THE WILLMORE FUNCTIONAL AND INSTABILITIES IN THE CAHN-HILLIARD EQUATION

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Abstract. We present stability/instability and asymptotic results for transition solutions of the Cahn-Hilliard equation. In particular the behavior of solutions of the Cahn-Hilliard equation in a neighborhood of an equilibrium is studied by exploring the Willmore functional. We show the convergence of the Willmore functional to zero. Furthermore linear and nonlinear instabilities of the Cahn-Hilliard equation locally in time turn out to correspond to increasing parts of the time evolution of the Willmore functional.

1. Introduction

We consider the Neumann boundary problem for the Cahn-Hilliard equation

\[
\begin{aligned}
&\quad \frac{\partial u}{\partial t} = \Delta(-\epsilon^2 \Delta u + F'(u)) \quad x \in \Omega, \\
&\quad \frac{\partial u}{\partial n} = \frac{\partial}{\partial n}(-\epsilon^2 \Delta u + F'(u)) = 0, \quad x \in \partial \Omega
\end{aligned}
\]

with \( x \in \Omega \subseteq \mathbb{R}^n \) and \( \epsilon > 0 \). \( F \) is a double well potential with \( F'(u) = \frac{1}{2}(u^3 - u) \). The Cahn-Hilliard equation models phase separation and subsequent phase coarsening of binary alloys, see [1] for a detailed description. Its solution has a time-conserved mean value

\[ \int_\Omega u(x,t)dx = \int_\Omega u(x,t=0)dx, \text{ for all } t > 0, \]

with decreasing energy

\[ E[u](t) = \int_\Omega \left( \frac{\epsilon^2}{2} |\nabla u(x,t)|^2 + F(u(x,t)) \right)dx. \]

We are interested in the properties of local stability, instability and asymptotics in time of smooth solutions of the Cahn-Hilliard equation in a neighborhood of a stationary solution \( u_0 \). In studying stability properties the motion of solutions near transition solutions, which continuously connect the two stable equilibria -1 and 1, plays an important role. In one dimension the so called kink solution \( u_0 = \tanh \frac{x}{\epsilon} \) is such a stationary solution of the Cahn-Hilliard equation. The analogue in two dimensions are the radially-symmetric bubble solutions.

In [2] Alikakos and Fusco proved spectral estimates of the linearized fourth order Cahn-Hilliard operator in two dimensions near bubble solutions. They proved that there is a two-dimensional manifold with exponentially small eigenvalues where the solutions asymptotically develop droplets on the boundary with a speed which is

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exponentially small. There are also some studies for the one-dimensional case in [3, 4, 5, 6].

In the context of the asymptotic behavior of the solutions in time [7] showed convergence to equilibria for solutions of (1) in two and three dimensions. In the one dimensional case Grinfeld and Novick-Cohen [8, 9] studied the asymptotic limit of stationary solutions carefully. In [10] the authors showed asymptotic stability for the kink solution for fixed $\epsilon = 1$, i.e. any small perturbation of $u_0$ in some suitable norm will decay to zero in time. The difficulty of stating the equilibria in multi-dimensions is that the limit set of the solutions can be large. However, some papers as [11, 12] still provide certain special types of equilibrium solutions.

In this paper the Willmore functional is proposed to study stability properties of solutions of the Cahn-Hilliard equation. The Willmore functional has its origin in geometry, see [13] for a discussion. A generalized version of this functional appears as the squared first variation of the free energy of the Cahn-Hilliard equation. The Willmore functional of the Cahn-Hilliard equation (1) is given by

$$W[u](t) = \frac{1}{4\epsilon} \int_{\Omega} \left( \epsilon \Delta u - \frac{1}{\epsilon} F'(u) \right)^2 dx.$$  \hspace{1cm} (2)

In section 2 we study the asymptotic limit of solutions of the Cahn-Hilliard equation by showing asymptotic decay of the Willmore functional in time. Because the Willmore functional generally does not decrease for all time but only for large $t$, there are difficulties in proving convergence. To overcome this we construct a non-negative functional balancing the Willmore functional with the energy functional so that the strongly decreasing property of the energy takes the main role controlling instabilities appearing in the Willmore functional. In Theorem 2.1 we finally prove that for the Cauchy problem and the Neumann boundary problem on a bounded domain the Willmore functional converges to zero as time tends to infinity in any dimension. We also show that the decay rate of $u_t$ in $H^{-2}(\Omega)$ is equal to the order of

$$\left[ \int_{t_0}^{t} \int_{\Omega} |\nabla(\epsilon^2 \Delta u - F'(u))|^2 dx ds \right]^2,$$

compare Remark 2.4. The main challenge of the proof of convergence in this paper is that we avoid to use Lojasiewicz inequality as before, say in [14].

In section 3 the linearized Cahn-Hilliard equation is considered. The important role of the Willmore functional in stability/instability analysis for the Cahn-Hilliard equation is motivated. The Willmore functional is asymptotically expanded in a stationary solution of the equation perturbed with an eigenvector of the linearized Cahn-Hilliard operator. It can be concluded that the Willmore functional decreases in time for eigenvectors corresponding to a negative eigenvalue and increases in the case of a positive eigenvalue. Roughly said this means that linear instabilities which correspond to positive eigenvalues of the Cahn-Hilliard equation can be detected by considering the evolution of the Willmore functional in time. Additionally spectral estimates for the linearized Cahn-Hilliard operator, mainly based on results in [3], [2], are presented. The behavior of the Willmore functional in time for the linearized Cahn-Hilliard operator seems to be similar as for the nonlinear Cahn-Hilliard operator where of course stability needs further rigorous analysis. For nonlinear instabilities this conjecture is discussed in various numerical examples in section 4.
2. A-priori Estimate of the Willmore Functional

It is natural to study stationary solutions of the Cahn-Hilliard equation by analyzing the energy functional

\[ E[u] = \int_\Omega \left( \frac{\epsilon^2}{2} |\nabla u|^2 + F(u) \right) dx. \]

The energy functional decreases in time since

\[ \frac{d}{dt} E[u] + \int_\Omega |\nabla(-\epsilon^2 \Delta u + F'(u))|^2 dx = 0. \]

There exists a series of works dealing with a sequence of functions minimizing the energy functional. The problem occurs when viscosity \( \epsilon \) goes to zero. In this case it is difficult to see the vanishing viscosity limit of the approximating sequence of functions minimizing the energy functional. The reason for this is that the term of reverse diffusion in the equation gradually affects the solution (the derivative in the energy seems to go to infinity and the limit function seems to be unregular). There seem to occur high frequent oscillations in the solution, not clearly visible in numerical computations. To look at this kind of phenomenon in an analytic way we introduce the Willmore functional as defined in (2) in the previous section. The Willmore functional is considered to describe the geometric boundary of two different stable states and the movement of curves under anisotropic flows.

In the following the long time asymptotic behavior of solutions of the Cahn-Hilliard equation is studied by exploring the Willmore functional. We consider the \( n \)-dimensional case of the Cahn-Hilliard equation. All following arguments hold true both for the Neumann boundary problem and the Cauchy problem with certain conditions on the spatial decay of the solutions. We begin by introducing some useful properties of the functionals in our setting of the stationary profile.

**Lemma 2.1.** For any test function \( \phi(x,t) \in C^\infty_0(\Omega \times (0,\infty)) \) we have

\[
\frac{d}{dt} \int_\Omega \phi \left( \frac{\epsilon^2}{2} |\nabla u|^2 + F(u) \right) dx + \int_\Omega \phi |\nabla(\epsilon^2 \Delta u - F'(u))|^2 dx
\]

\[
= \frac{d}{dt} \int_\Omega \phi \left( \frac{\epsilon^2}{2} |\nabla u|^2 + F(u) \right) dx + \frac{1}{2} \int_\Omega \Delta \phi F'(u) - \epsilon^2 \Delta u |^2 dx
\]

\[
- \epsilon^2 \int_\Omega \nabla(\nabla \phi \cdot \nabla u) \cdot \nabla(\epsilon^2 \Delta u - F'(u)) dx
\]

**Lemma 2.2.**

\[
\frac{d}{dt} \int_\Omega (\epsilon \Delta u - \frac{1}{\epsilon} F'(u))^2 dx + 2\epsilon^2 \int_\Omega |\Delta(\epsilon \Delta u - \frac{1}{\epsilon} F'(u))|^2 dx
\]

\[
= 2 \int_\Omega F''(u)(\epsilon \Delta u - \frac{1}{\epsilon} F'(u))\Delta(\epsilon \Delta u - \frac{1}{\epsilon} F'(u)) dx
\]

\[
\frac{d}{dt} \int_\Omega (\Delta(\epsilon \Delta u - \frac{1}{\epsilon} F'(u)))^2 dx + 2\epsilon^2 \int_\Omega (\Delta^2(\epsilon \Delta u - \frac{1}{\epsilon} F'(u)))^2 dx
\]

\[
= 2 \int_\Omega F''(u)\Delta(\epsilon \Delta u - \frac{1}{\epsilon} F'(u))\Delta^2(\epsilon \Delta u - \frac{1}{\epsilon} F'(u)) dx
\]

The proofs of Lemma 2.1 and 2.2 are straightforward.
Proposition 2.1. Let \( f, g \in C^1([0, \infty)) \) be nonnegative functions with \( g'(t) \leq 0 \) everywhere, \( \text{supp} \ f \subset \text{supp} \ g, \text{supp}_{t \in \text{supp} \ g} \frac{f}{g} \) and \( \text{supp}_{t \in \text{supp} \ g} \frac{f}{g} \) bounded, where \((f')^+(t) = \max\{f'(t), 0\}\). Let \( u \) be the solution of the Cahn-Hilliard equation with initial data \( u^0(x) \), either posed as a Cauchy problem in \( \Omega = \mathbb{R}^n \) or in a bounded domain \( \Omega \) with Neumann boundary conditions. For \( \Omega = \mathbb{R}^n \) further suppose
\[
e^2 \Delta u^0 - F'(u^0) = \nabla (e^2 \Delta u^0 - F'(u^0)) \text{ are spatially exponentially decaying as } |x| \to \infty \text{ and}
\]
\[
(2) \quad \int_{\Omega} F'(u) dx = 0 \text{ for all } t > 0.
\]
Then for a sufficiently large constant \( C \) we have
\[
\frac{d}{dt} \left[ \int_{\Omega} f(t)(e \Delta u - \frac{1}{\epsilon} F'(u))^2 dx + \frac{C}{\epsilon^2} \int_{\Omega} g(t)(\frac{\epsilon^2}{2} |\nabla u|^2 + F(u)) dx \right] \leq 0.
\]
In particular, \( e^{W[u]}(t) + CE[u](t) \leq e^{W[u^0]} + CE[u^0] \).

Remark 2.1. Note that the assumption (2) in Proposition 2.1 is no restriction on \( F \) and therefore reasonable. Since \( \int_{\Omega} F'(u) dx \) is a constant, we can rewrite the equation as
\[
u_t = \Delta(-e^2 \Delta u + F'(u) - \frac{1}{|\Omega|} \int_{\Omega} F'(u) dx).
\]
In the case of Neumann boundary conditions it follows
\[
\frac{\partial u}{\partial n} = \frac{\partial(-e^2 u + F'(u) - \frac{1}{|\Omega|} \int_{\Omega} F'(u) dx)}{\partial n} = 0.
\]
Thus we can set the integral to be zero.

Proof: Consider the functional
\[
U[u](t) = \int_{\Omega} f(t)(e \Delta u - \frac{1}{\epsilon} F'(u))^2 dx + \frac{C}{\epsilon^2} \int_{\Omega} g(t)(\frac{\epsilon^2}{2} |\nabla u|^2 + F(u)) dx.
\]
By using Lemma 2.1 and 2.2 we derive
\[
\frac{d}{dt} U[u](t) = f'(t) \int (e \Delta u - \frac{1}{\epsilon} F'(u))^2 dx + \frac{C}{\epsilon^2} g'(t) \int (\frac{\epsilon^2}{2} |\nabla u|^2 + F(u)) dx
\]
\[+ f(t) \left[ 2 \int F''(u)(e \Delta u - \frac{1}{\epsilon} F'(u)) \Delta(e \Delta u - \frac{1}{\epsilon} F'(u)) dx
\]
\[+ 2e^2 \int |\Delta(e \Delta u - \frac{1}{\epsilon} F'(u))|^2 dx \right] - \frac{C}{\epsilon^2} \int (\frac{\epsilon^2}{2} |\nabla(u - e^2 u + F'(u))|^2) dx \]
\[
(3) \quad \leq \frac{C}{\epsilon^2} g(t) \int |\nabla(-e^2 \Delta u + F'(u))|^2 dx.
\]
We need the following Lemma to deal with the last term in (3).

Lemma 2.3 (modified Poincare inequality on \( \mathbb{R}^d \)). Let \( \alpha > 0, C_1 > 0, C_2 > 0 \) be fixed constants. Then there exists a positive constant \( C_0(a, C_1, C_2) \) such that for any functions \( f \) in
\[
V_{a, C_1, C_2} = \{ f \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} f \, dx = 0, |f(x)| \leq C_1 e^{-a|x|} |f|_{L^2},
\]
\[|f'(x)| \leq C_2 e^{-a|x|} |f|_{L^2} \text{ for } x \in \mathbb{R}^d \},
\]
we have
\[
|f|_{L^2(\mathbb{R}^d)} \leq C_0 |\nabla f|_{L^2(\mathbb{R}^d)}.
\]
By using Lemma 2.3, the right side of (3) can be bounded by
\[
\left( f'(t) \right)^+ - \frac{C g(t)}{C_0} \int (\epsilon \Delta u - \frac{1}{\epsilon} F'(u))^2 dx + 2 f(t) \sup_{|u| \leq 2} |F''(u)| \int |\epsilon \Delta u - \frac{1}{\epsilon} F'(u)| \cdot |\Delta (\epsilon \Delta u - \frac{1}{\epsilon} F'(u))| dx
\]
\[\quad - 2 f(t) \epsilon^2 \int |\Delta (\epsilon \Delta u - \frac{1}{\epsilon} F'(u))|^2 dx + \frac{C}{\epsilon^2} g'(t) E[u](t).
\]
This expression is non-positive when \( C > \frac{C_0 f(t)}{2 \epsilon^2 g(t)} \), and some fixed \( b > 0 \), as \( t \) tends to zero, which contradicts the Poincare inequality.

Remark 2.2. Note that the Poincare inequality on \( \mathbb{R}^d \) is not valid in general. Consider for example the function \( h_a(x) = a^{-\frac{d+1}{2}} xe^{-a|x|^2} \) with \( a > 0 \) and some fixed \( b > 0 \). Then \( \int h_a^2 dx = O(1) \) and \( \int h_a^2 dx = O(a^{\frac{d}{2}}) \) as \( a \to 0^+ \). Therefore
\[
\frac{|h'_a|_{L^2}}{|h_a|_{L^2}} = O(a^{\frac{1}{2}}) \to 0
\]
as \( a \) tends to zero, which contradicts the Poincare inequality.

Finally we come to the main result of this section, namely

Theorem 2.1. Under the same assumptions as in Proposition 2.1 it follows
\[
\lim_{t \to \infty} W[u](t) = 0.
\]

Proof: Step 1: Let \( f(t) = \begin{cases} 
\frac{1+4t}{2}, & 0 \leq t \leq 1, \\
2 - t, & 1 \leq t \leq 2, \\
0, & t \geq 2.
\end{cases} \)
\[
g(t) = \begin{cases} 
1, & t \leq 2, \\
0, & t \geq 3.
\end{cases}
\]
Let \( C \) be a fix constant chosen as in Proposition 2.1. The functional
\[
U[u](t) \equiv 4 \epsilon f(t) W[u](t) + \frac{C}{\epsilon^2} g(t) E[u](t)
\]
is decreasing in time and \( U[u](t = 1) \leq U[u](t = 0) \). That is,
\[
W[u](t = 1) + \frac{C}{4 \epsilon^3} E[u](t = 1) \leq \frac{1}{2} W[u](t = 0) + \frac{C}{4 \epsilon^3} E[u](t = 0).
\]

Step 2: For each \( n \in \mathbb{N} \), setting \( f(t - n + 1) \) and \( g(t - n + 1) \) as in Proposition 2.1, we can again find inequalities for \( n - 1 \leq t \leq n \) and obtain
\[
W[u](t = n) + \frac{C}{4 \epsilon^3} E[u](t = n) \leq \frac{1}{2} W[u](t = n - 1) + \frac{C}{4 \epsilon^3} E[u](t = n - 1).
\]
Set \( \alpha_n = W[u](t = n) \). Then (4) can be rewritten as
\[
\alpha_n \leq \frac{1}{2} \alpha_{n-1} + \frac{C}{4 \epsilon^3} \int_{t=n-1}^{t=n} \int |\nabla (-\epsilon^2 \Delta u + F'(u))|^2 dx dt
\]
where we used the decay property (1) of the energy functional.
Step 3: We want to show that \( \alpha_n \) tends to zero as \( n \) tends to infinity. By an iterative argument we get

\[
\alpha_n \leq \left( \frac{1}{2} \right)^n \alpha_0 + \frac{C}{4e^3} \left( \frac{1}{2} \right)^{n-1} \int_0^1 \int_\Omega |\nabla(-\epsilon^2 \Delta u + F'(u))|^2 \, dx \, dt + \\
\left( \frac{1}{2} \right)^{n-2} \int_1^n \int_\Omega |\nabla(-\epsilon^2 \Delta u + F'(u))|^2 \, dx \, dt + \\
\cdots + \frac{1}{2} \int_{n-2}^{n-1} \int_\Omega |\nabla(-\epsilon^2 \Delta u + F'(u))|^2 \, dx \, dt + \\
\int_{n-1}^n \int_\Omega |\nabla(-\epsilon^2 \Delta u + F'(u))|^2 \, dx \, dt.
\]

Using a standard fact from analysis formulated in the following Lemma 2.4 it follows that \( \alpha_n \) converges to 0 for \( n \to \infty \).

**Lemma 2.4.** Let \((a_n), (b_n)\) are two nonnegative sequences such that their sums \( \sum_n a_n \) and \( \sum_n b_n \) are convergent. Then

\[
\lim_{n \to \infty} \sum_{i=0}^n a_i b_{n-i} = 0.
\]

We conclude our proof in step 4.

Step 4: It remains to prove that for any sequence \((t_n)\) tending to infinity, \( W[u](t_n) \) converges to zero. Therefore it suffices to prove that for any integer \( q \), \( W[u](t = \frac{q}{n}) \) converges to zero. Using Lemma 2.4 the proof is similar to Step 3 and we omit the details here.

**Remark 2.3.** Using Lemma 2.2 and Theorem 2.1 the asymptotic convergence of \( \| \Delta(\epsilon \Delta u - \frac{1}{\epsilon} F'(u)) \|_{L^2(\Omega)}^2 \) to zero can be shown and is left to the reader.

**Remark 2.4.** Note that the proof of Theorem 2.1 also gives us an expression for the decay of \( u_t \) in \( H^{-2} \), namely

\[
\| u_t \|_{H^{-2}} = \| \Delta(-\epsilon^2 \Delta u + F'(u)) \|_{H^{-2}} = C \| -\epsilon^2 \Delta u + F'(u) \|_{L^2} \to 0 \text{ as } t \to \infty,
\]

with a rate of order

\[
\sqrt{\int_{t_n}^{t} \int_\Omega |\nabla(-\epsilon^2 \Delta u - F'(u))|^2 \, dx \, ds}.
\]

According to Remark 2.3 the convergence of \( u_t \) to zero is also valid in \( L^2 \).

Finally we have shown that the Willmore functional asymptotically decreases to zero for the Cahn-Hilliard problems proposed in Proposition 2.1. This asymptotic convergence proves that \( u_t \to 0 \) for \( t \to \infty \) in \( H^{-2}(\Omega) \) and \( L^2(\Omega) \) respectively for every \( \epsilon > 0 \) and arbitrary dimension \( n \).
3. Linear Stability / Instability

In that section we consider the small time behavior of solutions of the Cahn-Hilliard equation. Additionally to the numerical examples in section 4 we here explore the evolution of small time instabilities of solutions analytically.

In section 3.1 we propose the analytic explanation of instabilities of the solutions by comparing the eigenvalues of the linearized operator with the evolution of the Willmore functional. In section 3.2 we discuss the eigenvalues of the linearized operator near stationary solutions.

3.1. Linear Stability Analysis using the Willmore Functional. In the following we provide a linear stability analysis around a stationary state \( u_0 \) satisfying

\[-\epsilon^2 \Delta u_0 + F'(u_0) = 0.\]

More precisely, we look for a solution of the form

\[ u(x, t) = u_0(x) + \delta v(x, t) + \mathcal{O}(\delta^2) \]

for sufficiently small \( 0 < \delta \ll 1 \) and some perturbation \( v(x, t) \). Due to mass conservation in the equation we may assume that \( v \) has mean zero for all time

\[ \int_{\Omega} v(x, t) \, dx = 0 \quad \forall t > 0. \]

Since \( u_0(x) \) solves also the non-stationary Cahn-Hilliard equation to zeroth order, we obtain the first-order expansion with respect to \( \delta \) via the linearized equation

(1)

\[ v_t = \Delta (-\epsilon^2 \Delta v + F''(u_0)v) := \Delta L_0 v. \]

We now compute the expansion of the Willmore functional. It can be expanded as

(2)

\[ W[u] = W[u_0] + \delta W'[u_0]v + \frac{\delta^2}{2} W''[u_0](v, v) + \mathcal{O}(\delta^3), \]

where the first and second order derivatives are taken as variations

\[ W'[u_0]v = \int \left[ (-\epsilon^2 \Delta u_0 + F'(u_0))(-\epsilon^2 \Delta v + F''(u_0)v) \right] dx \]

and

\[ W''[u_0](v, w) = \int (L_0 v)(L_0 w) dx + \int \left[ (-\epsilon^2 \Delta u_0 + F'(u_0))F''(u_0)v \right] w dx. \]

Since \( u_0 \) is a stationary solution, we have

\[ W'[u_0]v = 0 \quad \text{and} \quad W''[u_0](v, v) = \int (L_0 v)^2 dx. \]

Now let \( v_0 \) be an eigenfunction of the linearized fourth-order Cahn-Hilliard operator, i.e. there is \( \lambda \neq 0 \) such that

\[ \Delta (L_0 v_0) = \lambda v_0, \]

with Neumann boundary condition \( \frac{\partial v_0}{\partial n} |_{\partial \Omega} = 0 \) and mean zero of \( v_0 \). Note that \( \lambda \) is real since \( v_0 \) solves a symmetric eigenvalue problem in the scalar product of \( H^{-1} \), defined as the dual of \( H^1(\Omega) \cap \{ u : \int_{\Omega} u \, dx = 0 \} \).

The standard linear stability analysis yields that the perturbation of \( u_0 \) by \( \delta v_0 + \mathcal{O}(\delta^2) \) is linearly stable for \( \lambda < 0 \) and unstable for \( \lambda > 0 \). These two cases can
be translated directly into the local-in-time behavior of the Willmore functional, whose time derivative at time $t = 0$ is given by

$$\frac{d}{dt} W[u(t)]|_{t=0} = \delta W'[u_0][v|^t|_{t=0} + \delta^2 W''[u_0](v, v_t)|_{t=0} + O(\delta^3)$$

$$= \delta^2 \int (L_0 v_0)(L_0 v_t) dx + O(\delta^3)$$

$$= \delta^2 \int (L_0 v_0)(L_0 \Delta L_0 v_0) dx + O(\delta^3)$$

$$= \delta^2 \int (L_0 v_0)(L_0 (\lambda v_0)) dx + O(\delta^3)$$

$$= \lambda \delta^2 \int (L_0 v_0)^2 dx + O(\delta^3).$$

This means that to leading order, the time derivative of $W[u]$ has the same sign as $\lambda$, i.e., the Willmore functional is locally increasing in time in the unstable case, and locally decreasing in the stable case.

### 3.2. Spectral Estimates for the Linearized Operator.

In the subsequent analysis of transition solutions of the Cahn-Hilliard equation we explain linear stability/instability by eigenvalue estimates.

Alikakos, Bates and Fusco showed in [3] that there is exactly one unstable eigenvalue in the one-dimensional linearized eigenvalue problem. In 1994 Alikakos and Fusco [2] proved that the dimension of eigenfunctions of superslow eigenvalues of the linearized Cahn-Hilliard equation on $D \subseteq \mathbb{R}^n$ is at most $n$ for $n > 1$. We present these results here and refer to [3, 2] for details.

First we state properties of eigenvalues in the one-dimensional region $D = [0, 1]$. Since the periodically extended solution of the eigenvalue problem is integrable, we consider the problem linearized at an invariant manifold (formed by the movement of a self-similar solution with a parameter $\xi$) $u_\xi(x) = u_\xi(x - \xi) \in \mathcal{M}$:

$$\begin{cases} -\epsilon^2 H'''' + (F''(u_\xi) H)' = \lambda(\epsilon) H, & 0 < x < 1, \\ H = H'' = 0 & x = 0, 1. \end{cases}$$

The first eigenvalue is simple and exponentially small for small enough $\epsilon > 0$

$$0 < \lambda_1(\epsilon) = O \left( \frac{(u_{\xi x x}(0))^2}{\epsilon^i} \right) = O \left( \frac{e^{-2\nu \delta_{\xi}}}{\epsilon^i} \right),$$

where $\delta_{\xi}$ is a small positive constant given by the argument in the proof of (4) in [3] and $\nu$ is a generic constant, see [15]. The remaining spectrum is bounded from above by

$$\lambda_i(\epsilon) \leq -C < 0, i = 2, 3, \cdots$$

with $C$ independent of $\epsilon, \xi$. Both results are contained in [3].

Now we consider the eigenvalue problem of the linearized fourth order Cahn-Hilliard operator in $D \subseteq \mathbb{R}^2$. The results in higher dimensions are analogue to this case. Let $U(\eta)$ be the unique increasing bounded solution of $U'' = F(U) = 0$ on $\mathbb{R}$, and $V(\eta)$ a bounded function that satisfies the orthogonality condition

$$\int_{-\infty}^{\infty} f''(U(\eta)) \hat{U}^2(\eta) V(\eta) d\eta = 0,$$
where \( f(u) = F'(u) \). We consider a one-parameter family of functions \( u_\xi^\epsilon(x) \) represented by
\[
u_\xi^\epsilon(x) = \left\{ \begin{array}{ll}
U(x - \rho \epsilon) + \epsilon V(x - \rho \epsilon) + O(\epsilon^2), & |y - \rho| \leq \lambda, \\
q_\epsilon(x), & |y - \rho| > \lambda
\end{array} \right.
\]
where \( y = |x - \xi|, \rho > 0 \) and \( q_\epsilon(x) \) is an arbitrary function with \( f'(q_\epsilon(x)) \geq c > 0 \).

The function \( u_\xi^\epsilon(x) \) represents a bubble with center \( \xi \in D \) and radius \( \rho \).

The eigenvalue problem of the Cahn-Hilliard operator linearized in \( u_\xi^\epsilon(x) \) is given by
\[
-(\Delta - \epsilon^2 \Delta \phi + F''(u_\xi^\epsilon) \phi) = \lambda \phi, \quad x \in D \subseteq \mathbb{R}^2
\]
with Neumann boundary conditions
\[
\frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n}(-\epsilon^2 \Delta \phi + F''(u_\xi^\epsilon) \phi) = 0, \quad x \in \partial D.
\]
In [2] Alikakos and Fusco stated the following result.

**Theorem 3.1.** Let \( \lambda_1^\epsilon \leq \lambda_2^\epsilon \leq \lambda_3^\epsilon \leq \cdots \) be the eigenvalues of (5). Let \( \rho > 0, \delta > 0 \) be fixed. Then there exists an \( \epsilon_0 > 0 \) and constants \( C, C' > 0 \), independent of \( \epsilon \), such that for \( \epsilon < \epsilon_0 \) and \( \xi \in D \) with \( d(\xi, \partial D) > \delta \), the following estimates hold true:
\[
-C \epsilon^{-\frac{\xi}{2}} \leq \lambda_1^\epsilon \leq \lambda_2^\epsilon \leq C \epsilon^{-\frac{\xi}{2}} \leq \lambda_3^\epsilon \geq C' \epsilon.
\]

The first two eigenvalues \( \lambda_1^\epsilon, \lambda_2^\epsilon \) are superslow and the others are positive with \( \epsilon \) of order 1. This fact produces difficulties in the interpretation of our numerical examples. Nevertheless the positivity of superslow eigenvalues can be derived easily by the following:

For any smooth functions \( \phi(x) \in C_0^\infty(D) \) on a simply connected domain \( D \subset \mathbb{R}^n, n \geq 2 \) with mean zero
\[
\int_D \phi(x) dx = 0,
\]
we can find a unique vector valued function \( \psi(x) \in C_0^\infty(D)^n \) such that
\[
\text{div } \psi(x) = \phi(x)
\]
for any \( x \in D \). In particular \( \psi \) can be written as \( -\nabla(-\Delta_N)^{-1} \phi \), where \( \Delta_N \) denotes the Laplace operator with Neumann boundary conditions applied to mean zero functions.

Then the eigenvalue problem (5) can be written in terms of \( \psi \)
\[
\begin{cases}
\nabla(-\epsilon^2 \text{div } \psi + f'(u_\xi^\epsilon) \text{div } \psi) = \lambda(\epsilon) \psi & \text{in } D, \\
\psi = \frac{\partial}{\partial n} \text{div } \psi = 0 & \text{on } \partial D.
\end{cases}
\]

For simplicity we consider the cube \( D = [0, 1]^n \). By defining the eigenfunctions
\[
\psi(x) = (0, \cdots, H(x_i), \cdots, 0)
\]
for \( i = 1, 2, \cdots, n \), where \( H(y), y \in \mathbb{R} \) be the function constructed by (3), we find \( n \) positive eigenvalues of the linearized eigenvalue problem (6).
4. Nonlinear Stability / Instability

Motivated by the linear stability analysis in section 3.1 we introduce the definition of stability discussed in this paper as follows.

**Definition 4.1.** We call a function \( u(x,t) \) moving towards an unstable state at a time \( t_0 \) in the sense of the Willmore functional if

\[
\frac{d}{dt} W[u](t_0) > 0.
\]

Conversely, we say the function \( u(x,t) \) moves towards a stable state at a time \( t_0 \) in the sense of the Willmore functional if

\[
\frac{d}{dt} W[u](t_0) \leq 0.
\]

We consider the Cahn-Hilliard equation (1). Because of the nonconvexity of the Cahn-Hilliard equation the analytic analysis of its nonlinear operator becomes hard. The consideration of numerical examples could give us an idea about the behavior of solutions of the equation and their connection to the Willmore functional even for the nonlinear case. In the following a semi-implicit finite element discretization for the Cahn-Hilliard equation is shortly described and numerical examples are discussed.

4.1. Numerical Discretization. In the past 20 years numerical approximations of the solution of the Cahn-Hilliard equation have been studied by many authors, see [16] and [17] for further references. We use a semi-implicit approach in time and finite elements for the space discretization in this paper.

To discretize a fourth-order equation it is convenient to write it as a system of two equations of second order. In our case of the Cahn-Hilliard equation this results in the following system

\[
\begin{align*}
\dot{u} &= \Delta v \\
v &= -\epsilon^2 \Delta u + F'(u),
\end{align*}
\]

with Neumann boundary conditions

\[
\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}(-\epsilon^2 \Delta u + F'(u)) = 0, \quad x \in \partial \Omega.
\]

**Remark 4.1.** The numerical examples we are considering in one dimension only consist of transition solutions that are perturbed in a neighborhood of the middle of the domain \( \Omega \). Thus transient instabilities do not reach the boundary and the boundary values remain constant. Therefore it is also reasonable to use Dirichlet boundary conditions instead of a Neumann boundary in the one dimensional case.

For the time discretization we have to pay attention to the fourth-order of the equation and the nonlinearity \( F'(u) \). Discussing different possibilities of discretization we come to the following points

- Explicit schemes for fourth order equations restrict time steps to be of order \( O(h^4) \), where \( h \) is the spatial grid size.
- Fully implicit schemes are unconditionally stable. The disadvantage is the high computational effort for solving nonlinear equations.
Semi-implicit schemes are a compromise between explicit and implicit discretization. Shortly said, semi-implicit means that the equation is split in a convex and a concave part and discretized implicitly and explicitly respectively, see ([18],[19]). Therefore the restriction on the step sizes is less severe and we don’t have to solve nonlinear equations.

For these reasons we used the following semi-implicit approach

\[
\frac{u(t,x) - u(t - \Delta t, x)}{\Delta t} = \Delta v(t,x)
\]

\[
v(t,x) = -\epsilon^2 \Delta u(t,x) + F'(u(t - \Delta t, x)) + F''(u(t - \Delta t, x)) \cdot (u(t,x) - u(t - \Delta t, x)).
\]

where \( F'(u(t)) \) is linearized in the solution of the previous time step \( u(t - \Delta t) \).

For the space discretization we use linear finite elements on an equidistant grid.

### 4.2. Numerical Examples

In the following examples we consider the solution of the Cahn-Hilliard equation in one and two dimensions for different initial states in a neighborhood of a transition solution. In the one dimensional case the so called kink solution is given by \( u_0(x) = \tanh (\frac{x}{2\epsilon}) \). As a first approach in the one dimensional analysis we take as initial value \( u(x,t=0) = u_0(x) + p(x) \). The function \( p \) denotes a zero-mean perturbation, namely

\[
p(x) = \begin{cases} 
    a \cdot \sin \left( f \pi \frac{x}{C} \right) & x \in (-C \cdot \epsilon, C \cdot \epsilon) \\
    0 & \text{otherwise},
\end{cases}
\]

with amplitude \( a > 0 \), frequency \( f > 0 \) and support \( (-C \cdot \epsilon, C \cdot \epsilon) \) with \( C > 0 \).

Varying the parameter \( \epsilon \) and the support, the amplitude and the frequency of \( p \) we want to see how the solutions evolve in time. The behavior of the solutions is further compared with the evolution of the corresponding energy functional and the Willmore functional.

We begin with a fixed \( \epsilon = 0.1 \). For the first two examples in Figure 1 and 2 a fixed step size in space and time discretization was used. As spatial step size we took \( \Delta x = 0.05 \cdot \epsilon \) and for the size of the timesteps \( \Delta t = 10^3 \cdot \epsilon \). The parameters amplitude and frequency of the perturbation are also fixed equal 1. The difference between the two examples is the support of the perturbation and dependent on this is the convergence process.

In the first example an unstable state occurs. This means that over a certain time interval the difference between the solution \( u(.,t) \) and the stationary solution \( u_0 \) grows. The second example is stable in the numerical computations. In the first case of supporting interval \( (-15 \epsilon, 15 \epsilon) \) (Figure 1) we begin with an initial state having two peaks on both sides of zero. When time proceeds the peaks grow in the beginning, resulting in an unstable transient state. After this unstable state the solution converges to the kink solution for which convexity has to change in the end. In the case of supporting interval \( (-3 \epsilon, 3 \epsilon) \) in Figure 2 the solution converges uniformly to the kink solution without transitional state.

Comparing the time evolution of our first two examples with the corresponding energy functionals and Willmore functionals, we can see differences in the graphs of those. Talking first of all about the energy functional we can see that in the unstable case the energy functional has a saddle point, in the stable case it is concave. Considering the Willmore functional, instabilities seem to correspond to
peaks in the graph of the functional over time. The Willmore functional increases in time over a finite time interval in contrast to the stable case where it decreases for all times $t$. According to Definition 4.1 we say the solution is unstable in terms of the Willmore functional.

Another interesting phenomenon can be seen by starting with modified versions of the perturbation $p$. For example we could shift the sine perturbation to the left or to the right of zero to start with an unsymmetric initial state. The example in
Figure 2. (a)-(d): Evolution of the solution in time for $\epsilon = 0.1$ and a zero-mean perturbation supported on $(-3\epsilon, 3\epsilon)$ with $a = 1$, $f = 1$ and with corresponding energy functional (e) and Willmore functional (f).

Figure 3 is a result of a shift of the perturbation used in the example of Figure 1. The shift of the perturbation leads to a shift of the kink solution asymptotically in time as a consequence of conservation of mass. Considering again the Willmore functional an increase in time, visibly caused by a transitional instability, occurs. Again the solution is unstable in terms of the Willmore functional.

Considering the behavior of the perturbed solution in several numerical tests we can further make claims on how the perturbation has to look like so that an unstable
state occurs. For a fixed $\epsilon << 1$ we can see that the length of the supporting interval is of most relevant importance. Extending the supporting interval of the perturbation has an extension of the time interval and the size of the instability as a consequence, compare Figure 4 right diagram. Is the supporting interval too small no unstable state can be seen, compare Figure 2. Further it can be seen that also the amplitude and the frequency of the perturbation have an influence on the occurrence and size of the instability. With growing amplitude of the perturbation also the
time interval and the size of the instability change, compare Figure 4 left diagram. If the amplitude exceeds a certain threshold the solution will not converge to the kink solution anymore. As expected the more frequent the perturbation is the faster the solution converges to the kink solution. Therefore the higher the frequency of the perturbation the smaller the time interval of the instability. Because of the important role of the support of the perturbation it seems that perturbations with higher frequency bring us no additional information for our study of the instability.

By changing the parameter $\epsilon$ one can see that with decreasing $\epsilon$ the time interval of the instability decreases. In Figure 5 the point in time of the maximal amplitude of the instability is shown for different $\epsilon$. The size of the instability stays approximately the same.
In the two dimensional case the analog to the kink solutions are the so called bubble solutions. Like the kink solutions they are characterized by a transition area of order $\epsilon$ from the steady states -1 and 1 but are additionally radially symmetric, see [20] for details. In Figure 6 the evolution of a solution of the two dimensional Cahn-Hilliard equation over a finite time interval is shown. In this example we used equidistant space and time discretization $\Delta x = \Delta y = \epsilon$ and $\Delta t = \epsilon^4$. As initial value we took a radial-symmetric bubble solution perturbed with a sinus wave in $x_1$-direction. For a better comparison with the one dimensional case a vertical cut
in $x_2 = 0.5$ of the solution is shown. Again the solution exhibits a local growth of amplitude before converging uniformly to the stationary solution. Especially near $x_1 = 0.5$ the solution incipiently tends away from the bubble solution. As predicted this phenomenon causes an increase of the Willmore functional in a short time interval before converging to 0. It seems that the definition of instability in terms of the Willmore functional makes sense, also in the two dimensional case.

5. Conclusion and Outlook

We considered local instability and asymptotic behavior of the Cahn-Hilliard equation in the neighborhood of certain transition solutions. We found that studying instabilities of the Cahn-Hilliard equation is closely connected to studying the temporal decay behavior of the Willmore functional.

In section 2 the large-time limit of the Willmore functional was shown to be zero in a general setting. With this result also the convergence of $u_t \to 0$ in $H^{-2}(\Omega)$ and in $L^2(\Omega)$ respectively for solutions $u$ of the Cahn-Hilliard equation was proved. We find that the Willmore functional is a good surveying quantity to study structures of instability patterns of the solutions. The asymptotic decay rate in arbitrary dimensions is still a challenging problem. In section 3 spectral estimates for the linearized fourth order Cahn-Hilliard operator were made. A formal computation for the Willmore functional, asymptotically expanded in the eigenvectors, strengthened our conjecture about the connection between the Willmore functional and the stability of solutions of the Cahn-Hilliard equation. Motivated by the linear analysis, in section 4 nonlinear instabilities of transition solutions were found numerically and compared with the behavior of the corresponding Willmore functional in the one and two dimensional case. We found the Willmore functional to be decreasing if the solution converges to the equilibrium state without transitional instability and having peaks when instabilities occur.

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