varepsilon''$$

with boundary data $u_\varepsilon' = 0$ in the strong sense. Then, classical arguments (i.e., testing the equation with an appropriately chosen test function) yield that for any open interval I and $\delta > 0$ we have

$$\int_I ((u_\varepsilon')^2+\varepsilon^2)^{p/2}\, dx \leqslant \int_{I_\delta} |f'|^p\, dx + O(\varepsilon^{p-1}) \tag{4.14}$$

for all $p \in (1,2]$ (for a precise argument see [**25**]). Here, I_δ is the interval which is a δ-neighbourhood of I.

We now want to pass to the limit $p \to 1$. First, consider $f \in W^{1,2}((a,b))$. By the boundary condition, the derivative $|u'|$ is small close to a and b. Then, taking small $\delta > 0$, I such that $I_\delta = (a,b)$ and $p = 2$ in estimate (4.14), we get that

$$\int_a^b |u_\varepsilon'|^2\, dx \leqslant \varepsilon + \int_{a+\delta}^{b-\delta} ((u_\varepsilon')^2+\varepsilon^2)\, dx \leqslant \int_a^b |f'|^2\, dx + O(\varepsilon),$$

and consequently u_ε has a subsequence which converges weakly in $W^{1,2}((a,b))$ and uniformly in $C([a,b])$ to some $u \in W^{1,2}((a,b))$. Since it is clear that the functional E_ε Γ-converges to E, u is the unique minimiser of E (note that this argument already implies that $W^{1,2}$ regularity is preserved). Then, passing to the limit with $\varepsilon \to 0$ in estimate (4.14), we get

$$\int_I |u'|^p\, dx \leqslant \limsup_{\varepsilon\to 0} \int_I ((u_\varepsilon')^2+\varepsilon^2)^{p/2}\, dx \leqslant \int_{I_\delta} |f'|^p\, dx.$$

After passing to the limit $p \to 1$ and $\delta \to 0$, we get

$$\int_I |u'|\, dx \leqslant \int_I |f'|\, dx, \tag{4.15}$$

since f' lies in $L^2((a,b))$ as as such it gives zero measure to points.

Our next goal is to prove an estimate analogous to (4.15) for general $f \in BV((a,b))$. Since $f \in L^2((a,b))$, by the Meyers-Serrin approximation theorem (Theorem 1.16) we can find smooth functions f_n such that $f_n \to f$ in $L^2((a,b))$ and strictly in $BV((a,b))$. Consider a second regularisation of E, i.e.,

$$E_n(u) = \frac{\lambda}{2} \int_a^b |u - f_n|^2\, dx + \int_a^b |u'|.$$

Let $u_k \in BV((a,b))$ be the unique minimiser of E_n. Then, comparing the energy of u_k with the zero function, we get

$$\int_a^b |u_k'| \leqslant E_n(u_k) \leqslant E_n(0) = \int_a^b |f_n|^2\, dx,$$

which is uniformly bounded. Thus, u_k is uniformly bounded in $BV((a,b))$, and it converges (on a subsequence) in $L^2((a,b))$ and weakly* in $BV((a,b))$ to some function $u \in BV((a,b))$. Clearly, E_n Γ-converges to E as $f_n \to f$ in $L^2((a,b))$, so u is the unique minimiser of E. Since f_n converges strictly to f, passing to the limit in estimate (4.15) yields

$$\int_I |u'| \leqslant \int_I |f'|$$

whenever $|f'|(\partial I) = 0$.

This concludes the proof up to standard measure-theoretic arguments: write $I = B(x,r)$, and consider any open set $V \subset (a,b)$. Then, by the Besikovitch covering theorem (see [**23**]) one can write

$$V = \bigcup_{i=1}^{\infty} B(x_i, r_i) \cup N,$$

where the balls $B(x_i,r_i)$ are pairwise disjoint, satisfy $|f'|(\partial B(x_i,r_i)) = 0$, and $|u'|(N) = 0$. Thus,

$$|u'|(V) = \sum_{i=1}^{\infty} |u'|(B(x_i,r_i)) \leqslant \sum_{i=1}^{\infty} |f'|(B(x_i,r_i)) \leqslant |f'|(V).$$

To pass to a general open set, by approximation properties of Borel measures, given a Borel set $A \subset (a,b)$ and $\delta > 0$ one can find an open set $V \subset (a,b)$ with $A \subset V$ and $|f'|(V \backslash A) \leqslant \delta$. Therefore,

$$|u'|(A) \leqslant |u'|(V) \leqslant |f'|(V) \leqslant |f'|(A) + \delta,$$

which concludes the proof once we pass with $\delta \to 0$. □

Since this result is local (i.e., it holds for any Borel set A), we immediately get that many regularity properties of the initial data are inherited by the solution - let us list here some most important consequences.

Corollary 4.22. *In the notation of the previous Theorem, we have:*

(a) $f \in W^{1,p}((a,b))$ *implies* $u \in W^{1,p}((a,b))$*;*
(b) $f \in SBV((a,b))$ *implies* $u \in SBV((a,b))$*;*
(c) $J_u \subset J_f$*, i.e., no new discontinuities are formed;*
(d) The size of the jumps of u *is smaller than the size of jumps of* f*.*

Due to the construction from the previous Section, this implies similar results for the total variation flow (formally for now, we will discuss at length the total variation flow in the final lecture).

Corollary 4.23. *Suppose that* $u : \Omega \times (0,T) \to \mathbb{R}$ *is a solution of the total variation flow*

$$u_t = \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

with Neumann boundary conditions and initial data $u_0 \in L^2(\Omega)$*. Then, for a.e.* $t \in (0,T)$*:*

(a) $u_0 \in W^{1,p}((a,b))$ *implies* $u(t) \in W^{1,p}((a,b))$*;*
(b) $u_0 \in SBV((a,b))$ *implies* $u(t) \in SBV((a,b))$*;*
(c) $J_{u(t)} \subset J_{u_0}$*, i.e., no new discontinuities are formed;*
(d) The size of the jumps of $u(t)$ *is smaller than the size of jumps of* u*.*

However, the one-dimensional proof presented above fails in higher dimensions, essentially due to the fact that we do not have C^1 regularity for the approximating sequence. However, the part of the result concerning the jump set of the solution is valid in higher dimensions (at least for $f \in BV(\Omega) \cap L^N(\Omega)$), using a level-set argument similar to the one from the second lecture, see [**13, 15**].

Theorem 4.24. *Suppose that* $u \in BV(\Omega) \cap L^2(\Omega)$ *is a minimiser of* E *for* $f \in BV(\Omega) \cap L^N(\Omega)$*. Then,* $J_u \subset J_f$ *and the size of the jumps of* u *is smaller than the size of jumps of* f*.*

Similarly, suppose that $u : \Omega \times (0,T) \to \mathbb{R}$ *is a solution of the total variation flow*

$$u_t = \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

with Neumann boundary conditions and initial data $u_0 \in BV(\Omega) \cap L^N(\Omega)$*. Then, for a.e.* $t \in (0,T)$ *we have* $J_{u(t)} \subset J_{u_0}$ *and the size of the jumps of* $u(t)$ *is smaller than the size of jumps of* u*.*

Further reading

The subdifferential of the total variation, and consequently the subdifferential of the ROF functional, was characterised for the first time in [**2**]; see also the monograph [**3**]. The method presented here, quite simpler than the original approach, is relatively new and first appeared in [**30**] in a more general context. The results concerning the jump set of the ROF functional (and the total variation flow) first appeared in [**13**] and [**15**]; we present the newer one-dimensional results from [**25**] as a model case, because the methods used are far simpler.

CHAPTER 5

Total variation flow

Again, we assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^N$. Consider the following Neumann problem

$$
\begin{cases} u_t(t,x) = \Delta_1 u & \text{in } \ (0,T)\times\Omega; \\ \frac{\partial u}{\partial \nu}(t) = 0 & \text{on } \ (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \ \Omega, \end{cases} \tag{5.1}
$$

where $u_0 \in L^2(\Omega)$. Here, Δ_1 denotes the 1-Laplacian operator, i.e.

$$
\Delta_1(u) = \operatorname{div}\left(\frac{Du}{|Du|}\right).
$$

As discussed in the last lecture, this equation arises as a continuum version of an iteration scheme involving the Rudin-Osher-Fatemi functional. The goal of this lecture is to introduce a notion of weak solutions and study some qualitative properties of this equation.

5.1. Semigroup approach to evolution equations

We now present the basic results concerning the semigroup approach to gradient flows of convex functionals in Hilbert spaces; the most classical reference is [**10**]. Let H be a separable Hilbert space. For $1 \leqslant p < \infty$, we denote

$$
L^p(a,b;H) := \left\{ u : [a,b] \to H \text{ measurable such that } \int_a^b \|u(t)\|_H^p \, dt < \infty \right\}
$$

and

$$
W^{1,p}(a,b;H) := \Big\{ u \in L^p(a,b;H) \text{ and } \exists\, v \in L^p(a,b;H) :
$$
$$
u(t) - u(a) = \int_a^t v(s)\, ds \quad \forall\, t \in (a,b) \Big\}.
$$

If $u \in W^{1,p}(a,b;H)$, it is differentiable in time for almost all $t \in (a,b)$ and

$$
u(t) - u(a) = \int_a^t u'(s)\, ds \quad \forall\, t \in (a,b).
$$

We also set $W^{1,p}_{\text{loc}}(0,T;H)$ to be the space of all functions u with the following property: for all $0 < a < b < T$, we have that $u \in W^{1,p}(a,b;H)$.

Consider the abstract Cauchy problem

$$(5.2) \qquad \begin{cases} u'(t) + \partial\mathcal{F}(u(t)) \ni 0 & t \in (0,T), \\ u(0) = u_0, & u_0 \in H. \end{cases}$$

DEFINITION 5.1. We say that $u \in C([0,T];H)$ is a *strong solution* of problem (5.2), if the following conditions hold: $u \in W^{1,2}_{\mathrm{loc}}(0,T;H)$; for almost all $t \in (0,T)$ we have $u(t) \in D(\partial\mathcal{F})$; and it satisfies (5.2).

THEOREM 5.2 (Brezis-Komura theorem). *Let $\mathcal{F} : H \to (-\infty, \infty]$ be a proper, convex, and lower semi-continuous functional. Given $u_0 \in \overline{D(\mathcal{F})}$, there exists a unique strong solution of the abstract Cauchy problem* (5.2). *Moreover, we have that $\sqrt{t} \cdot u'(t) \in L^2(0,T;H)$, and $u \in W^{1,2}(0,T;H)$ whenever $u_0 \in D(\mathcal{F})$.*

We refer to [**10**] for a summary of main additional properties of solutions; let us only briefly mention the semigroup property, the T-contraction property, and the regularity of time derivative. If we denote by $S(t)u_0$ the unique strong solution $u(t)$ of the abstract Cauchy problem (5.2) for initial data u_0, then $S(t) : \overline{D(\mathcal{F})} \to H$ is a continuous semigroup satisfying the T-contraction property

$$\|(S(t)u_0 - S(t)v_0)\|_H \leqslant \|u_0 - v_0\|_H$$

for all $u_0, v_0 \in \overline{D(\mathcal{F})}$ and $t > 0$. Furthermore, we have that $u' \in L^2_{\mathrm{loc}}(0,T;H)$, and the function $t \mapsto \mathcal{F}(u(t))$ is convex, decreasing, and locally Lipschitz with the derivative (defined for a.e. $t > 0$)

$$\frac{d}{dt}\mathcal{F}(u(t)) = -\Big\|u'(t)\Big\|_H^2 = -\Big\|\partial^-\mathcal{F}(u(t))\Big\|_H^2,$$

where $\partial^-\mathcal{F}(\cdot)$ denotes the element of minimal norm in $\partial\mathcal{F}(\cdot)$. In fact,

$$u'(t) = \partial^-\mathcal{F}(u(t)).$$

Moreover, whenever $u_0 \in D(\partial\mathcal{F})$, we have that $u' \in L^\infty(0,T;H)$ and

$$\Big\|u'(t)\Big\|_H \leqslant \Big\|\partial^-\mathcal{F}(u_0)\Big\|_H$$

for all $t \in (0,T)$.

5.2. The total variation flow

Consider the energy functional $\mathcal{F} : L^2(\Omega) \to [0, +\infty]$ associated with problem (5.1) and defined by

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_\Omega |Du| & \text{if } u \in BV(\Omega) \cap L^2(\Omega); \\ +\infty & \text{if } u \in L^2(\Omega)\backslash BV(\Omega). \end{cases}$$

The functional $\mathcal{F}$ is lower semicontinuous with respect to convergence in $L^2(\Omega)$. Clearly, $\mathcal{F}$ is convex; thus, by the Brezis-Komura theorem (Theorem 5.2) there exists a unique strong solution of the abstract Cauchy problem

$$\begin{cases} 0 \in u'(t) + \partial\mathcal{F}(u(t)) & \text{for } t \in [0,T]; \\ u(0) = u_0. \end{cases}$$

Recall that the subdifferential of $\mathcal{F}$ in $L^2(\Omega)$ can be characterised using the following operator $\mathcal{A}$ (see Theorem 4.16).

DEFINITION 5.3. We say that $(u, v) \in \mathcal{A}$ if and only if $u, v \in L^2(\Omega)$, $u \in BV(\Omega)$ and there exists a vector field $\mathbf{z} \in X_2(\Omega)$ such that the following conditions hold:

$$\|\mathbf{z}\|_\infty \leqslant 1;$$

$$(\mathbf{z}, Du) = |Du| \quad \text{as measures;}$$

$$-\mathrm{div}(\mathbf{z}) = v \quad \text{in } \Omega;$$

$$[\mathbf{z}, \nu^\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Then, $\mathcal{A} = \partial\mathcal{F}$. In light of this, we can give the following definition of solutions to the Neumann problem (5.1).

DEFINITION 5.4. Given $u_0 \in L^2(\Omega)$, we say that u is a *weak solution* to the Neumann problem (5.1) in $[0,T]$, if $u \in C([0,T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega))$, $u(0,\cdot) = u_0$, and for almost all $t \in (0,T)$

$$0 \in u_t(t,\cdot) + \mathcal{A}u(t,\cdot).$$

In other words, for almost all $t \in (0,T)$ we have $u(t) \in BV(\Omega)$ and there exist vector fields $\mathbf{z}(t) \in X_2(\Omega)$ such that the following conditions hold:

$$\|\mathbf{z}(t)\|_\infty \leqslant 1;$$

$$(\mathbf{z}(t), Du(t)) = |Du(t)| \quad \text{as measures;}$$

$$u_t(t) = \mathrm{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$[\mathbf{z}(t), \nu^\Omega] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

With this definition, since $\mathcal{A}$ coincides with $\partial\mathcal{F}$, by the Brezis-Komura theorem (Theorem 5.2) we get the following existence and uniqueness result.

THEOREM 5.5. *For every $u_0 \in L^2(\Omega)$ there exists a unique weak solution $u \in C([0,T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega))$ to the Neumann problem* (5.1) *with initial datum u_0.*

The next exercise concerns an equivalent characterisation of weak solutions in terms of an integral equality satisfied on almost every time slice.

EXERCISE 5.6. Let $u_0 \in L^2(\Omega)$ and assume that

$$u \in C([0,T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega))$$

satisfies $u(0,\cdot) = u_0$. Show that u is a weak solution to the Neumann problem (5.1) if and only if for almost all $t \in (0,T)$ we have $u(t) \in BV(\Omega)$ and there exists a vector field $\mathbf{z} \in X_2(\Omega)$ such that $\|\mathbf{z}\|_\infty \leqslant 1$, $u_t(t) = \operatorname{div}(\mathbf{z}(t))$ in the sense of distributions and

$$\int_\Omega |Du(t)| + \int_\Omega u_t(t)(u(t) - v)\,dx = \int_\Omega (\mathbf{z}(t), Dv)$$

for every $v \in BV(\Omega) \cap L^2(\Omega)$.

MEMO 13. *A multivalued operator $\mathcal{A} \subset L^2(\Omega) \times L^2(\Omega)$ is called completely accretive if and only if the following condition is satisfied (see* [**3, 7**]*):*

$$\int_\Omega T(u^1 - u^2)(v^1 - v^2)\,dx \geqslant 0 \tag{5.3}$$

for every $(u^1, v^1), (u^2, v^2) \in \mathcal{A}$ and all functions $T \in C^\infty(\mathbb{R})$ such that $0 \leqslant T' \leqslant 1$, T' has compact support, and $x = 0$ is not contained in the support of T.

If $\mathcal{A}$ additionally satisfies the range condition, we have the following contraction and maximum principle in any L^q space, where $1 \leqslant q \leqslant +\infty$: for $u_{1,0}, u_{2,0} \in \overline{D(\mathcal{A})}$ and denoting by u_i the unique solution of the problem

$$\begin{cases} \dfrac{du_i(t)}{dt} + \mathcal{A}u_i(t) \ni 0, & t \in (0, \infty) \\ u_i(0) = u_{i,0} \end{cases}$$

for $i = 1, 2$, we have

$$\|(u_1(t) - u_2(t))^+\|_{L^q(\Omega)} \leqslant \|(u_{1,0} - u_{2,0})^+\|_{L^q(\Omega)} \quad \forall\, 0 < t < T.$$

LEMMA 5.7. *The operator $\mathcal{A}$ is completely accretive.*

PROOF. We need to show that condition (5.3) holds for all $T \in C^\infty(\mathbb{R})$ satisfying the above conditions, i.e. such that $0 \leqslant T' \leqslant 1$, T' has compact support, and $x = 0$ is not contained in the support of T. For $j = 1, 2$, let $(u^j, v^j) \in \mathcal{A}$ and let $\mathbf{z}^j$ be the associated vector fields. Observe that $T(u^1 - u^2) \in BV(\Omega)$. Since $\|\mathbf{z}^1\|_\infty \leqslant 1$ and $\|\mathbf{z}^2\|_\infty \leqslant 1$, by Proposition 3.2 for every Borel set $B \subset \Omega$ we have

$$\begin{aligned}\int_B (\mathbf{z}^1 - \mathbf{z}^2, & D(u^1 - u^2)) \\ &= \int_B |Du^1| - \int_B (\mathbf{z}^1, Du^2) + \int_B |Du^2| - \int_B (\mathbf{z}^2, Du^1) \geqslant 0.\end{aligned}$$

By definition of the Radon-Nikodym derivative $\theta(\mathbf{z}^1 - \mathbf{z}^2, D(u^1 - u^2), x)$ we get

$$\int_B \theta(\mathbf{z}^1 - \mathbf{z}^2, D(u^1 - u^2), x)\, d|D(u^1 - u^2)| = \int_B (\mathbf{z}^1 - \mathbf{z}^2, D(u^1 - u^2)) \geqslant 0$$

for all Borel sets $B \subset \Omega$. Therefore,

$$\theta(\mathbf{z}^1 - \mathbf{z}^2, D(u^1 - u^2), x) \geqslant 0 \qquad |D(u^1 - u^2)| - \text{a.e. on } \Omega$$

and since $|DT(u^1 - u^2)|$ is absolutely continuous with respect to $|D(u^1 - u^2)|$, we also have

$$\theta(\mathbf{z}^1 - \mathbf{z}^2, D(u^1 - u^2), x) \geqslant 0 \qquad |DT(u^1 - u^2)| - \text{a.e. on } \Omega.$$

By Proposition 3.17, we get that

$$\theta(\mathbf{z}^1 - \mathbf{z}^2, DT(u^1 - u^2), x) \geqslant 0 \qquad |DT(u^1 - u^2)| - \text{a.e. on } \Omega,$$

so

$$\text{(5.4)} \qquad \int_\Omega (\mathbf{z}^1 - \mathbf{z}^2, DT(u^1 - u^2)) \\ = \int_\Omega \theta(\mathbf{z}^1 - \mathbf{z}^2, DT(u^1 - u^2), x)\, d|DT(u^1 - u^2)| \geqslant 0.$$

To conclude that the operator $\mathcal{A}$ is completely accretive, we now apply the anisotropic Gauss-Green formula (Theorem 3.9) and use the estimate (5.4) to get

$$\int_\Omega T(u^1 - u^2)(v^1 - v^2)\, dx = -\int_\Omega T(u^1 - u^2)(\mathrm{div}(\mathbf{z}^1) - \mathrm{div}(\mathbf{z}^2))\, dx \\ = \int_\Omega (\mathbf{z}^1 - \mathbf{z}^2, DT(u^1 - u^2)) \geqslant 0,$$

so $\mathcal{A}$ satisfies the condition (5.3) and thus is completely accretive. □

Exercise 5.8. Prove that for a smooth function $T : \mathbb{R} \to \mathbb{R}$ the measure $|DT(u)|$ is absolutely continuous with respect to $|Du|$.

As a consequence of the complete accretivity of the operator $\mathcal{A}$, we get the following comparison principle.

Theorem 5.9. *For all $r \in [1, \infty]$, if u_1, u_2 are weak solutions to* (5.1) *for the initial data $u_{1,0}, u_{2,0} \in L^2(\Omega) \cap L^r(\Omega)$ respectively, then*

$$\|(u_1(t) - u_2(t))^+\|_r \leqslant \|(u_{1,0} - u_{2,0})^+\|_r.$$

A similar construction, using a mix of techniques from the proof above and the Euler-Lagrange characterisation of solutions to the least gradient problem, leads to a characterisation of solutions to the Dirichlet problem for the total variation flow, i.e.,

$$\text{(5.5)} \qquad \begin{cases} u_t(t,x) = \Delta_1 u & \text{in } (0,T) \times \Omega; \\ u(t) = h & \text{on } (0,T) \times \partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $u_0 \in L^2(\Omega)$. The corresponding energy functional is $\mathcal{F}_h : L^2(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}_h(u) := \begin{cases} \displaystyle\int_\Omega |Du| + \int_{\partial\Omega} |u - h|\, d\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega) \cap L^2(\Omega); \\ +\infty & \text{if } u \in L^2(\Omega) \backslash BV(\Omega). \end{cases}$$

We leave the proof in the form of the following series of exercises.

Exercise 5.10. We say that $(u, v) \in \mathcal{A}_h$ if and only if $u, v \in L^2(\Omega)$, $u \in BV(\Omega)$ and there exists a vector field $\mathbf{z} \in X_2(\Omega)$ such that the following conditions hold:

$$\|\mathbf{z}\|_\infty \leqslant 1;$$

$$(\mathbf{z}, Du) = |Du| \quad \text{as measures};$$

$$-\mathrm{div}(\mathbf{z}) = v \quad \text{in } \Omega;$$

$$[\mathbf{z}, \nu^\Omega] \in \operatorname{sign}(h-u) \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega,$$

where sign denotes the multivalued sign function. Show that $\mathcal{A}_h \subset \partial\mathcal{F}_h$ (so in particular it is monotone).

Then, we need to prove that $\mathcal{A}_h = \partial\mathcal{F}_h$ in a similar way as in Theorem 4.16. Observe that the range condition for the operator $\mathcal{A}_h$ boils down to proving existence of $u \in BV(\Omega)$ and $\mathbf{z} \in X_2(\Omega)$ such that the following conditions hold:

$$\|\mathbf{z}\|_\infty \leqslant 1;$$

$$(\mathbf{z}, Du) = |Du| \quad \text{as measures;}$$

$$-\operatorname{div}(\mathbf{z}) = g - u \quad \text{in } \Omega;$$

$$[\mathbf{z}, \nu^\Omega] \in \operatorname{sign}(h-u) \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

Exercise 5.11. Consider the same spaces, operators, and functionals as in Step 2 of the proof of Theorem 4.16, with the only difference in the definition of E_0; set

$$E_0(v_0) = \int_{\partial\Omega} |v_0 - h| \, d\mathcal{H}^{N-1}.$$

Then, show that

$$E_0^*(v_0^*) = \begin{cases} \displaystyle\int_{\partial\Omega} h\, v_0^* \, d\mathcal{H}^{N-1} & \text{if } |v_0^*| \leqslant 1 \quad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega; \\ +\infty & \text{otherwise,} \end{cases}$$

and check that with this change Steps 2 and 3 of the proof are correct.

Exercise 5.12. Prove that in the setting analogous to Lemma 5.7, we have

$$\int_{\partial\Omega} -T(u^1 - u^2)[\mathbf{z}^1 - \mathbf{z}^2, \nu^\Omega] \, d\mathcal{H}^{N-1} \geqslant 0,$$

from which follows the complete accretivity of $\mathcal{A}_h$.

Exercise 5.13. Show that in the case of the operator $\mathcal{A}_h$, the first subdifferentiability property (4.1) gives also an estimate for the boundary behaviour of u, and we may conclude using a similar argument as in Steps 4 and 5 of the proof that $\mathcal{A}_h = \partial\mathcal{F}_h$.

From this, using the Brezis-Komura theorem, we deduce existence of a unique weak solution to the Dirichlet problem for the total variation flow (5.5) in the following sense: given $u_0 \in L^2(\Omega)$, we say that u is a *weak solution* to the Dirichlet problem (5.5) in $[0,T]$, if $u \in C([0,T]; L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega))$, $u(0,\cdot) = u_0$, and for almost all $t \in (0,T)$

$$0 \in u_t(t,\cdot) + \mathcal{A}_h u(t,\cdot).$$

In other words, for almost all $t \in (0,T)$ we have $u(t) \in BV(\Omega)$ and there exist vector fields $\mathbf{z}(t) \in X_2(\Omega)$ such that the following conditions hold:

$$\|\mathbf{z}(t)\|_\infty \leqslant 1;$$

$$(\mathbf{z}(t), Du(t)) = |Du(t)| \quad \text{as measures;}$$

$$u_t(t) = \mathrm{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega);$$

$$[\mathbf{z}(t), \nu^\Omega] \in \mathrm{sign}(h - u) \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega.$$

By complete accretivity of the operator $\mathcal{A}_h$, it also satisfies the comparison principle similar to the one above, i.e., for all $r \in [1, \infty]$, if u_1, u_2 are weak solutions to (5.5) for the initial data $u_{1,0}, u_{2,0} \in L^2(\Omega) \cap L^r(\Omega)$ respectively, then

$$\|(u_1(t) - u_2(t))^+\|_r \leqslant \|(u_{1,0} - u_{2,0})^+\|_r.$$

5.3. Asymptotic behaviour

Let us first see an explicit example of the evolution by the total variation flow. For simplicity, we consider the Dirichlet problem with zero boundary data.

EXAMPLE 5.14. Let $N \geqslant 2$ and take $\Omega \subset \mathbb{R}^N$ such that $B(0, r) \Subset \Omega$ and let $h \equiv 0$. Then, consider the initial data

$$u_0 = k\chi_{B(0,r)}.$$

We will show that the unique solution to the Dirichlet problem (5.5) is given by

$$u(x, t) = \mathrm{sign}(k)\bigg(|k| - \frac{\mathcal{H}^{N-1}(\partial B(0, r))}{\mathcal{L}^N(B(0, r))} t\bigg)^+ \chi_{B(0,r)}(x).$$

In particular, $\chi_{B(0,r)}$ is a (nonlinear) eigenfunction of the total variation flow. Equivalently, we have

$$u(x, t) = \mathrm{sign}(k)\frac{N}{r}\bigg(\frac{|k|r}{N} - t\bigg)^+ \chi_{B(0,r)}(x). \tag{5.6}$$

Before we start proving that formula (5.6) holds, let us note the following observations: the initial condition is satisfied; it changes linearly in time; the solution reaches zero in finite time; the shape of the solution does not change, i.e., the jump set remains the same until the extinction time and the size of the jump decreases; in particular, the total variation flow has no smoothing effect.

Without loss of generality, suppose that $k > 0$. Let us look for solutions to problem (5.5) of the form $u(x, t) = \alpha(t)\chi_{B(0,r)}(x)$. Our (non-rigorous) motivation is as follows: since the initial data are radial, we expect the solutions to be radial; the right-hand side of the PDE (5.5) is, for characteristic functions, the mean curvature of the set, so since the boundary of a ball has constant mean curvature we expect that the whole ball will evolve in the same way; the exterior of the ball, which has a constant value of u_0, should evolve in the same way; and the zero Dirichlet datum should entail that the value of u is zero near the boundary.

For such a function u the Dirichlet boundary condition $[\mathbf{z}, \nu^\Omega] \in \mathrm{sign}(h - u)$ is automatically satisfied for any $\mathbf{z} \in X_2(\Omega)$ with $\|\mathbf{z}\|_\infty \leqslant 1$; let us find such vector field $\mathbf{z}$ which satisfies the other two conditions, i.e.,

$$u'(t) = \mathrm{div}(\mathbf{z}(t)) \tag{5.7}$$

and

$$\int_\Omega (\mathbf{z}(t), Du(t)) = \int_\Omega |Du(t)|.$$

From the second condition, we infer that $\mathbf{z}$ should be equal to $-\nu^{B(0,r)}$ on $\partial B(0,r)$. By the first condition, $\mathrm{div}(\mathbf{z}(t))$ should be the same for all points in $B(0,r)$; thus, a good candidate for $\mathbf{z}$ is

$$\mathbf{z}(x,t) = -\frac{x}{r}.$$

For such $\mathbf{z}$, integrating equation (5.7) over $B(0,r)$ and applying the Gauss-Green formula (Theorem 3.9) gives

$$\begin{aligned}\alpha'(t)\,\mathcal{L}^N(B(0,r)) &= \int_{B(0,r)} \mathrm{div}(\mathbf{z}(t))\,dx \\ &= \int_{\partial B(0,r)} \mathbf{z}(t)\cdot\nu^{B(0,r)}\,d\mathcal{H}^{N-1} = -\mathcal{H}^{N-1}(\partial B(0,r)).\end{aligned}$$

Therefore,

$$\alpha'(t) = -\frac{\mathcal{H}^{N-1}(\partial B(0,r))}{\mathcal{L}^N(B(0,r))} = -\frac{N}{r},$$

and consequently

$$\alpha(t) = k - \frac{N}{r}t.$$

Observe that this formula makes sense until the extinction time $T_{\mathrm{ex}} = \frac{kr}{N}$. We also need to construct the vector field $\mathbf{z}$ outside of $B(0,r)$; to this end, observe that since $u' \equiv 0$ on $\Omega\backslash B(0,r)$, considering radial vector fields $\mathbf{z}$, i.e., $\mathbf{z} = \rho(|x|)\frac{x}{|x|}$, we have

$$0 = \mathrm{div}(\mathbf{z}(t)) = \nabla\rho(|x|)\cdot\frac{x}{|x|} + \rho(|x|)\mathrm{div}\left(\frac{x}{|x|}\right) = \rho'(|x|) + \rho(|x|)\frac{N-1}{|x|}.$$

Solving this equation on (r,∞) with the initial condition $\rho(r) = -1$ gives the unique solution

$$\rho(s) = -r^{N-1}s^{1-N},$$

so

$$\mathbf{z}(x,t) = -r^{N-1}\frac{x}{|x|^N}.$$

Again, we use this formula on $(0,T_{\mathrm{ex}})$. Since $u(x,t)\equiv 0$ for $t \geqslant T_{\mathrm{ex}}$, by taking $\mathbf{z}\equiv 0$ we see that the conditions are satisfied. Therefore, by the above computations the vector field

$$\mathbf{z}(x,t) = \begin{cases} -\dfrac{x}{r} & \text{if } x\in B(0,r) \text{ and } t < T_{\mathrm{ex}}; \\ -r^{N-1}\dfrac{x}{|x|^N} & \text{if } x\notin B(0,r) \text{ and } t < T_{\mathrm{ex}}; \\ 0 & \text{if } t \geqslant T_{\mathrm{ex}} \end{cases}$$

safisfies the desired conditions for $u(x,t)$ given by equation (5.6).

Observe that in the above example both pieces $B(0,r)$ and $B(0,R)\backslash B(0,r)$ move linearly in time, after some time $T_1 > 0$ the values of u on both sets become equal, and then the two sets move together until the extinction time, when the solution becomes identically zero. This is qualitatively different than e.g. for the heat flow, where the solution for positive initial data stays positive for all times.

Exercise 5.15. Using a similar argument, show that a solution to the homogeneous Dirichlet problem for the total variation flow on Ω with initial data

$$u_0 = k\chi_{B(0,R)\backslash B(0,r)}$$

with $r < R$, $k > 0$, and $B(0,R) \Subset \Omega$ is equal to

$$u(x,t) = \left(k - \frac{\mathcal{H}^{N-1}(\partial B(0,R) \cup \partial B(0,r))}{\mathcal{L}^N(B(0,R)\backslash B(0,r))}t\right)\chi_{B(0,R)\backslash B(0,r)}(x) \\ + \frac{\mathcal{H}^{N-1}(\partial B(0,r))}{\mathcal{L}^N(B(0,r))}t\chi_{B(0,r)}(x)$$

for

$$t < T_1 := \left(\frac{\mathcal{H}^{N-1}(\partial B(0,R) \cup \partial B(0,r))}{\mathcal{L}^N(B(0,R)\backslash B(0,r))} + \frac{\mathcal{H}^{N-1}(\partial B(0,r))}{\mathcal{L}^N(B(0,r))}\right)^{-1} k,$$

and for $t \geqslant T_1$ the solution evolves as in the previous Example. Hint: look for solutions of the form $u(x,t) = \alpha(t)\chi_{B(0,r)} + \beta(t)\chi_{B(0,R)\backslash B(0,r)}$.

Exercise 5.16. Using a similar argument, show that a solution to the Neumann problem for the total variation flow on $B(0,R)$ with initial data

$$u_0 = k\chi_{B(0,r)}$$

with $r < R$ and $k > 0$ is equal to

$$u(x,t) = \left(k - \frac{N}{r}t\right)\chi_{B(0,r)} + \frac{Nr^{N-1}}{R^N - r^N}t\chi_{B(0,R)\backslash B(0,r)}$$

for $t < T_{\text{ex}} := (\frac{N}{r} + N\frac{r^{N-1}}{R^N - r^N})^{-1}k$. For $t > T_{\text{ex}}$, the solution is constant and equals

$$u(x,t) \equiv k - \frac{N}{r}T_{\text{ex}}.$$

Hint: again look for solutions of the form $u(x,t) = \alpha(t)\chi_{B(0,r)} + \beta(t)\chi_{B(0,R)\backslash B(0,r)}$.

Observe that in the above example both pieces $B(0,r)$ and $B(0,R)\backslash B(0,r)$ move linearly in time, at the extinction time the values of u on the two sets become equal (and the solution is equal to the mean value of the initial data), and then the evolution stops.

We now give an explicit bound for the extinction time of the solutions. As a first step, let us see that for the Neumann problem the mean value of the solution is preserved.

Lemma 5.17. *Let $u : \Omega \times (0,T)$ be a weak solution to problem* (5.1)*. Then, for a.e. $t \in (0,T)$ we have*

$$\int_\Omega u(t)\,dx = \int_\Omega u_0\,dx.$$

Proof. By definition of the weak solution, for a.e. $t \in (0,T)$ there exists $\mathbf{z} \in X_2(\Omega)$ with the properties given in Definition 5.4. Therefore,

$$\int_\Omega u_t\,dx = \int_\Omega \operatorname{div}(\mathbf{z}(t))\,dx = \int_\Omega [\mathbf{z}(t), \nu^\Omega]\,d\mathcal{H}^{N-1} = 0,$$

and integrating this equation over time yields the claim. $\square$

We conclude this lecture with the following theorem describing the asymptotic behaviour of solutions.

THEOREM 5.18. *Suppose that* $u : \Omega \times (0,T)$ *is a weak solution to problem* (5.1). *Then, if we denote*

$$T_{\mathrm{ext}}(u_0) = \inf \Big\{ \tau > 0 : \; u(t) = (u_0)_\Omega \;\; \textit{for all } t > \tau \Big\},$$

we have that $T_{\mathrm{ext}}(u_0) < \infty$ *and there exists a constant* $C = C(\Omega)$ *such that* $T_{\mathrm{ext}}(u_0) \leqslant C \cdot \|u_0 - (u_0)_\Omega\|_{L^2(\Omega)}$.

Similarly, if $u : \Omega \times (0,T)$ *is a weak solution to problem* (5.5) *for* $h = 0$, *and we denote*

$$T_{\mathrm{ext}}(u_0) = \inf \Big\{ \tau > 0 : \; u(t) = 0 \;\; \textit{for all } t > \tau \Big\},$$

we have that $T_{\mathrm{ext}}(u_0) < \infty$ *and there exists a constant* $C = C(\Omega)$ *such that* $T_{\mathrm{ext}}(u_0) \leqslant C \cdot \|u_0\|_{L^2(\Omega)}$.

PROOF. We show the result for the Neumann problem (the other proof is very similar). Let u be a weak solution to problem (5.1) and consider the function $\frac{1}{2}\int_\Omega |u-(u_0)_\Omega|^2 \, dx$, which is absolutely continuous in time on $(0,T)$ by the regularity of u. Then, we compute its time derivative, i.e.,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t)-(u_0)_\Omega|^2 \, dx = \int_\Omega (u(t)-(u_0)_\Omega)\, u_t \, dx = \int_\Omega (u(t)-(u_0)_\Omega) \operatorname{div}(\mathbf{z}(t)) \, dx.$$

But, the term with the constant is equal to zero by the Gauss-Green formula (Theorem 3.9) and the Neumann boundary condition. Similarly,

$$\int_\Omega u(t) \operatorname{div}(\mathbf{z}(t)) \, dx = -\int_\Omega (\mathbf{z}, Du(t)) = -\int_\Omega |Du(t)|,$$

therefore, by the Poincaré inequality,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t) - (u_0)_\Omega|^2 \, dx \leqslant -\int_\Omega |Du(t)| \leqslant -C\bigg(\int_\Omega |u(t) - (u_0)_\Omega|^2 \, dx \bigg)^{1/2}.$$

Thus, the function $\|u(t) - (u_0)_\Omega\|_{L^2(\Omega)}$ satisfies a differential inequality

$$\frac{d}{dt}\|u(t) - (u_0)_\Omega\|^2_{L^2(\Omega)} \leqslant -C\|u(t) - (u_0)_\Omega\|_{L^2(\Omega)}.$$

Therefore, when the right-hand side is nonzero, we get

$$\frac{d}{dt}\|u(t) - (u_0)_\Omega\|_{L^2(\Omega)} \leqslant -C,$$

and consequently,

$$\|u(t) - (u_0)_\Omega\|_{L^2(\Omega)} \leqslant \|u_0 - (u_0)_\Omega\|_{L^2(\Omega)} - Ct,$$

so the extinction time is at most equal to $C(\Omega)\|u_0 - (u_0)_\Omega\|_{L^2(\Omega)}$. □

EXERCISE 5.19. Modify the proof above to cover the case of the homogeneous Dirichlet problem.

On a final note, let us mention that this behaviour is a typical feature of convex p-homogeneous functionals in Hilbert spaces with $p < 2$; it was shown in a recent paper [**11**] with a method based on the above argument.

Further reading

A classical reference on the total variation flow is the monograph [**3**], where existence of solutions is obtained by an approximation of p-Laplace type.

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