The Asymptotic Behavior of Globally Smooth Solutions of the Multidimensional Isentropic Hydrodynamic Model for Semiconductors

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Abstract: In this paper we study the asymptotic behavior of globally smooth solutions of the Cauchy problem for the multidimensional isentropic hydrodynamic model for semiconductors in \(\mathbb{R}^d\). We prove that smooth solutions (close to equilibrium) of the problem converge to a stationary solution exponentially fast as \(t \to +\infty\).

Keywords: Multidimensional hydrodynamic model, semiconductors, asymptotic behavior, globally smooth solution

AMS Subject Classifications: 35L65, 76X05, 35M10, 35L70, 35Q60

1 Introduction

The isentropic hydrodynamic model describing the electron flow in a unipolar semiconductor is given by [14, 20]:

\[
\begin{aligned}
&n_t + \nabla \cdot (nu) = 0, \\
u_t + (u \cdot \nabla)u + \frac{1}{n} \nabla p(n) = \nabla \Phi - \frac{u}{\tau}, \\
&\Delta \Phi = n - b(x), \Phi \to 0 \quad \text{as} \quad |x| \to +\infty
\end{aligned}
\]

for \((x,t) \in \mathbb{R}^d \times [0, +\infty), \quad d = 2, 3\), where \(n, u, \Phi\) denote the electron density, electron velocity and the electrostatic potential, respectively. The constant \(\tau > 0\) is the velocity...
relaxation time, the function $b(x)$ denotes the prescribed density of positive charged background ions (doping profile), and the pressure-density function $p = p(n)$ has the property that $n^2 p'(n)$ is strictly increasing function from $[0, +\infty)$ onto $[0, +\infty)$. A usual hypothesis is

$$p(n) = a^2 n^\gamma, \quad n > 0, \quad a \neq 0, \quad \gamma \geq 1. \quad (1.2)$$

The model (1.1)-(1.2) is a simplified multidimensional hydrodynamic model which was analyzed by Degond and Markowich [4] for the first time in the stationary case. For the one-dimensional case, the Cauchy problem and the initial boundary value problem of (1.1) has been extensively studied by many authors in the literature (see, e. g., [2, 3, 6, 11-13, 16, 18, 19, 23]). In the stationary case, Degond-Markowich [3] proved the existence and uniqueness of steady-state solutions in the subsonic case. Gamba [6] discussed the existence and uniqueness of steady-state solutions in the transonic case. In the dynamic case, Zhang [23] and Marcati-Natalini [18] investigated the global existence of weak solutions of the one-dimensional initial-boundary value problem and the Cauchy problem respectively by using the tools of compensated compactness. The corresponding results on the zero relaxation limit have been obtained also in [13] and [18]. Luo, Natalini and Xin [16] and Hsiao and Yang [12] investigated the asymptotic behavior of smooth solutions of the Cauchy problem and the initial boundary value problem of (1.1) respectively and proved that under appropriate conditions on the doping function $b(x)$ and the pressure function $p(n)$ the corresponding steady state solutions of the simplified hydrodynamic model and the drift-diffusion model are exponentially (locally) asymptotically stable.

It is, of course, more important and more interesting to study the system (1.1)-(1.2) in the multidimensional case, but very little is known so far, due to the serious difficulties in establishing the global existence of weak or smooth solutions and other related problems. Besides the local classical solutions obtained in [15, 17], only steady state solutions in the subsonic case and in the dynamic case solutions with geometrical structure (symmetry) or without vorticity were studied in [2, 4, 5, 7, 10, 11] respectively. Chen and Wang [2] proved the existence of global weak solutions with geometrical structure of the system (1.1). Hsiao and Wang [10, 11] considered the smooth solutions of the system and established the global existence and asymptotic behavior of the spherically symmetrical solution of (1.1) with $\gamma = 1$ and $\Omega = \{x \in R^d : 0 < R_1 \leq |x| \leq R_2 < +\infty\}$ in [10] and $\gamma > 1$ and $\Omega = \{x \in R^d : |x| \geq R_1 > 0\}$ in [11], respectively. Engelberg, Liu and Tadmor [5] also studied the critical threshold phenomena of one-dimensional and multi-dimensional pressureless Euler-Poisson equations with geometrical symmetry in the multi-dimensional case and with and without relaxation. Recently, Guo [7] investigated the irrotational Euler-Poisson equation system without relaxation and without geometrical symmetry and demonstrated that the smooth, irrotational initial data which are small perturbations of a fluid at rest lead to globally smooth, irrotational solutions of the Euler-Poisson system. However, the method used in [7] does not work for the case of the rotational fluid.

Here we deal with the general $d$-dimensional problem without geometrical assumptions. For simplicity, we only discuss the case when the doping profile is a positive constant.

Introducing the electric field $\mathbf{e}$ and the quasi-Fermi electrostatic potential $h(n)$ by
\[ e = \nabla \Phi \] and \[ h'(n) = \frac{1}{n} p'(n) \], respectively, the system (1.1) reduces to
\[
\begin{align*}
& n_t + \nabla \cdot (nu) = 0, \\
& u_t + (u \cdot \nabla)u + \nabla h(n) = e - \frac{n}{r}, \\
& \nabla \cdot e = n - b(x)
\end{align*}
\] (1.3)
for \( x \in R^d, t > 0 \).

We consider smooth solutions for the Cauchy problem (1.3) with
\[
n(x, 0) = n_0(x), u(x, 0) = u_0(x), x \in R^d, d = 2, 3.
\] (1.4)
Using Green’s formulation, it follows from (1.1) that
\[
e(x, t) = \nabla \Phi(x, t) = \nabla \Delta^{-1}(n_0(x) - b(x)) - \nabla \Delta^{-1} \nabla \cdot \int_0^t (nu)(x, s)ds.
\] (1.5)
Thus, (1.1) can be reduced to the form of the conservation law with a non-local term
\[
\begin{align*}
& n_t + \nabla \cdot (nu) = 0, \\
& u_t + (u \cdot \nabla)u + \nabla h(n) = \nabla \Delta^{-1}(n_0(x) - b(x)) - \nabla \Delta^{-1} \nabla \cdot \int_0^t (nu)(x, s)ds - \frac{n}{r}.
\end{align*}
\]

Before stating our main results of this paper, we give a local existence theorem for the Cauchy problem (1.3)-(1.4), which can be established by a standard contraction mapping argument and the proof of which will therefore be omitted here, see e.g. [1, 15, 17].

**Proposition 1.1 (Local existence)** Assume that \( b(x) = B > 0 \) (positive constant) and \( n(x, 0) - B \in H^3(R^d), u(x, 0) \in H^3(R^d), \nabla \Phi(x, 0) \in H^3(R^d) \). Then there exists a unique smooth solution \((n(x, t), u(x, t), e(x, t))\) of (1.3)-(1.4) satisfying
\[
n, u, e \in C^1(R^d \times [0, T_{max})
\]
and
\[
n(x, t) - B, u(x, t), e(x, t) \in L^\infty(0, T; H^3(R^d))
\]
defined on a maximal interval of existence \([0, T_{max})\) with \( T_{max} > 0 \). Moreover, if \( T_{max} < \infty \), then
\[
\| (n(\cdot, t) - B, u(\cdot, t), e(\cdot, t)) \|_{L^2(R^d)}^2 + \| (n_t, u_t, e_t)(\cdot, t) \|_{L^2(R^d)}^2 + \int_0^t \| (n(\cdot, \tau) - B, u(\cdot, \tau), e(\cdot, \tau)) \|_{L^2(R^d)}^2 + \| (n_t, u_t, e_t)(\cdot, \tau) \|_{L^2(R^d)}^2 d\tau \to \infty, \quad t \to T_{max}.
\]

**Remark 1.1** Since the non-local term \( \nabla \Delta^{-1} \nabla \cdot \int_0^t (nu)(x, s)ds \) is a sum of products of Riesz transforms of \( \int_0^t (nu)(x, s)ds \), we have, by the \( L^2 \) boundedness of the Riesz transformation, see [22],
\[
\| \nabla \Delta^{-1} \nabla \cdot \int_0^t (nu)(x, s)ds \|_{L^2(R^d)} \leq C \| \int_0^t (nu)(x, s)ds \|_{L^2(R^d)}
\]
for some constant $C > 0$. Noticing this crucial fact, the proof of the local existence in Proposition 1.1 become more transparent.

Now we state our main result on the global existence and large time behavior of the solutions as follows.

**Theorem 1.2** Assume that $b(x) = B > 0$ and (1.2) holds. Assume that $n(\cdot, 0) - B \in H^3(\mathbb{R}^d)$, $u(\cdot, 0) \in H^3(\mathbb{R}^d)$ and $e(\cdot, 0) \in H^3(\mathbb{R}^d)$. Then there exist positive constants $\delta_0$, depending only on $B$, such that if $\|n(\cdot, 0) - B, u(\cdot, 0), e(\cdot, 0)\|_{H^3(\mathbb{R}^d)} + \|(n_t, u_t, e_t)(\cdot, 0)\|_{H^3(\mathbb{R}^d)} \leq \delta_0$, then there exists a unique global smooth solution $(n, u, e)$ of (1.3)-(1.4). Moreover,

$$
\|(n(\cdot, t) - B, u(\cdot, t), e(\cdot, t))\|^2_{H^3(\mathbb{R}^d)} + \|(n_t, u_t, e_t)(\cdot, t)\|^2_{H^2(\mathbb{R}^d)} \\
\leq C_1 \|(n(\cdot, 0) - B, u(\cdot, 0), e(\cdot, 0))\|^2_{H^3(\mathbb{R}^d)} + \|(n_t, u_t, e_t)(\cdot, 0)\|^2_{H^2(\mathbb{R}^d)} \exp(-\alpha_1 t)
$$

with positive constants $\alpha_1$ and $C_1$.

**Remark 1.2** The parameter $\delta_0$, which measures the allowed deviation from equilibrium, may depend upon the relaxation time $\tau$.

**Remark 1.3** The proof of Theorem 1.2 is based on the classical energy type estimates in higher order Sobolev spaces. Here the main idea in using the relaxation term for obtaining global existence (close to equilibrium) of smooth solution to (1.3)-(1.4) is to use Sobolev estimate techniques for higher order derivatives to overcome the difficulties arising from three nonlinear terms (convection, pressure and electric field) in the energy flux.

Throughout this paper we denote various generic constants by $a_i, b_i, c_i, C, C_1, D, D_i, K_i$ and $\lambda$, only dependent on $B$. Repeated indices mean summation from 1 to $d$. $H^m(\mathbb{R}^d)$, $m \in \mathbb{Z}_+$, denotes the usual Sobolev space of order $m$ equipped with the norm

$$
\|g\|_{H^m(\mathbb{R}^d)} = \sum_{0 \leq |\alpha| \leq m} \|\partial_\alpha g\|,
$$

where $\| \cdot \| \equiv \| \cdot \|_{L^2(\mathbb{R}^d)}$ and $\partial_\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d}$ with $\sum_{i=1}^N \alpha_i = |\alpha|$ and $\partial_{x_i} = \partial_{x_i}$. The Euclidean norm and inner product for $\mathbb{R}^d$ are denoted by $| \cdot |$ and $\langle \cdot , \cdot \rangle$, resp., $a \cdot b$ for $a, b \in \mathbb{R}^d$. For a vector valued function $f = (f_1, \cdots, f_m)$ and a normed space $X$ of scalar functions with the norm $\| \cdot \|$, $f \in X$ means that each component of $f$ is in $X$; We put $\|f\| := \|f_1\| + \|f_2\| + \cdots + \|f_m\|$, $\partial f = \partial f = (\partial_1 f_1, \cdots, \partial_d f_d)$ and $\partial_2 f = \partial_2 f = \partial_2 (\partial_2^{-1} f)$. In particular, $\partial_{x_i} N_i = (\partial_{x_1} N_i, \partial_{x_2} N_i, \cdots, \partial_{x_d} N_i)$, $\partial_{x_i}^2 N_i = (\partial_{x_1}^2 N_i, \partial_{x_1} \partial_{x_i} N_i, \cdots, \partial_{x_d}^2 N_i)$ and $\|\partial_{x_i}^2 u\|^2 = \sum_{i,j,k,h=1}^d \int_{\mathbb{R}^4} |\partial_{x_i} \partial_{x_j} \partial_{x_k} u_{i,j,h}|^2 dx$ etc. Moreover, the integration domain $\mathbb{R}^d$ will generally be omitted without any ambiguity.

We shall also make repeated use of various standard inequalities. The first one of these is

$$
|ab| \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \epsilon > 0. \tag{1.6}
$$

As a consequence of Young’s inequality and Gagliardo-Nirenberg’s inequality (see [21]), we have

$$
\|u\|_{L^p} \leq C(d, p)\|u\|^{\frac{d}{d-1}} \frac{\|\nabla u\|^{d-1}}{\|\nabla u\|} \tag{1.7}
$$

for $u \in H^1$, $p \geq 2$ when $d = 2$ and $p \in [2, 6]$ when $d = 3$, where $C(d, p)$ in (1.7) is a positive constant depending only on $p$ and $d$. 
2 Hydrodynamic Model

In this section we shall prove Theorem 1.2 by using the energy method.

Set
\[ n(x, t) = B + N(x, t), \]  
\[ e(x, t) = \nabla \Phi(x, t). \]  

Then the function \((N, u, e)\) satisfies the system
\[
\begin{align*}
N_t + \text{div}((B + N)u) &= 0, \\
\dot{u} + (u \cdot \nabla)u + \nabla(h(B + N) - h(B)) &= e - u, \\
\text{div} &= N.
\end{align*}
\]  

(Here, without loss of generality, we have set \(\tau = 1\).

To prove Theorem 1.2, we first establish the following a priori estimates.

**Proposition 2.1** There exist positive constants \(\delta_1, \alpha\) and \(C\), depending only on \(B\), such that for any \(T > 0\), if
\[
\sup_{0 \leq t \leq T} \left( \|(N, u, e)(\cdot, t)\|_{H^3(R^d)} + \|(N_t, u_t, e_t)(\cdot, t)\|_{H^2(R^d)} \right) \leq \delta_1,
\]  
then
\[
\|(N, u, e)(\cdot, t)\|_{H^3(R^d)}^2 + \|(N_t, u_t, e_t)(\cdot, t)\|_{H^2(R^d)}^2 \leq C \left( \|(N, u, e)(\cdot, 0)\|_{H^3(R^d)}^2 + \|(N_t, u_t, e_t)(\cdot, 0)\|_{H^2(R^d)}^2 \right) \exp(-\alpha t)
\]  
for any \(t \in [0, T]\).

**Proof** First, by using a priori assumption (2.4) and the Sobolev inequality, we have
\[
\sup_{x \in R^d} |(N, \partial_x N, N_t, u, \partial_x u, u_t, e, \partial_x e, e_t)| \leq C \left( \|(N, u, e)(\cdot, t)\|_{H^3(R^d)} + \|(N_t, u_t, e_t)(\cdot, t)\|_{H^2(R^d)} \right) \leq C\delta_1.
\]  
By (1.6), (2.3)\(_1\) and (2.6), it is easy to obtain
\[
\|\partial_x^i N_t\| \leq C \sum_{k=1}^{i+1} (\|\partial_x^k u\| + \|\partial_x^k N\|), i = 0, 1, 2.
\]  

Take the derivative \(\partial_x^j, j = 1, 2, 3\) of (2.3)\(_3\) and multiply the resulting equation by \(\partial_x^j \Phi\), and then integrate it over \(R^d\). Integration by parts, using \(e = \nabla \Phi\), (2.6) and (1.6), gives
\[
\int |\partial_x^j e|^2 dx = \int \partial_x^{j-1} N \partial_x^{-1} \text{div} e dx \leq \frac{1}{2} \int |\partial_x^j e|^2 dx + C \int |\partial_x^{j-1} N|^2 dx,
\]  
i.e.
\[
\|\partial_x^j e\|^2 \leq C \|\partial_x^{j-1} N\|^2, j = 1, 2, 3.
\]
Similarly, we have
\[ \|e_t\| \leq C(\|u\| + \|N\|), \]  
\[ \|\partial_x^m e_t\| \leq C \sum_{k=0}^m (\|\partial_x^k u\| + \|\partial_x^k N\|), \quad m = 1, 2. \]  
(2.9)

Now we take \( \partial_t^l \) on the both sides of (2.3) to find
\[ \partial_t^l u_t + \partial_t^l (u \cdot \nabla u) + \partial_t^l \nabla (h(n) - h(B)) = \partial_t^l e - \partial_t^l u. \]  
(2.10)

(Here and in the following, for convenience, we still use \( n \) in stead of \( B + N \).

Multiply (2.10) with \( l = 0, 1 \) by \( \partial_t^l u \) and integrate it over \( \mathbb{R}^d \). Integration by parts leads to
\[ \frac{1}{2} \frac{d}{dt} \|\partial_t^l u\|^2 + \|\partial_t^l u\|^2 - \int \partial_t^l (h(n) - h(B)) \partial_t^l \text{div} u dx \]  
\[ + \int \partial_t^l \Phi \partial_t^l \text{div} u dx + \int \partial_t^l (u \cdot \nabla u) \partial_t^l u dx = 0, l = 0, 1. \]  
(2.11)

We shall now estimate the integrals in (2.11).

It follows from (2.6) that there exists \( \delta_1 > 0 \) such that
\[ \frac{B}{2} \leq n \leq 2B. \]  
(2.12)

Using (2.12) and (1.2), we have that there exist \( \delta_1 > 0 \) such that
\[ 0 < D_1 \leq h'(n) \leq D_2 < \infty; \quad |h^{(k)}(n)| \leq D_3 < \infty \]  
(2.13)

for any positive integer \( k \).

Thus, we have
\[ - \int (h(n) - h(B)) \text{div} u dx = \int (h(n) - h(B)) \frac{N_t + u \cdot \nabla N}{n} dx \]  
\[ = \frac{d}{dt} \int_0^N \frac{h(s + B) - h(B)}{s + B} ds dx + \int \frac{h'(B + \theta N)}{n} N_t u \cdot \nabla N dx \]  
\[ \geq \frac{d}{dt} \int_0^N \frac{h(s + B) - h(B)}{s + B} ds dx - C \delta_1 \int (|u|^2 + |\nabla N|^2) dx, \]  
\[ - \int \partial_t (h(n) - h(B)) \partial_t \text{div} u dx \]  
\[ = \int h'(n) \partial_t N \frac{1}{n} (N_t + (u \cdot \nabla N)_t) - \frac{N_t}{n^2} (N_t + u \cdot \nabla N) dx \]  
\[ \geq \frac{d}{dt} \int \frac{h'(n)}{2n} N_t^2 dx - C \delta_1 \int N_t^2 dx - \int \frac{h'(n)u}{n} \cdot \nabla (N_t^2) dx \]  
\[ \geq \frac{d}{dt} \int \frac{h'(n)}{2n} N_t^2 dx - C \delta_1 \int N_t^2 dx \]  
(2.15)

for some positive constant \( \theta : 0 < \theta < 1 \), with the help of the smallness of \( |N|, |N_t|, |u| \) 
and \( \nabla N \), as can be ensured by (2.6) and the smallness of \( \delta_1 \).
By (2.3) and (2.3)$_3$, we have, for $l = 0, 1$,

$$
\int \partial_t \Phi \partial_t \text{div} u dx = -\frac{1}{\gamma} \int \partial_t \Phi \partial_t (N_l + \text{div}(Nu)) dx
$$

$$
= -\frac{1}{\gamma} \int \partial_t \Phi \partial_t \text{div} e_t dx + \frac{1}{\gamma} \int \partial_t \text{div} e_t (Vu) dx
$$

$$
= \frac{1}{2\pi} \int \|\partial_t e_t\|^2 + \frac{1}{\gamma} \int \partial_t \text{div} e_t (Nu) dx
$$

$$
\geq \frac{1}{2B} \int \|\partial_t e_t\|^2 - c\delta \int (|\partial_t e_t|^2 + |u|^2 + N^2 + |u_t|^2 + N_t^2) dx
$$

and

$$
\int \partial_t (u \cdot \nabla u) \partial_t \text{div} u dx \geq -C\delta_1 \int (|\partial_t u|^2 + |\nabla u|^2) dx.
$$

(2.16)

Notice that in the above we use the different formulation of \text{div} u, i.e. \text{div} u = -\frac{N_t + u \nabla N}{n} and \text{div} u = -\frac{N_t + \text{div}(Nu)}{B}$, in estimating the nonlinear pressure term integral and the electric field term integral, respectively, whose advantages will be reflected in establishing high order energy estimates.

Thus, (2.11), together with (2.14)-(2.17), implies

$$
\frac{1}{2} \int \|u\|^2 + |u_t|^2 + \int_0^T \frac{h(t) + h(B)}{s + B} ds + \frac{h'(n)}{n} N_t^2 + \frac{1}{\gamma} \int (|e|^2 + |e_t|^2) dx
$$

$$
+ C \int (|u|^2 + |u_t|^2) |dx \leq C\delta_1 \|(N, \nabla N, N_t, \nabla u)\|^2.
$$

(2.18)

Now we estimate $\|\nabla u\|^2$. Noting the fact that the electric field is irrotational and there exists a specially dependent relation (2.3)$_3$ between the electric field and the electron density, we use the formulation $\|\nabla u\|^2 = \|\text{curl} u\|^2 + \|\text{div} u\|^2$ to estimate $\|\nabla u\|^2$ in order to simplify our proof.

Take \text{div} of (2.3)$_2$ and multiply the resulting equation by \text{div} u, then integrate over $R^d$, using integration by parts, to obtain

$$
\frac{1}{2} \frac{d}{dt} \|\text{div} u\|^2 + \|\text{div} u\|^2 = -\int \text{div} \nabla (h(n) - h(B)) \text{div} u dx
$$

$$
+ \int \text{div} \text{div} u dx - \int \text{div} (u \cdot \nabla u) \text{div} u dx.
$$

(2.19)

Using (2.3)$_1$, (1.6) and the smallness of $|N_l|, |\nabla N|, |u|$ and $|\nabla u|,

$$
-\int \text{div} \nabla (h(n) - h(B)) \text{div} u dx = \int \nabla (h(n) - h(B)) \text{div} u dx
$$

$$
= -\int \nabla (h(n) - h(B)) \nabla (\frac{N_t + u \nabla N}{n}) dx
$$

$$
= -\int h'(n) \nabla N [\frac{1}{n} (\nabla N_t + \nabla (u \cdot \nabla N)) - \frac{\nabla N}{n^2} (N_t + \text{div}(Nu))] dx
$$

$$
\leq -\frac{1}{2n} \int h'(n) |\nabla N|^2 dx + \frac{1}{2} \int \partial_t \frac{h'(n)}{n} |\nabla N|^2 dx
$$

$$
- \int \frac{h'(n)}{n} u \cdot \nabla (\frac{|\nabla N|^2}{2}) dx + C\delta_1 \int |\nabla N|^2 dx
$$

$$
\leq -\frac{1}{2n} \int h'(n) |\nabla N|^2 dx + C\delta_1 \int |\nabla N|^2 dx,
$$

(2.20)
\[ \int \text{div}(\text{div}u)dx = \int N \text{div}u dx \]

\[ = -\frac{1}{B} \int N(N_t + \text{div}(N(u)))dx = -\frac{1}{B} \frac{d}{dt} \int N^2 dx + \frac{1}{B} \int N \cdot \nabla N dx \]

\[ \leq -\frac{1}{2B} \frac{d}{dt} \|N\|^2 + C\delta_1 \|N\|_{H^1} \] (2.21)

and

\[ -\int \text{div}(u \cdot \nabla u)dx = -\int \partial_i(w^i \partial_j u^j) \partial_k u^k dx \]

\[ = -\int \partial_i u^i \partial_j w^j \text{div}u dx + \int \frac{1}{2} (\text{div}u)^3 dx \leq C\delta_1 \int \| \nabla u \|^2 dx. \] (2.22)

Thus, (2.19) together with (2.20)-(2.22), implies

\[ \frac{1}{2} \frac{d}{dt} \| \text{div}u \|^2 + \| \text{div}u \|^2 \leq C\delta_1 \| (N, \nabla N, \nabla u) \|^2 \] (2.23)

Similarly, we take \( \text{curl} \) of (2.3)_2 and multiply the resulting equation by \( \text{curl} u \) in \( L^2(R^d) \) to get

\[ \frac{1}{2} \frac{d}{dt} \| \text{curl} u \|^2 + \| \text{curl} u \|^2 = -\int \text{curl}(u \cdot \nabla u)\text{curl}u dx. \] (2.24)

Direct calculations and integration by parts give

\[ \int \text{curl}(u \cdot \nabla u)\text{curl}u dx \]

\[ = -\int (\partial_k(w^i \partial_j u^j - \partial_l(w^i \partial_j u^k))(\partial_k u^l - \partial_l u^k)) dx \]

\[ = -\int (\partial_k w^i \partial_j u^j - \partial_l u^i \partial_j u^k)(\partial_k u^l - \partial_l u^k) dx - \int w^i \partial_j \left( \frac{\text{curl}u^2}{2} \right) dx \]

\[ \leq C\delta_1 \| \nabla u \|^2. \] (2.25)

Thus, (2.24), together with (2.25), implies

\[ \frac{1}{2} \frac{d}{dt} \| \text{curl} u \|^2 + \| \text{curl} u \|^2 \leq C\delta_1 \| \nabla u \|^2. \] (2.26)

Combining (2.23) and (2.26), we have, with the help of smallness of \( \delta_1 \), that

\[ \frac{d}{dt} \int \| \nabla u \|^2 + \frac{1}{B} |N|^2 + \frac{h'(n)}{n} |\nabla N|^2 dx + C\| \nabla u \|^2 \leq C\delta_1 \| (N, \nabla N) \|^2. \] (2.27)

On the other hand, multiply (2.3)_2 by \( \nabla N \) in \( L^2(R^d) \) to get

\[ \int h'(n)|\nabla N|^2 dx = \int e\nabla N dx - \int (u_t + u)\nabla N dx - \int u \cdot \nabla u \nabla N dx. \] (2.28)

Using (2.3)_3 and (1.6), one easily finds

\[ \int e\nabla N dx = -\int \text{div}e\nabla N dx = -\|N\|^2, \] (2.29)
\[ -\int \mathbf{u} \cdot \nabla \mathbf{u} \nabla N \, dx \leq C\delta_1\| (\nabla N, \nabla \mathbf{u}) \|^2 \]  
\hspace{1cm} (2.30)

and
\[ -\int (\mathbf{u}_t + \mathbf{u}) \nabla N \, dx \leq \epsilon \| \nabla N \|^2 + C \| (\mathbf{u}, \mathbf{u}_t) \|^2 . \]  
\hspace{1cm} (2.31)

Thus, (2.28), together with (2.29)-(2.31), implies
\[ \| (N, \nabla N) \|^2 \leq C \| (\mathbf{u}_t, \nabla \mathbf{u}) \|^2 \]  
\hspace{1cm} (2.32)

with the help of (2.13) and the smallness of \( \delta_1 \) and \( \epsilon \).

Combining (2.18), (2.27), (2.32), (2.7) with \( i = 0 \) and (2.9), we have
\[ \frac{1}{2} \frac{d}{dt} \int |\mathbf{u}|^2 + |\mathbf{u}_t|^2 + \int_0^t \frac{h(s+B)-h(B)}{s+B} \, ds + \frac{h'(n)}{n} N_j^2 + \frac{\lambda}{B} (|e|^2 + |e_t|^2) + |\nabla \mathbf{u}|^2 \]  
\hspace{1cm} + \frac{1}{B} |N|^2 + \frac{h'(n)}{n} |\nabla N|^2 \, dx + C \int |(N, \nabla N, \mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}, \mathbf{e}_t)|^2 \, dx \leq 0 .
\hspace{1cm} (2.33)

By (2.3)_2 and (2.6), we have
\[ \| e \|^2 \leq C \| (\mathbf{u}, \mathbf{u}_t, \nabla N, \nabla \mathbf{u}) \|^2 . \]  
\hspace{1cm} (2.34)

Using (2.3)_1 and (2.3)_3, we have
\[ -\int \partial_x e \partial_x \mathbf{u} \, dx = -\int \partial_x \nabla \Phi \partial_x \mathbf{u} \, dx \]  
\hspace{1cm} + \int \partial_x \Phi \partial_x (N_t + \text{div}(N \mathbf{u})) \, dx \]  
\hspace{1cm} = \frac{1}{B} \int \partial_x e \partial_x \mathbf{e}_t \, dx + \frac{1}{B} \int \partial_x e \partial_x (N \mathbf{u}) \, dx \]  
\hspace{1cm} = \frac{1}{2B} \frac{d}{dt} \| \partial_x e \|^2 + \frac{1}{B} \int \partial_x e \partial_x (N \mathbf{u}) \, dx ,
\hspace{1cm}
which gives
\[ \frac{d}{dt} \int \frac{1}{2B} |\partial_x e|^2 \, dx \leq C \| (\partial_x e, \partial_x \mathbf{u}, \partial_x N) \|^2 . \]  
\hspace{1cm} (2.35)

Thus, (2.33), together with (2.34), (2.35) and (2.8) with \( j = 1 \), implies
\[ \frac{d}{dt} \int |\mathbf{u}|^2 + |\mathbf{u}_t|^2 + \int_0^t \frac{h(s+B)-h(B)}{s+B} \, ds + \frac{h'(n)}{n} N_j^2 + |e|^2 + \frac{\lambda}{B} (|\nabla \mathbf{e}|^2 + |e_t|^2) + |\nabla \mathbf{u}|^2 \]  
\hspace{1cm} + \frac{1}{B} |N|^2 + \frac{h'(n)}{n} |\nabla N|^2 \, dx + C \int |(N, \nabla N, \mathbf{u}, \mathbf{u}_t, \nabla \mathbf{e}, \nabla \mathbf{e}_t)|^2 \, dx \leq 0 \]  
\hspace{1cm} (2.36)

for some positive constant \( \lambda > 0 \).

The next step is to get the estimates for the higher order derivatives. Differentiate (2.3)_2 with respect to \( x \) twice and multiply by \( \partial_x^2 \mathbf{u} \) and integrate the resulting equation over \( R^d \). Then integrations by parts give
\[ \frac{d}{dt} \int \frac{1}{2} |\partial_x^2 \mathbf{u}|^2 \, dx + \int |\partial_x^2 \mathbf{u}|^2 \, dx - \int \partial_x^2 (h(n) - h(B)) \partial_x^2 (\text{div} \mathbf{u}) \, dx \]  
\hspace{1cm} + \int \partial_x^2 \Phi \partial_x^2 (\text{div} \mathbf{u}) \, dx + \int \partial_x^2 (\mathbf{u} \cdot \nabla \mathbf{u}) \partial_x^2 \mathbf{u} \, dx = 0 \]  
\hspace{1cm} (2.37)
We shall now estimate the integrals of (2.37).

Using (2.3), (1.2), (2.6), (2.13), (2.7) and (2.3), we have

\[- \int \partial_x^2(h(n) - h(B)) \partial_x^2(div u) dx \]
\[= \int \partial_x^2(h(n) - h(B)) \partial_x^2(N + u \nabla N) dx \]
\[\geq \int \frac{\mu(n)}{n} \partial_x^2 N (\partial_x^2 N_t + \alpha \partial_x \partial_x^2 N) dx \]
\[+ \int \frac{\mu(n)}{n} \partial_m N \partial_x N (\partial_n \partial_x N_t + \alpha \partial_x \partial_m \partial_x N_t) dx \]
\[- \mu_1 \int (|\partial_x^2 N|^2 + |\partial_x N|^2 + |\partial_x^2 u|^2) dx \]
\[= \frac{d}{dt} \int \left( \frac{\mu(n)}{2} |\partial_x^2 N|^2 + \frac{\mu(n)}{2} \partial_m N \partial_x N \partial_x N \partial_x \partial_x N dx \right. \]
\[\left. - \int \left( \frac{\mu(n)}{n} |\partial_x^2 N|^2 - \frac{\mu(n)}{n} \partial_m N \partial_x N \partial_x N \partial_x \partial_x N dx \right. \right. \]
\[\left. \left. + \int \frac{\mu(n)}{n} \partial_x^2 N \partial_x \partial_x N + \frac{\mu(n)}{n} \partial_m N \partial_x \partial_x N \partial_x \partial_x N dx \right. \right. \]
\[\left. \left. - \mu_1 \int (|\partial_x^2 N|^2 + |\partial_x N|^2 + |\partial_x^2 u|^2) dx \right. \right. \]
\[\geq \frac{d}{dt} \int \left( \frac{\mu(n)}{2} |\partial_x^2 N|^2 + \frac{\mu(n)}{2} \partial_m N \partial_x N \partial_x \partial_x N dx \right. \]
\[\left. \left. - \mu_1 \int (|\partial_x^2 N|^2 + |\partial_x N|^2 + |\partial_x^2 u|^2) dx \right. \right. \]
\[\geq \frac{d}{dt} \int \left( \frac{\mu(n)}{2} |\partial_x^2 N|^2 + \frac{\mu(n)}{2} \partial_m N \partial_x N \partial_x \partial_x N dx \right. \]
\[\left. \left. - \mu_1 \int (|\partial_x^2 N|^2 + |\partial_x N|^2 + |\partial_x^2 u|^2) dx \right. \right. \]
\[= \frac{d}{dt} \int \partial_x^2 \Phi \partial_x^2 (div u) dx \]
\[= - \frac{1}{B} \int \partial_x^2 \Phi \partial_x^2 (N_t + div(N u)) dx \]
\[= - \left( \frac{1}{B} \int \partial_x^2 e \partial_x^2 e dx + \frac{1}{B} \int \partial_x^2 e \partial_x^2 (N u) dx \right. \]
\[\geq \frac{1}{2B} \int \partial_x^2 e^2 dx - C \delta_1 \| (\partial_x^2 e, \partial_x^2 N, \partial_x^2 u, \partial_x N, \partial_x u) \|^2. \]

As in (2.25), we easily get

\[\int \partial_x^2 (u \cdot \nabla u) \partial_x^2 u dx \geq -C \delta_1 \| \partial_x^2 u \|^2. \]

Thus, (2.37), together with (2.38)-(2.40), implies

\[\frac{d}{dt} \int \left( \frac{1}{2} |\partial_x^2 u|^2 + \frac{1}{2B} |\partial_x^2 e|^2 + \frac{\mu(n)}{2n} |\partial_x^2 N|^2 + \frac{\mu(n)}{2n} \partial_m N \partial_x N \partial_x \partial_x N dx \right. \]
\[\left. + C \int |\partial_x^2 u|^2 dx - C \delta_1 \int (|\partial_x N|^2 + |\partial_x^2 N|^2 + |\partial_x u|^2 + |\partial_x^2 e|^2) dx \leq 0. \]

Differentiate (2.3)_2 with respect to x and multiply the resulting equation by \( \partial_x \nabla N \). Integration by parts gives

\[\| (\partial_x N, \partial_x^2 N) \|^2 \leq C \| (\partial_x u, \partial_x^2 u) \|^2. \]
Take \( \partial_x \partial_t \) to (2.3)\(_2\) and multiply the resulting equation by \( \partial_x u_t \) in \( L^2(\mathbb{R}^d) \) to get

\[
\frac{1}{2} \frac{d}{dt} \| \partial_x u_t \|^2 + \| \partial_x u_t \|^2 - \int [\partial_x (h(n) - h(B))]_t \partial_x \Phi_t \| \partial_x \partial_t u_t \| dx \\
+ \int \partial_x (u \cdot \nabla u) \partial_x u_t \| dx = 0.
\]  

(2.43)

Using (2.3)\(_1\), (2.3)\(_3\), (1.6) and (2.7), by some tedious but straightforward calculations, we obtain

\[
- \int [\partial_x (h(n) - h(B))]_t \partial_x \Phi_t \| \partial_x \partial_t u_t \| dx \\
= \int \partial_x (h'(n)N_t) \partial_x (\frac{N+u \cdot \nabla u}{n})_t dx - \frac{1}{2} \int \partial_x \Phi_t \partial_x (N_t + \text{div}(N u))_t dx \\
\geq \frac{d}{dt} \int \frac{h'(n)}{2n} |\partial_x N_t|^2 + \frac{1}{2} |\partial_x e_t|^2 \| dx + C \delta_1 \| (\| \partial_x e_t \|, \| \partial_x N_t \|)^2
\]

(2.44)

and

\[
\int \partial_x (u \cdot \nabla u) \partial_x u_t \| dx \geq -C \delta_1 \| (\partial_x u_t, \partial_x^2 u_t) \|^2.
\]  

(2.45)

Thus, (2.43), together with (2.9) with \( m = 1 \), (2.44) and (2.45), implies

\[
\frac{d}{dt} \int \frac{1}{2} |\partial_x u_t|^2 + \frac{h'(n)}{2n} |\partial_x N_t|^2 + \frac{1}{2} |\partial_x e_t|^2 \| dx + C \| (\partial_x u_t, \partial_x e_t) \|^2 \\
\leq C \delta_1 \| (\partial_x^2 u, \partial_x^2 N) \|^2 + C \| (\partial_x u_t, \partial_x N_t) \|^2
\]  

(2.46)

with the help of the smallness of \( \delta_1 \).

Combining (2.41), (2.42), (2.46), (2.7) with \( i = 1 \) and (2.8) with \( j = 2 \), we obtain

\[
\frac{d}{dt} \int \frac{1}{2} |\partial_x^2 u|^2 + \frac{1}{2} |\partial_x^2 e|^2 + \frac{h'(n)}{2n} |\partial_x^2 N|^2 + \frac{h'(n)}{n} |\partial_m N_t \partial_N \partial_m N_t + \frac{1}{2} |\partial_x u_t|^2 \\
+ \frac{h'(n)}{2n} |\partial_x N_t|^2 + \frac{1}{2} |\partial_x e_t|^2 \| dx + C \| (\partial_x N_t, \partial_x^2 N, \partial_x^2 u, \partial_x^2 e, \partial_x u_t, \partial_x e_t) \|^2
\]

(2.47)

\[
\leq C \| (\partial_x u, \partial_x N) \|^2.
\]

We now turn to the estimates of the third derivatives. Notice that in obtaining the estimates on the first and the second derivatives, we have used the smallness of \( |(N, \partial_x N, u, \partial_x u, e, \partial_x e)| \) and \( |(N_t, u_t, e_t)| \), which are guaranteed by (2.6) and the smallness of \( \delta_1 \). However, the above arguments do not work for the third derivatives because we can not obtain the smallness of \( |(\partial_x^2 N, \partial_x^2 u, \partial_x^2 e)| \) and \( |(\partial_x N_t, \partial_x u_t, \partial_x e_t)| \). Hence we give a detailed discussion.

Differentiate (2.3)\(_2\) with respect to \( x \) three times and multiply the resulting equation by \( \partial_x^3 u \). Integration by parts gives

\[
\frac{1}{2} \frac{d}{dt} \| \partial_x^3 u \|^2 + \| \partial_x^3 u \|^2 - \int \partial_x^3 (h(n) - h(B)) \partial_x^3 \text{div} u dx \\
- \int \partial_x^3 e \partial_x^3 u dx + \int \partial_x^3 (u \cdot \nabla u) \partial_x^3 u dx = 0.
\]  

(2.48)
We will estimate each integral in (2.48).

First, by (2.3)\(_1\), we have

\[
div \mathbf{u} = -\frac{N_t + \mathbf{u} \cdot \nabla N}{n}, \quad div \mathbf{u} = \frac{-N_t + \text{div}(N \mathbf{u})}{B}.
\]

(2.49)

In the following, we will use the first formulation of \(div \mathbf{u}\) to estimate nonlinear pressure term integral in order to avoid the presence of the fourth derivatives of the velocity \(\mathbf{u}\) (because now we are being in three order energy estimates) while use the second one of \(div \mathbf{u}\) to estimate the electric field term integral in order to use irrotationity of the electric field, i.e., \(\mathbf{e} = \nabla \Phi\), and the specially dependent relation between the electric field and the electron density.

Now we estimate the most difficult nonlinear pressure term integral in (2.48).

Using (2.49)\(_1\) and by direct calculations, we have

\[
I_1 = -\int \partial_x^2(h(n) - h(B))\partial_x^2 \text{div} \mathbf{u} \, dx
\]

\[
= \int \partial_x^2(h(n) - h(B))\partial_x^2(N_t + \mathbf{u} \cdot \nabla N) \, dx
\]

\[
= \int \{h'(n)\partial_x^3 N + h''(n)(\partial_t N \partial_m \partial_t N + \cdots) + h'''(n)\partial_t N \partial_m N \partial_t N\}
\]

\[
\left\{\frac{1}{n}(\partial_x^3 N_t + \mathbf{u} \cdot \nabla \partial_x^3 N + \partial_t \mathbf{u} \cdot \nabla \partial_t N + \cdots) + \partial_t \partial_m \mathbf{u} \cdot \nabla \partial_t N + \cdots + \partial_x^3 \mathbf{u} \cdot \nabla N\right\}
\]

\[
- \frac{\partial_t \partial_m N_t}{n^2}(\partial_t \partial_m \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \partial_m \partial_t N + \cdots)
\]

\[
- \left(\frac{\partial_t \partial_m N_t}{n^2} - \frac{2\partial_t \partial_t N \partial_m N}{n^3}\right)(\partial_t N_t + \mathbf{u} \cdot \nabla \partial_t N + \partial_t \mathbf{u} \cdot \nabla N) - \cdots
\]

\[
+ \left(-\frac{\partial_x^3 N}{n^2} + \frac{2(\partial_t \partial_m N \partial_t N + \cdots)}{n^2} - \frac{6\partial_t \partial_m N \partial_t N \partial_t N}{n^4}\right)(N_t + \mathbf{u} \cdot \nabla N)\} \, dx
\]

\[= I_{11} + I_{12} + I_{13} + I_1^{(R)},\]

(2.50)

where

\[
I_{11} = \int \left\{\frac{h'(n)}{n} \partial_x^3 N \partial_x^3 N_t + \frac{1}{n} \partial_x^3 N_t [h''(n)(\partial_t N \partial_m \partial_t N + \partial_m N \partial_t \partial_t N + \partial_t N \partial_t \partial_m N)] + h'''(n)\partial_t N \partial_m N \partial_t N\right\} \, dx,
\]

\[
I_{12} = \int \left\{\frac{h'(n)}{n} \mathbf{u} \cdot \nabla \left(\frac{\partial_x^3 N^2}{2}\right)\right\}
\]

\[
+ \frac{1}{n} [h''(n)(\partial_t N \partial_m \partial_t N + \partial_m N \partial_t \partial_t N + \partial_t N \partial_t \partial_m N) + h'''(n)\partial_t N \partial_m \partial_t N] \mathbf{u} \cdot \nabla \partial_x^3 N\} \, dx,
\]

\[
I_{13} = \int f_1 f_2 f_3 f_4 \, dx
\]

\[= \int \{\text{the sums of all terms without smallness of the maximum norm of exact three multipliers}\} \, dx\]
(in the following we will give its exact form) and \(I_1^{(R)} = I_1 - I_{11} - I_{12} - I_{13} = \int fgh \, dx\) denotes the remained terms integral in which the maximum norm of one of the functions \(f, g, h\) is small enough and the integral of the other functions contribute to \(\| (\partial_x N, \partial_x u) \|_{H^2(R^d)}^2 + \| \partial_x N_t \|_{H^1(R^d)}^2\). Here and in the following we will use \(\cdots\) to denote those terms which are ones of the same structure as the first term before them.

Now we estimate each integral in (2.50).

\[
I_1^{(R)} \geq -C\delta_1(\| \partial_x N, \partial_x u \|_{H^2(R^d)}^2 + \| \partial_x N_t \|_{H^1(R^d)}^2), \tag{2.51}
\]

\[
I_{11} \geq \frac{d}{dt} \int \left( \frac{h''(n)}{2n} |\partial_x N|^2 + \frac{1}{n} h''(n) (\partial_h N \partial_m \partial_t N + \partial_m N \partial_h \partial_t N + \partial_t N \partial_h \partial_m N) + h''(n) \partial_h N \partial_m N \partial_t N \partial_h \partial_m N \right) dx \tag{2.52}
\]

\[
-I_1^{(R)} \leq \int \left[ \frac{h''(n)}{n} (\partial_h N \partial_m \partial_t N + \cdots) \partial_h \partial_m N \partial_t N \right] dx.
\]

For \(I_{12}\), by integration by parts, we have

\[
I_{12} = -\int \left\{ dv \left( \frac{h''(n)}{n} u \right) |\partial_t N|^2 + \partial_h \partial_m \partial_t N \nabla \left( \frac{1}{n} h''(n) (\partial_h N \partial_m \partial_t N + \partial_m N \partial_h \partial_t N + \partial_t N \partial_h \partial_m N) \right) \right\} dx \tag{2.53}
\]

\[
\geq -C\delta_1 \| \partial_x N \|_{H^2(R^d)}^2 + I_{12}^{(R)},
\]

where

\[
I_{12}^{(R)} = \int \left[ \frac{h''(n)}{n} (u \cdot \nabla \partial_h N \partial_m \partial_t N + \cdots) \partial_h \partial_m N \partial_t N \right] dx.
\]

The rest is to deal with the integrals \(I_{11}^{(R)}, I_{12}^{(R)}\) and \(I_{13}\), which is harder to deal with due to the loss of smallness of the maximum norm of exact three multipliers. This difficulty caused by the dimension of the position space can be overcome by employing Young’s inequality and Gagliardo-Nirenberg’s inequality.

Denote \(I_2 = I_{11}^{(R)} + I_{12}^{(R)} + I_{13}\), then by direct calculations and using Young’s inequality (1.6), we have

\[
I_2 = \int \cdot dx \left\{ \frac{h''(n)}{n} (\partial_h \partial_m u \cdot \nabla \partial_t N + \cdots) \partial_h \partial_m \partial_t N \right\} - \frac{h''(n)}{n} \partial_h \partial_m \partial_t N \partial_h N_t + \partial_h \partial_m \partial_t N \partial_h \partial_m \partial_t N - \cdots
\]

\[
+ \frac{h''(n)}{n} (\partial_h N_t \partial_m \partial_t N \partial_h \partial_m \partial_t N + \cdots + u \cdot \nabla \partial_t N \partial_h \partial_m \partial_t N - \cdots + \partial_h \partial_m \partial_t N \partial_h N \partial_m \partial_t N - \cdots)
\]

\[
\geq -\epsilon \| (\partial_x^2 N, \partial_x^3 N) \|^2 - C(\epsilon) \int (|\partial_x^2 u|^4 + |\partial_x^2 N|^4 + |\partial_x N_t|^4) \, dx,
\]
where $\epsilon$ is a positive constant independent of $\delta_1$ (only depending on $B$).

Using (2.7), (2.6), (1.7), (2.4) and $d = 2, 3$, we have

$$
\begin{align*}
&f(|\partial_x^3 u|^4 + |\partial_x^2 N|^4 + |\partial_x N|^4)dx \\
&\leq C f(|\partial_x^2 u|^4 + |\partial_x^2 N|^4)dx + C\delta_1 \|\partial_x u\|^2 \\
&\leq C \|\partial_x^2 u\|^{4-d} \|\partial_x \partial_x^2 u\|^d + C \|\partial_x^2 N\|^{4-d} \|\partial_x \partial_x^2 N\|^d + C\delta_1 \|\partial_x u\|^2 \\
&\leq C \|\partial_x^2 u\|^{4-d} \|\partial_x \partial_x^2 u\|^{d-2} \|\partial_x \partial_x^2 u\|^2 \\
&+ C \|\partial_x^2 N\|^{4-d} \|\partial_x \partial_x^2 N\|^{d-2} \|\partial_x \partial_x^2 N\|^2 + C\delta_1 \|\partial_x u\|^2 \\
&\leq C\delta_1 \|\partial_x \partial_x^2 u\|^2 + C\delta_1 \|\partial_x \partial_x^2 N\|^2 + C\delta_1 \|\partial_x u\|^2.
\end{align*}
$$

(2.55)

In the above last inequality we have used the smallness of $\|(N, u)\|_{H^3(B^i)}$, see (2.4).

Thus, (2.54), together with (2.55), gives

$$
I_2 \geq -\epsilon \|(\partial_x^2 N, \partial_x^2 N)\|^2 - C\delta_1 \|\partial_x \partial_x^2 u\|^2 - C\delta_1 \|\partial_x \partial_x^2 N\|^2 - C\delta_1 \|\partial_x u\|^2.
$$

(2.56)

Combining (2.50), (2.51), (2.52), (2.53) and (2.56), we have

$$
\begin{align*}
-f \partial_x^3 (h(n) - h(B)) \partial_x^3 \text{div} u dx \\
\geq \frac{4}{d} \int \left\{ \frac{h(n)}{2n} |\partial_x^3 N|^2 + \frac{1}{n} h''(n) (\partial_t N \partial_m \partial_t N + \partial_m N \partial_t \partial_t N + \partial_t N \partial_m \partial_t N) \right. \\
+ h''(n) \partial_t N \partial_m \partial_t N \left[ \partial_t \partial_m \partial_t N \right] dx \\
-C\delta_1 \|(\partial_x^3 N, \partial_x^2 N, \partial_x N, \partial_x^2 u, \partial_x^3 u, \partial_x u)\| - \epsilon \|\partial_x^3 N\|^2
\end{align*}
$$

(2.57)

with the help of (2.7).

As in (2.39), by using (2.49)$_2$, we have

$$
\begin{align*}
-f \partial_x^3 \text{e} \partial_x^3 u dx &= f \partial_x^3 \text{e} \partial_x^3 \text{div} u dx \\
&= -\frac{1}{B} f \partial_x^3 \text{e} \partial_x^3 (N_t + \text{div}(Nu)) dx \\
&= \frac{1}{B} f \partial_x^3 \text{e} \partial_x^3 u_t dx + \frac{1}{B} f \partial_x^3 \text{e} \partial_x^3 (Nu) dx \\
&\geq \frac{1}{2B} \frac{d}{dt} \|\partial_x^3 \text{e}\|^2 - C\delta_1 \|(\partial_x^3 N, \partial_x^2 N, \partial_x^2 u, \partial_x^3 u, \partial_x^3 e)\|^2.
\end{align*}
$$

(2.58)
By direct calculations and using integration by parts, we have

\[ f \partial_x^2 (u \cdot \nabla u) \partial_x^2 u dx \]

\[ = - f (\partial_h \partial_m \partial_t u \cdot \nabla u + \partial_m \partial_t u \cdot \nabla \partial_h u + \cdots \]

\[ + \partial_t u \cdot \nabla \partial_h \partial_m u + \cdots + u \cdot \nabla \partial_h \partial_m \partial_t u ) \partial_h \partial_m \partial_t u dx \]

(2.59)

\[ \geq - C \delta_1 \int |\partial_x^2 u|^2 dx - \epsilon \int |\partial_x^2 u|^2 dx - C(\epsilon) \int |\partial_x^2 u|^4 dx \]

\[ \geq -(C \delta_1 + \epsilon) \| \partial_x^2 u \|^2 - C \delta_1 \| \partial_x u \|^2 \]

with the help of Cauchy-Schwarz’s inequality (1.6), (2.55) and the smallness of \(|u|\) and \(|\nabla u|\).

Thus, (2.48), together with (2.57), (2.58) and (2.59), implies

\[ \frac{d}{dt} \int \left\{ \frac{1}{2} |\partial_x^2 u|^2 + \frac{1}{2n} |\partial_x^2 e|^2 + \frac{h''(n)}{2n} |\partial_x^2 N|^2 + \frac{1}{n} [h^n(n)(\partial_h N \partial_m \partial_t N)

\[ + \partial_m N \partial_h \partial_t N + \partial_h N \partial_m \partial_t N] + h''(n) \partial_h N \partial_m \partial_t N \partial_h \partial_m \partial_t N \right\} dx \]

(2.60)

\[ + C \| \partial_x^2 u \|^2 - C \delta_1 \| (\partial_x^2 N, \partial_x^3 N, \partial_x u, \partial_x^2 u, \partial_x^3 u) \|^2 - \epsilon \| \partial_x^2 u \|^2 \leq 0 \]

with the help of the smallness of \( \epsilon \) and \( \delta_1 \).

As in (2.42), one easily obtain

\[ \| (\partial_x^2 N, \partial_x^3 N) \|^2 \leq C \| (\partial_x u, \partial_x^2 u, \partial_x^3 u) \|^2. \] (2.61)

Similarly to (2.43), we have

\[ \frac{1}{2} \frac{d}{dt} \| \partial_x^2 u \|^2 + \| \partial_x^2 u \|^2 - \int |\partial_x^2 (h(n) - h(B))_t - \partial_x^2 \Phi_t | \partial_x^2 u dx \]

\[ + \int \partial_x^2 (u \cdot \nabla u) \partial_x^2 u dx = 0. \] (2.62)

As in (2.44) and (2.57), by some tedious but straightforward calculations, we can obtain

\[ - \int (\partial_x^2 (h(n) - h(B))_t - \partial_x^2 \Phi_t | \partial_x^2 u dx \]

\[ = \int \partial_x^2 (h'(n) N_t) \partial_x^2 (\frac{N_t + u \cdot \nabla N}{n})_t dx - \frac{1}{B} \int \partial_x^2 \Phi_t \partial_x^2 (N_t + \text{div}(N u))_t dx \]

\[ \geq \frac{d}{dt} \int \frac{h'(n)}{2n} |\partial_x^2 N_t|^2 + \frac{1}{2B} |\partial_x^2 e_t|^2 dx \]

(2.63)

\[ - C \delta_1 \| (\partial_x^2 u, \partial_x^3 u, \partial_x^2 u, \partial_x u, \partial_x^2 N, \partial_x N, \partial_x^2 N, \partial_x^3 N) \|^2 \]

\[ \geq \frac{d}{dt} \int \frac{h'(n)}{2n} |\partial_x^2 N_t|^2 + \frac{1}{2B} |\partial_x^2 e_t|^2 dx \]

\[ - C \delta_1 \| (\partial_x u, \partial_x^2 u, \partial_x^3 u, \partial_x u, \partial_x^2 u, \partial_x N, \partial_x^2 N, \partial_x^3 N, \partial_x e_t) \|^2 \]

and

\[ \int \partial_x^2 (u \cdot \nabla u) \partial_x^2 u dx \geq -C \delta_1 \| (\partial_x u, \partial_x^2 u, \partial_x^3 u) \|^2. \] (2.64)
Thus, (2.62), together with (2.63) and (2.64), implies

$$
\frac{d}{dt} \int \left( \frac{1}{2} |\partial_x^2 u|^2 + \frac{n}{2} |\partial_x^2 N|^2 + \frac{1}{2B} |\partial_x^2 e_t|^2 \right) dx + C \|\partial_x^2 u\|^2 \\
\leq C \delta_1 \|\partial_x^2 u, \partial_x^2 N, \partial_x^2 e_t\|^2 + C \|\partial_x u, \partial_x u_t, \partial_x^2 u, \partial_x N, \partial_x^2 N\|^2.
$$

(2.65)

Combining (2.60), (2.61), (2.65), (2.7) with $i = 2$, (2.8) with $j = 3$ and (2.9) with $m = 2$, we can obtain

$$
\frac{d}{dt} \int \left\{ \frac{1}{2} |\partial_x^2 u|^2 + \frac{1}{2B} |\partial_x^2 e_t|^2 + \frac{h'(n)}{2n} |\partial_x^2 N|^2 + \frac{1}{n} |h''(n)| (\partial_x N \partial_x N_t + \partial_x N \partial_x N_t) \right. \\
+ \frac{1}{2} |\partial_x^2 u_t|^2 + \frac{h'(n)}{2n} |\partial_x^2 N_t|^2 + \frac{1}{2B} |\partial_x^2 e_t|^2 \left. \right\} dx \\
+ C \|\partial_x^2 N_t, \partial_x^2 N, \partial_x^2 u, \partial_x^2 u_t, \partial_x^2 e_t\|^2 \leq C \|\partial_x u, \partial_x^2 u, \partial_x N, \partial_x^2 N\|^2.
$$

(2.66)

with the help of the smallness of $\epsilon$ and $\delta_1$.

Finally, combining (2.36), (2.47) and (2.66), we have, with the help of the smallness of $\delta_1$, that

$$
\frac{d}{dt} \int G dx + C \|(N, u, e)\|_{H^s}^2 + \|(N_t, u_t, e_t)\|_{H^2}^2 \leq 0,
$$

(2.67)

where

$$
G = A_1 \{ A_2 |u|^2 + |u_t|^2 + \int_0^N \frac{h'(B)}{s+B} ds + \frac{h'(n)}{n} N_t^2 + |e|^2 + \lambda \left( |\nabla e|^2 + |e_t|^2 \right) \\
+ |\nabla u|^2 + \frac{1}{B} |N|^2 + \frac{h'(n)}{n} |\nabla N|^2 \} \left\{ \frac{1}{2} |\partial_x^2 u|^2 + \frac{1}{2B} |\partial_x^2 e_t|^2 + \frac{h'(n)}{2n} |\partial_x^2 N|^2 \right. \\
+ \frac{h''(n)}{n} \partial_x N \partial_x N_t + \frac{1}{2} |\partial_x^2 u_t|^2 + \frac{h'(n)}{2n} |\partial_x^2 N_t|^2 + \frac{1}{2B} |\partial_x e_t|^2 \left. \right\} \\
+ \left\{ \frac{1}{2} |\partial_x^2 u|^2 + \frac{1}{2B} |\partial_x^2 e_t|^2 + \frac{h'(n)}{2n} |\partial_x^2 N|^2 \right. \\
+ \frac{1}{n} |h''(n)(\partial_x N) \partial_x N_t + \partial_x N \partial_x N_t + \partial_x N \partial_x N_t + \partial_x N \partial_x N_t + \partial_x N \partial_x N_t + \partial_x N \partial_x N_t \right\} \left( \partial_x N \partial_x N_t + \partial_x N \partial_x N_t + \partial_x N \partial_x N_t \right) dx \\
+ \frac{1}{2} |\partial_x^2 u_t|^2 + \frac{h'(n)}{2n} |\partial_x^2 N_t|^2 + \frac{1}{2B} |\partial_x e_t|^2 \right\}
$$

for some positive constants $A_1$ and $A_2$.

Using Young’s inequality (1.6), estimates (2.6) and the properties (2.13) of the entropy function $h(n)$, we have

$$
| \int \left\{ \frac{h''(n)}{n} \partial_x N \partial_x N_t + \frac{1}{n} |h''(n)(\partial_x N) \partial_x N_t + \partial_x N \partial_x N_t + \partial_x N \partial_x N_t + \partial_x N \partial_x N_t + \partial_x N \partial_x N_t \right\} dx | \\
\leq C \delta_1 \|N\|_{H^1(R^d)}^2.
$$
Thus, it is easy to see that $G$ satisfies

$$c\left(\left\| (N, u, e)(\cdot, t) \right\|^2_{H^3} + \left\| (N_t, u_t, e_t)(\cdot, t) \right\|^2_{H^2}\right) \leq \int G dx$$

(2.68)

for some positive constants $C > c$ and $A_1, A_2 > 0$.

Thus, (2.67) and (2.68) implies (2.5). And Proposition 2.1 is proved.

Theorem 1.2 follows from the standard argument by using the local existence theorem (Proposition (1.1)) and the a priori estimates given in Proposition 2.1, for detail, see e.g. [8].

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References


