Quasineutral limit of a nonlinear drift diffusion model for semiconductors

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\textbf{Abstract:} The limit of vanishing Debye length (charge neutral limit ) in a nonlinear bipolar drift-diffusion model for semiconductors without pn-junction (i.e. with a unipolar background charge ) is studied. The quasineutral limit (zero-Debye-length limit) is performed rigorously by using the so-called entropy functional which yields appropriate uniform estimates.

\textbf{Keywords:} Quasineutral limit, nonlinear drift-diffusion equations, entropy method

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1 Introduction and Formal Asymptotics

The scaled semiconductor drift-diffusion equations read

\[
\begin{aligned}
n^\lambda_t &= \mu_d \text{div}(\nabla (n^\lambda)^m + n^\lambda E^\lambda) \\
p^\lambda_t &= \mu_p \text{div}(\nabla (p^\lambda)^m - p^\lambda E^\lambda) \\
-\lambda^2 \text{div}E^\lambda &= n^\lambda - p^\lambda - C
\end{aligned}
\]

with \( x \in \Omega \subset \mathbb{R}^d \), \( \Omega \) bounded with smooth boundary, \( t \geq 0 \) and \( E^\lambda = -\nabla \Phi^\lambda \). The unknowns \( n^\lambda, p^\lambda, E^\lambda, \Phi^\lambda \) are the electron density, the hole density, the electric field and the electric potential, respectively. The given function \( C = C(x) \) is the doping profile

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describing fixed background charges. The dimensionless positive parameters \( \mu_n, \mu_p \) and \( \lambda \) are the scaled mobilities of electrons and holes and the scaled Debye length, respectively. The constants \( \gamma_n, \gamma_p > 0 \) are the adiabatic or isothermal (if \( \gamma_n = \gamma_p = 1 \)) exponents. The regime \( 0 < \gamma_n < 1 \) (or \( 0 < \gamma_n < 1 \)) describes a fast diffusion process in the electron (hole) density, whereas \( \gamma_n > 1 \) (or \( \gamma_n > 1 \)) is related to slow diffusion.

We consider an insulated semiconductor modeled by the initial-boundary value problem for (1.1) subject to the boundary and initial conditions:

\[
(\nabla(n^\lambda)^{\gamma_n} + n^\lambda E^\lambda) \cdot \nu = 0,
\]

\[
(\nabla(p^\lambda)^{\gamma_p} - p^\lambda E^\lambda) \cdot \nu = 0,
\]

\[
E^\lambda \cdot \nu = 0, \quad x \in \partial \Omega,
\]

\[
n^\lambda(t = 0, x) = n_0^\lambda(x), \quad p^\lambda(t = 0, x) = p_0^\lambda(x), \quad x \in \Omega,
\]

where \( \nu \) is the normal vector along the boundary \( \partial \Omega \).

A necessary solvability condition for the Poisson equation (1.1) subject to the Neumann boundary conditions for the field in (1.2) is global charge neutrality,

\[
\int_\Omega (n^\lambda - p^\lambda - C) dx = 0.
\]

Since the total numbers of electrons and holes are conserved, it is sufficient to require the corresponding condition for the initial data:

\[
\int_\Omega (n_0^\lambda - p_0^\lambda - C) dx = 0. \tag{1.4}
\]

We are mainly interested in the behavior of the solutions of the problem (1.1)-(1.4) in the vanishing Debye length limit \( \lambda \to 0 \). It is important to mention that the quasineutral limit is a well-known challenging problem for the (bipolar) hydrodynamic model and for the kinetic Vlasov Poisson model. In both cases there exist only partial results concerning the quasineutral limit [BG, CG, G1, G2, GLMS], but the full problem is still unsolved. Therefore, it is natural to study the quasineutral limit on the level of the drift-diffusion model. The existence analysis of the bipolar drift-diffusion problem in the isothermal case was done by [GaI] and in the case \( 1 < \gamma_n, \gamma_p \) by [J]. To our knowledge, in the fast diffusion regime there are no existence results available.

Before stating our main results, we perform the quasineutral limit \( \lambda \to 0 \) formally in the system (1.1). Setting \( \lambda = 0 \) in (1.1) we obtain the system

\[
\begin{aligned}
\frac{d n}{dt} &= \mu_n div (\nabla n^{\gamma_n} + n E) \\
\frac{d p}{dt} &= \mu_p div (\nabla p^{\gamma_p} - p E) \\
0 &= n - p - C,
\end{aligned}
\]

where \( n, p, E = -\nabla \Phi \) are the formal limits of \( n^\lambda, p^\lambda, E^\lambda \) as \( \lambda \to 0 \).

Because of the singular perturbation character of the problem (the Poisson equation becomes an algebraic equation in the limit) we cannot a priori expect that all initial and
boundary conditions hold for the limiting problem. However, by the conservation of the continuity equations the property of zero flux through the boundary will prevail in the limit:

$$\nabla n^\gamma + nE \cdot \nu = 0, \quad \nabla (p^\gamma - pE) \cdot \nu = 0 \quad \text{on} \quad \partial \Omega,$$

while the boundary condition for the electric field $E^\lambda$ does not. Initial conditions for the limiting problem are satisfied in the sense of $H^{-1}(\Omega)$.

Simple manipulations of (1.5) give the parabolic-elliptic system (see [GLMS])

$$\begin{cases}
\left( \frac{1}{\mu_n} + \frac{1}{\mu_p} \right) p_t = div(\nabla ((p + C)^\gamma_n + p^\gamma) + CE) \\
- div\left( (\mu_n + \mu_p) p + \mu_n C \right) E = div(\nabla (\mu_n (p + C)^\gamma_n - \mu_p p^\gamma)).
\end{cases}$$

If, further, $C \equiv 0$ the limiting problem becomes

$$p_t = D div(\nabla (p^\gamma + p^\gamma)), \quad D = \frac{\mu_n \mu_p}{\mu_n + \mu_p}.$$ (1.8)

This fact is well-known in plasma physics, see [C, G1, G2, JP].

The quasineutral limit $\lambda \to 0$ has been carried out formally in [R]. However, there are very few rigorous results concerning the quasineutral limit in the drift-diffusion equations for semiconductors [G1, G2, GLMS, R]. To our knowledge the first and only rigorous result for the standard ( isothermal $\gamma_n = \gamma_p = 1$) drift-diffusion case was obtained in [GLMS], where the two cases were discussed: either $C \equiv 0$ or $C$ does not change sign. For the isentropic case $\gamma_n, \gamma_p > 1$, Gasser [G1, G2] performed this limit rigorously under two assumptions, namely, the initial data have to be equal ($n_0 = p_0$) and $C \equiv 0$. A similar analysis for related models of plasmas physics has been carried out in [BGS, CG, JP].

We study the quasineutral limit rigorously in the present paper for the general case $\gamma_n > 0, \gamma_p > 0$ when the doping profile is a constant or does not change sign, generalizing the results of [GLMS] to nonlinear adiabatic diffusion. Also, in this paper we employ multiplier techniques instead of the invariant region method used in [GLMS] in order to obtain lower and upper bounds on the densities.

The paper is organized in the following way. Section 2 is devoted to a priori estimates and section 3 gives rigorous results of the quasineutral limit.

## 2 A Priori Estimates.

We present a priori estimates on the solutions of (1.1)-(1.3) in this section. The main tools are the entropy method and energy arguments.

**Lemma 2.1** ([G1, G2]) Let $n^\lambda_0, p^\lambda_0 \in L^q, 1 \leq q \leq +\infty$ be nonnegative and uniformly in $\lambda$. Let $C \in L^\infty(\Omega)$. Then the solutions of (1.1)-(1.3) satisfy $n^\lambda, p^\lambda \geq 0$,

$$\|n^\lambda(t, \cdot)\|_{L^1(\Omega)} = \|n^\lambda_0(\cdot)\|_{L^1(\Omega)}, \quad \|p^\lambda(t, \cdot)\|_{L^1(\Omega)} = \|p^\lambda_0(\cdot)\|_{L^1(\Omega)}.$$ (2.1)

Moreover, there exists a positive constant $M(\lambda)$ such that

$$\|n^\lambda(t, \cdot)\|_{L^2(\Omega)} + \|p^\lambda(t, \cdot)\|_{L^4(\Omega)} \leq M(\lambda).$$ (2.2)
Since our results are strongly based on these estimates we repeat the proof in appendix 1. To establish some estimates uniformly in \( \lambda \), we introduce the entropy

\[
e^\lambda(t) = \int_{\Omega} \left( n^\lambda \frac{(n^\lambda)^{\gamma_n-1}}{\gamma_n-1} + p^\lambda \frac{(p^\lambda)^{\gamma_p-1}}{\gamma_p-1} + \frac{\lambda^2}{2} |E^\lambda|^2 \right) dx + e_0,
\]

where \( \left( \frac{\lambda^{\gamma_n-1}}{\gamma_n-1} \right) |_{\gamma_n=1} := lnA \) for \( A > 0 \) and the constant \( e_0 \) is chosen such that the entropy \( e^\lambda(t) \) is a nonnegative quantity. An easy calculation gives (see appendix 2 for details)

**Lemma 2.2** Let \( t > 0 \). Then

\[
\frac{d}{dt} e^\lambda(t) = - \int_{\Omega} \left( \mu_n \frac{\nabla (n^\lambda)^{\gamma_n} + n^\lambda E^\lambda}{n^\lambda} + \mu_p \frac{\nabla (p^\lambda)^{\gamma_p} - p^\lambda E^\lambda}{p^\lambda} \right) dx
\]

holds for the solutions of system (1.1)-(1.3).

As we will see in the proof of theorem 3.1 this gives \( L^\infty(\Omega) \) \((L^{p^\nu}(\Omega))\) estimates on \( n^\gamma_n \) \((p^{p^\nu})\). Under additional assumption \((C = const)\) we obtain the following \( L^\infty(\Omega) \) estimates.

**Lemma 2.3** Assume that \( C = const \). Assume \( n^0_0, p^0_0 \in L^\infty(\Omega) \) uniformly in \( \lambda \in (0, 1] \), \( n^\lambda_0 \geq \delta + C \geq 0, p^\lambda_0 \geq \delta \) for some positive constant \( \delta \) independent of \( \lambda \). Then there exist positive constants \( M_1, M_2 \) independent of \( \lambda \) such that the solution of (1.1)-(1.3) satisfies

\[
0 < M_1 \leq n^\lambda, p^\lambda \leq M_2 < +\infty.
\]

**Proof** Choose positive constants \( b_\alpha \) independent of \( \lambda \), where \( \alpha = +, - \) such that \( b_+ \geq \| p^0(\cdot) \|_{L^\infty} \) and \( b_+ + C \geq \| n^0(\cdot) \|_{L^\infty} \) and \( b_- = \delta \leq p^\lambda_0 \). Set \( a_\alpha = b_\alpha + C \). Denote \( z_+ = \max\{ z, 0 \} \) and \( z_- = \min\{ z, 0 \} \).

Using \((n^\lambda-a_\alpha)_\alpha, (p^\lambda-b_\alpha)_\alpha)\) as test functions in Eqns (1.1)1,2 respectively and employing the Poisson equation (1.1)3, it turns out that

\[
\frac{1}{2} \int_{\Omega} (n^\lambda - a_\alpha)^2 dx + \int_{\Omega} \mu_n \nabla (n^\lambda)^{\gamma_n} \cdot \nabla (n^\lambda - a_\alpha) dx ds
\]

\[
= -\mu_n \int_{\Omega} n^\lambda E^\lambda \cdot \nabla (n^\lambda - a_\alpha) dx ds
\]

\[
= \mu_n \int_{\Omega} ((n^\lambda - a_\alpha)_\alpha^2 + a_\alpha (n^\lambda - a_\alpha)) div E^\lambda dx ds
\]

\[
= -\frac{\mu_n}{\lambda} \int_{\Omega} ((n^\lambda - a_\alpha)_\alpha^2 + a_\alpha (n^\lambda - a_\alpha)) (n^\lambda - p^\lambda - C) dx ds
\]

\[
= -\frac{\mu_n}{\lambda} \int_{\Omega} ((n^\lambda - a_\alpha)_\alpha^2 + a_\alpha (n^\lambda - a_\alpha) ((n^\lambda - a_\alpha) - (p^\lambda - b_\alpha))) dx ds.
\]

Similarly

\[
\frac{1}{2} \int_{\Omega} (p^\lambda - b_\alpha)^2 dx + \int_{\Omega} \mu_p \nabla (p^\lambda)^{\gamma_p} \cdot \nabla (p^\lambda - b_\alpha) dx ds
\]

\[
= \frac{\mu_p}{\lambda} \int_{\Omega} ((p^\lambda - b_\alpha)_\alpha^2 + b_\alpha (p^\lambda - b_\alpha) ((n^\lambda - a_\alpha) - (p^\lambda - b_\alpha))) dx ds
\]
Multiplying (2.4) by $\frac{\mu_n a}{\mu_p a}$ and adding (2.3), we find

\begin{align*}
\frac{1}{2} \int_\Omega ((n^\lambda - a_\alpha)^2 + \frac{\mu_n a}{\mu_p a} (p^\lambda - b_\alpha)^2) dx \\
+ \int_0^t \int_\Omega (\mu_n \nabla (n^\lambda)^\gamma \cdot \nabla (n^\lambda - a_\alpha) + \frac{\mu_n a}{\mu_p a} \nabla (p^\lambda)^\gamma \cdot \nabla (p^\lambda - b_\alpha))_a dx ds \\
= -\frac{\mu_n a}{\mu_p a} \int_0^t \int_\Omega ((n^\lambda - a_\alpha) a - (p^\lambda - b_\alpha)_a)((n^\lambda - a_\alpha) - (p^\lambda - b_\alpha)) dx ds \\
-\frac{1}{\lambda} \int_0^t \int_\Omega (\mu_n (n^\lambda - a_\alpha) - \frac{\mu_n a}{\mu_p a} (p^\lambda - b_\alpha)_a)((n^\lambda - a_\alpha) - (p^\lambda - b_\alpha)) dx ds.
\end{align*}

(2.5)

Since the function $z \mapsto z_\alpha$ is non-decreasing, we obtain, with the help of (2.5) and Lemma 2.1, that there exists a positive constant $M_3(\lambda)$ such that

\begin{align*}
\int_\Omega ((n^\lambda - a_\alpha)^2 + (p^\lambda - b_\alpha)^2) dx \\
\leq M_3(\lambda) \int_0^t \int_\Omega ((n^\lambda - a_\alpha)^2 + (p^\lambda - b_\alpha)^2) dx ds.
\end{align*}

Applying the Gronwall’s inequality, we have

\[ \int_\Omega ((n^\lambda - a_\alpha)^2 + (p^\lambda - b_\alpha)^2) dx = 0, \]

which gives the results. This completes the proof of Lemma 2.3.

In case of non-constant doping profile we can show the following a.e.-bounds on the solutions.

**Lemma 2.4** Let $C(x) \geq C_\geq > 0$ and $n^\lambda_0 \geq \delta > 0$ in $\Omega$ for some $\delta$ independent of $\lambda$. Then there is a constant $M > 0$ independent of $\lambda$ such that $n^\lambda \geq M$ on $R^+ \times \Omega$. If, alternatively, $C(x) \leq -C < 0$ and $p^\lambda_0 \geq \delta > 0$, then $p^\lambda \geq M > 0$ on $R^+ \times \Omega$.

**Proof** As in the proof of Lemma 2.3 we multiply the equation for $n^\lambda$ in (1.1) by the test function $(n^\lambda - a_-)_-$ and obtain

\begin{align*}
\frac{1}{2} \int_\Omega (n^\lambda - a_-)^2 dx - \frac{1}{2} \big| \int_\Omega (n^\lambda - a_-)^2 |_{t=0} dx \\
+ \int_0^t \int_\Omega \mu_n \nabla (n^\lambda)^\gamma \cdot \nabla (n^\lambda - a_\alpha)_a dx ds \\
= -\frac{\mu_n a}{\mu_p a} \int_0^t \int_\Omega (n^\lambda - a_-) (n^\lambda - p^\lambda - C)_a dx ds \\
-\frac{1}{\lambda} \int_0^t \int_\Omega a_- (n^\lambda - a_-)_- ((n^\lambda - a_-) + (a_- - p^\lambda - C)) dx ds.
\end{align*}

(2.6)

We take $a_- = \min(C_\geq, \delta)$ and thus $n^\lambda(0, x) - a_- \leq 0, a_- - p^\lambda - C \leq 0$ hold. Therefore, the second integral on the right hand side of (2.6), with the help of $zz_- \geq 0$, is non-positive. The first integral on the right hand side can be estimated by (using Lemma 2.1)

\[ -\frac{\mu_n}{\lambda^2} \int_0^t \int_\Omega (n^\lambda - a_-)^2 (n^\lambda - p^\lambda - C) dx ds \leq M_4(\lambda) \int_0^t \int_\Omega (n^\lambda - a_-)^2 dx ds. \]

Thus, we have

\[ \frac{1}{2} \int_\Omega (n^\lambda - a_-)^2 dx \leq M_4(\lambda) \int_0^t \int_\Omega (n^\lambda - a_-)^2 dx ds. \]
Applying the Gronwall’s inequality, we have

$$\int_{\Omega} (n^\lambda - a_-)^2 \, dx = 0,$$

which gives the assertion for $n^\lambda$. The same method can be applied to the $p^\lambda$ equation in (1.1) and then the assertion for $p^\lambda$ holds. This completes the proof of Lemma 2.4.

3 The Limit $\lambda \to 0$

We establish the main result on the quasineutral limit of (1.1)-(1.3) in this section.

**Theorem 3.1** Assume the initial data $n_0^\lambda, p_0^\lambda \geq 0$ are such that the initial entropy $e^\lambda(t = 0)$ is uniformly bounded as $\lambda \to 0$ and that there are functions $n_0, p_0 \in L^\infty(\Omega)$ such that $n_0^\lambda \to n_0, p_0^\lambda \to p_0$ strongly in $L^\infty(\Omega)$ as $\lambda \to 0$. Also, let one of the following assumptions hold:

(A) (i) $C(x) \equiv \text{const.}$,

(ii) there exists a positive constant $\delta$ independent of $\lambda$ such that $n_0^\lambda \geq \delta + C > 0, p_0^\lambda \geq \delta$ in $\Omega$,

(iii) $0 < \gamma_n, \gamma_p < \infty$;

(B) (i) there exists a positive constant $C$ such that $C(x) \geq C > 0$ (or $C(x) \leq -C < 0$) and $C(x) \in W^{1, \infty}(\Omega)$,

(ii) $n_0^\lambda, p_0^\lambda$ are bounded away from 0 uniformly as $\lambda \to 0$,

(iii) $1 \leq \gamma_p \leq \frac{3}{2}, \quad \frac{2d}{d+1} < \gamma_n$ (or $1 \leq \gamma_n \leq \frac{3}{2}, \quad \frac{2d}{d+1} < \gamma_p$).

Let $T > 0$ and $Q_T = (0, T) \times \Omega$. Then, as $\lambda \to 0$ the following convergences hold (after extracting subsequences):

$$\begin{align*}
&\cdot n^\lambda \to n \text{ strongly in } L^{q_n}(Q_T); \\
&\cdot p^\lambda \to p \text{ strongly in } L^{q_p}(Q_T); \\
&\cdot E^\lambda \to E \text{ weakly in } L^s(Q_T); \\
&\cdot n^\lambda - p^\lambda - C = O(\lambda) \text{ in } L^2(Q_T)
\end{align*}$$

where $q_n, q_p, s > 1$ depend on $\gamma_n$ and $\gamma_p$. Furthermore, the limit $(n, p, E)$ satisfies the system (1.5)-(1.6) in $D'(Q_T)$ and the initial data $n(t = 0, x) = n_0(x), p(t = 0, x) = p_0(x)$ in the sense of $H^{-1}(\Omega)$.

**Proof**

**Step 1: uniform estimates.**

Using the assumptions on the initial data and the entropy inequality in Lemma 2.2, we get

$$\frac{\nabla(n^\lambda)^m + n^\lambda E^\lambda}{n^\lambda^{\frac{3}{2}}}, \quad \frac{\nabla(p^\lambda)^n - p^\lambda E^\lambda}{(p^\lambda)^{\frac{3}{2}}} \in L^2(Q_T) \text{ uniformly in } \lambda.$$ 

(3.2)
Also, with
\[
\frac{(n^\lambda)^m}{\gamma_n - 1} \leq \left| n^\lambda \frac{(n^\lambda)^{m-1}}{\gamma_n - 1} \right| + \frac{n^\lambda}{\gamma_n - 1}, \quad \gamma_n > 1
\]
we conclude
\[
n^\lambda \in L^\infty((0, t); L^{s_1}(\Omega)), 1 \leq s_1 \leq \gamma_n,
\]
\[
p^\lambda \in L^\infty((0, t); L^\tau(\Omega)), 1 \leq \tau \leq \gamma_p
\]
uniformly in \(\lambda\).

Then, we multiply the Poisson equation by \((n^\lambda - p^\lambda - C)/\lambda^2\), use the equations (1.1) and obtain
\[
\int_0^t \Omega \left( \frac{(n^\lambda)^{2-\gamma_n}}{\gamma_n} + \frac{(p^\lambda)^{2-\gamma_p}}{\gamma_p} \right) (E^\lambda)^2 + \frac{(n^\lambda - p^\lambda - C)^2}{\lambda^2} \right) dx ds \\
= \int_0^t \Omega \left( \frac{(n^\lambda)^{2-\gamma_n}}{\gamma_n} \cdot \nabla (n^\lambda)^{\gamma_n + n^\lambda E^\lambda} + \frac{(p^\lambda)^{2-\gamma_p}}{\gamma_p} \cdot \nabla (p^\lambda)^{\gamma_p - p^\lambda E^\lambda} \right) \cdot E^\lambda dx ds \\
- \int_0^t \Omega \left( \frac{(n^\lambda)^{2-\gamma_n}}{\gamma_n} \cdot \nabla (n^\lambda)^{\gamma_n + n^\lambda E^\lambda} + \frac{(p^\lambda)^{2-\gamma_p}}{\gamma_p} \cdot \nabla (p^\lambda)^{\gamma_p - p^\lambda E^\lambda} \right) \cdot |E^\lambda| dx ds \\
+ \int_0^t \Omega \left| E^\lambda \right| \cdot |\nabla C| dx ds.
\]

For the first integral on the right hand side of (3.4), we have, with the help of Young’s inequality, that
\[
\int_0^t \Omega \left( \frac{(n^\lambda)^{2-\gamma_n}}{\gamma_n} \cdot \nabla (n^\lambda)^{\gamma_n + n^\lambda E^\lambda} + \frac{(p^\lambda)^{2-\gamma_p}}{\gamma_p} \cdot \nabla (p^\lambda)^{\gamma_p - p^\lambda E^\lambda} \right) \cdot |E^\lambda| dx ds \\
\leq \sigma \int_0^t \Omega \left( (n^\lambda)^{2-\gamma_n} + (p^\lambda)^{2-\gamma_p} \right) |E^\lambda|^2 dx ds \\
+ \frac{1}{4\sigma} \left( \frac{1}{\gamma_n^2} + \frac{1}{\gamma_p^2} \right) \int_0^t \Omega \left( \frac{\nabla (n^\lambda)^{\gamma_n + n^\lambda E^\lambda}}{n^\lambda} + \frac{\nabla (p^\lambda)^{\gamma_p - p^\lambda E^\lambda}}{p^\lambda} \right) dx ds
\]
for some \(\sigma > 0\) independent of \(\lambda\).

The lower bound on \(n^\lambda \geq M > 0\) (for \(C(x) \geq \underline{C} > 0\)) (see Lemma 2.4) gives (and \(p^\lambda \geq 0\)) by simple algebraic considerations the relation
\[
(n^\lambda)^{2-\gamma_n} + (p^\lambda)^{2-\gamma_p} \leq M_5 ((n^\lambda)^{2-\gamma_n} + (p^\lambda)^{2-\gamma_p}), \quad 1 \leq \gamma_p \leq \frac{3}{2}, \quad 1 \leq \gamma_n.
\]

Here and in the following, \(M_5\) denotes the generic constant independent of \(\lambda\).

Thus, (3.5), together with (3.6), gives
\[
\int_0^t \Omega \left( \frac{(n^\lambda)^{2-\gamma_n}}{\gamma_n} \cdot \nabla (n^\lambda)^{\gamma_n + n^\lambda E^\lambda} + \frac{(p^\lambda)^{2-\gamma_p}}{\gamma_p} \cdot \nabla (p^\lambda)^{\gamma_p - p^\lambda E^\lambda} \right) \cdot |E^\lambda| dx ds \\
\leq M_5 \sigma \int_0^t \Omega \left( (n^\lambda)^{2-\gamma_n} + (p^\lambda)^{2-\gamma_p} \right) |E^\lambda|^2 dx ds \\
+ \frac{1}{4\sigma} \left( \frac{1}{\gamma_n^2} + \frac{1}{\gamma_p^2} \right) \int_0^t \Omega \left( \frac{\nabla (n^\lambda)^{\gamma_n + n^\lambda E^\lambda}}{n^\lambda} + \frac{\nabla (p^\lambda)^{\gamma_p - p^\lambda E^\lambda}}{p^\lambda} \right) dx ds
\]
for some $\sigma > 0$ independent of $\lambda$.

Now we estimate the last integral on the right hand side of (3.4). For constant doping profile, we have

$$\int_0^t \int_\Omega |E^\lambda| |\nabla C| dx ds = 0. \quad (3.8)$$

For nonconstant doping profile we have, with the help of Young’s inequality and Lemma 2.4, that

$$\int_0^t \int_\Omega |E^\lambda| |\nabla C| dx ds \leq M_5 \frac{||\nabla C||_\infty}{(\min_{Q_T}(n^\lambda+p^\lambda))^{\frac{1}{2}}} \int_0^t \int_\Omega ((n^\lambda)^{\frac{1}{2}} + (p^\lambda)^{\frac{1}{2}})|E^\lambda| dx ds$$

$$\leq M_5 M^{-\frac{1}{2}}||\nabla C||_\infty \int_0^t \int_\Omega ((n^\lambda)^{\frac{2-\gamma}{2}} + (p^\lambda)^{\frac{2-\gamma}{2}})|E^\lambda| dx ds$$

$$\leq M_5 M^{-\frac{1}{2}}||\nabla C||_\infty (\sigma \int_0^t \int_\Omega ((n^\lambda)^{2-\gamma n} + (p^\lambda)^{2-\gamma n})|E^\lambda|^2 dx ds + \frac{1}{4\sigma} \int_0^t \int_\Omega ((n^\lambda)^{\gamma n-1} + (p^\lambda)^{\gamma n-1}) dx ds)$$

(3.9)

for some $\sigma > 0$ independent of $\lambda$.

Thus, using (3.4), (3.7), (3.8) and (3.9), we have

$$\int_0^t \int_\Omega ((n^\lambda)^{\gamma m} + (p^\lambda)^{\gamma m})|E^\lambda|^2 + \frac{1}{\lambda^2} dx ds$$

$$\leq M_5 \sigma \int_0^t \int_\Omega ((n^\lambda)^{2-\gamma n} + (p^\lambda)^{2-\gamma n})|E^\lambda|^2 dx ds$$

$$+ \frac{1}{4\sigma} (\frac{1}{\gamma m} + \frac{1}{\gamma m}) \int_0^t \int_\Omega (\frac{|\nabla (n^\lambda)|}{m^\gamma m} + \frac{|\nabla (p^\lambda)|}{p^\gamma m^\gamma m} + \frac{p^\lambda}{p^\lambda}) dx ds$$

$$+ M_5 M^{-\frac{1}{2}}||\nabla C||_\infty (\sigma \int_0^t \int_\Omega ((n^\lambda)^{2-\gamma n} + (p^\lambda)^{2-\gamma n})|E^\lambda|^2 dx ds$$

$$+ \frac{1}{4\sigma} \int_0^t \int_\Omega ((n^\lambda)^{\gamma n-1} + (p^\lambda)^{\gamma n-1}) dx ds)$$

(3.10)

for some $\sigma > 0$ independent of $\lambda$.

Choosing $\sigma$ small enough and using (3.3) and Lemma 2.2, we obtain from (3.10) that

$$\int_0^t \int_\Omega ((n^\lambda)^{2-\gamma n} + (p^\lambda)^{2-\gamma n})|E^\lambda|^2 dx ds \leq M_5,$$

$$\int_0^t \int_\Omega \frac{(n^\lambda-p^\lambda-C)^2}{\lambda^2} dx ds \leq M_5$$

uniformly in $\lambda$.

**Step 2: convergence of the densities.**

In this step we prove strong convergence of $n^\lambda$ and $p^\lambda$ in $L^q(T)$ and $L^p(T)$, respectively.

From (3.2) we have

$$\frac{\gamma m}{\gamma m - \frac{1}{2}} \nabla (n^\lambda)^{\gamma m - \frac{1}{2}} + (n^\lambda)^{\frac{1}{2}} E^\lambda \in L^2(Q_T).$$

Also, we have

$$\| (n^\lambda)^{\frac{1}{2}} E^\lambda \|_{L^{\frac{2m}{2m-1}}(Q_T)} \leq \| (n^\lambda)^{\frac{1-2m}{2m-1}} E^\lambda \|_{L^2(Q_T)} \| (n^\lambda)^{\frac{2m-1}{2m-1}} \|_{L^{\frac{2m}{2m-1}}(Q_T)}$$
(with $1 < \frac{2\gamma_n}{2\gamma_n - 1} \leq 2$) and therefore $\nabla (n^\lambda)^{\gamma_n - \frac{1}{2}} \in L^{\frac{2\gamma_n}{2\gamma_n - 1}}(Q_T)$. In the range $1 \leq \gamma_n \leq \frac{3}{2}$ we have

$$\| \nabla n^\lambda \|_{L^\gamma(Q_T)} \leq M_5 \| (n^\lambda)^{\frac{3}{2} - \gamma_n} \|_{L^{\frac{2\gamma_n}{2\gamma_n - 1}}(Q_T)} \| \nabla (n^\lambda)^{\gamma_n - \frac{1}{2}} \|_{L^{\frac{2\gamma_n}{2\gamma_n - 1}}(Q_T)}$$

and analogously for $\nabla p^\lambda$.

In the range $\frac{3}{2} \leq \gamma_n$, using the lower bound on $n^\lambda$, we have

$$\| \nabla n^\lambda \|_{L^{\gamma(Q_T)}} \leq \frac{M_5}{(\min_{Q_T} n^\lambda)^{\frac{3}{2} - \gamma_n}} \| \nabla (n^\lambda)^{\gamma_n - \frac{1}{2}} \|_{L^{\frac{2\gamma_n}{2\gamma_n - 1}}(Q_T)}.$$

Therefore

$$n^\lambda \in L^1((0, t); W^{1,1}(\Omega)) \text{ uniformly in } \lambda \quad (3.11)$$

and

$$(n^\lambda)_t \in L^1((0, t); W^{-1,1}(\Omega)) \text{ uniformly in } \lambda, \quad (3.12)$$

where we used

$$\| \nabla (n^\lambda)^{\gamma_n} + n^\lambda E^\lambda \|_{L^1(Q_T)} \leq \| \frac{\nabla (n^\lambda)^{\gamma_n} + n^\lambda E^\lambda}{(n^\lambda)^{\frac{3}{2}}} \|_{L^2(Q_T)} \| (n^\lambda)^{\frac{1}{2}} \|_{L^2(Q_T)}.$$

From (3.11) and (3.12) we conclude by [Si]

$$\{ n^\lambda \} \text{ compact in } L^1((0, t); L^\rho(\Omega)), 1 \leq \rho < \begin{cases} \frac{d}{d-1}, d > 1; \\ \infty, d = 1. \end{cases}$$

Therefore

$$n^\lambda \to n \text{ a.e. in } L^1(Q_T). \quad (3.13)$$

Using

$$\| n^\lambda |E^\lambda| \|_{L^1(Q_T)} \leq \| (n^\lambda)^{1 - \frac{2\rho}{\gamma_n}} |E^\lambda| \|_{L^2(Q_T)} \| (n^\lambda)^{\frac{2\rho}{\gamma_n}} \|_{L^2(Q_T)},$$

we obtain

$$\nabla (n^\lambda)^{\gamma_n}, (n^\lambda)^{\gamma_n} \in L^1(Q_T),$$

$$(n^\lambda)^{\gamma_n} \in L^1((0, t); W^{1,1}(\Omega)) \subset L^1((0, t); L^\rho(\Omega)), 1 \leq \rho \leq \begin{cases} \frac{d}{d-1}, d > 1; \\ \infty, d = 1. \end{cases}$$

uniformly in $\lambda$.

Therefore, with (3.3) we have

$$n^\lambda \in L^\infty((0, t); L^\gamma(\Omega)) \cap L^\gamma((0, t); L^\gamma(\Omega)) \text{ uniformly in } \lambda,$$

and by interpolation

$$n^\lambda \in L^{\gamma(2\gamma_n - 1)}(Q_T) \text{ uniformly in } \lambda.$$
This gives with (3.13)
\[ n^\lambda \to n \text{ strongly in } L^\alpha(Q_T), \ 1 \leq \alpha < \gamma_n \frac{d+1}{d}. \]

Similarly
\[ p^\lambda \to p \text{ strongly in } L^\beta(Q_T), \ 1 \leq \beta < \gamma_p \frac{d+1}{d}. \]

**Step 3: convergence of the electric field related terms.**

Next we have to deal with the other nonlinearities \((n^\lambda - p^\lambda - C)E^\lambda\) and \(n^\lambda E^\lambda, p^\lambda E^\lambda\). For this purpose we have to estimate the electric field.

In the range \(\gamma_n \leq 2\) using the lower bound we have
\[ \|E^\lambda\|^2_{L^2(Q_T)} \leq \frac{1}{(\min_{Q_T} n^\lambda)^{2-\gamma_n}} \| (n^\lambda)^{2-\gamma_n} |E^\lambda|^2 \|_{L^1(Q_T)} \]

whereas in the range \(\gamma_n > 2\) we conclude
\[ \|E^\lambda\|_{L^{\frac{2m}{\gamma_n-1}}(Q_T)} \leq \| (n^\lambda)^{1-\frac{2\gamma_n}{2m}} \|^{\frac{2_m-2}{2_m}}_{L^2(Q_T)} \| (n^\lambda)^{2-\gamma_n} \| L^{\frac{2\gamma_n}{\gamma_n-1}}(Q_T). \]

Since \(\frac{\gamma_n}{\gamma_n-1} < 2\) for \(\gamma_n > 2\) and \(n^\lambda - p^\lambda - C \in L^2(Q_t)\) only this estimate is not sufficient.

The entropy inequality gives
\[ \lambda E^\lambda \in L^\infty((0, \infty); L^2(\Omega)) \text{ uniformly in } \lambda. \]

Dividing the Poisson equation by \(\lambda\) and using the \(L^2\)-estimate on \(\frac{n^\lambda - p^\lambda - C}{\lambda}\) we conclude
\[ \lambda |E^\lambda| \in L^2((0, \infty); H^1(\Omega)) \text{ uniformly in } \lambda \]

and therefore
\[ \lambda |E^\lambda| \in L^\infty((0, +\infty); L^2(\Omega)) \cap L^2((0, \infty); L^{s_2}(\Omega)) \text{ uniformly in } \lambda, \]

where
\[ 2 \leq s_2 \leq \begin{cases} \frac{2d}{d-2}, d > 2; \\ \infty, d = 1, 2. \end{cases} \]

By interpolation we obtain
\[ \lambda |E^\lambda| \in L^{s_3}(Q_T) \text{ uniformly in } \lambda, \ s_3 = \begin{cases} \frac{2d+2}{d}, d > 2; \\ 4, d = 1, 2. \end{cases} \]

Combining this result with the estimate on \(E^\lambda\) (for \(\gamma_n > 2\)) we obtain
\[ \lambda^{s_4} |E^\lambda| \in L^2(Q_T) \text{ uniformly in } \lambda \text{ for some } s_4: \ 0 < s_4 < 1. \]

These estimates on \(E^\lambda\) allow for \(\gamma_n \geq 2\) the conclusion
\[ |E^\lambda|(n^\lambda - p^\lambda - C) = \lambda^{1-s_4}(\lambda^{s_4} |E^\lambda|) \left(\frac{n^\lambda - p^\lambda - C}{\lambda} \right) \to 0 \text{ strongly in } L^1(Q_T). \]
For $\gamma_n \leq 2$ we simply use the $L^2$ bound

$$|E^\lambda|(n^\lambda - p^\lambda - C) = \lambda |E^\lambda|^\frac{n^\lambda - p^\lambda - C}{\lambda} \to 0 \text{ strongly in } L^1(Q_T).$$

We now turn to $n^\lambda E^\lambda$ and $p^\lambda E^\lambda$. Due to

$$p^\lambda E^\lambda = n^\lambda E^\lambda - (n^\lambda - p^\lambda - C) E^\lambda - CE^\lambda,$$

it suffices to treat $n^\lambda E^\lambda$. In the case $\gamma_n \geq 2$ we use the estimate on $E^\lambda$ and the estimate on $n^\lambda ((\gamma_n^{-1})^\prime = \gamma_n)$

$$n^\lambda |E^\lambda| \to nE \text{ in } \mathcal{D}'(Q_T).$$

For $\gamma_n \leq 2$, we use the $L^2$-estimate on $E^\lambda$ and the $L^\alpha$-estimate ($\alpha < \gamma_n \frac{d+1}{d}$) on $n^\lambda$ we have

$$n^\lambda E^\lambda \to nE \text{ in } \mathcal{D}'(Q_T)$$

with the help of $\frac{2d}{d+1} < \gamma_n \leq 2$.

Therefore we can perform all limits in the nonlinear terms and conclude the proof.

**Remark 3.1** In the case of constant doping profile the proof is much simpler due to the lower and upper $L^\infty$ uniform bounds on $n^\lambda$ and $p^\lambda$ (see Lemma 2.3). Moreover, $n^\lambda \to n, p^\lambda \to p$ strongly in $L^q(Q_T)$ for any $1 \leq q < \infty$ and $E^\lambda \to E$ weakly in $L^2(Q_T)$.

**Remark 3.2** If the nonnegative initial data $n_0^\lambda, p_0^\lambda$ satisfy $n_0^\lambda \in L^\infty(\Omega), p_0^\lambda \in L^\infty(\Omega)$ uniformly in $\lambda$ and $n_0^\lambda - p_0^\lambda - C = \lambda d^\lambda$, where $d^\lambda \in L^2(\Omega)$ uniformly in $\lambda$, then the initial entropy $e^\lambda(t = 0)$ is bounded uniformly in $\lambda$.

**Remark 3.3** The uniqueness of the solution of (1.5)-(1.6) with general $\gamma_n > 0, \gamma_p > 0$ is an open problem. Obviously, in the case of uniqueness all the convergences are true for the whole sequences and not only for extracted subsequences.

**Conclusion**

In this paper we have performed a quasineutral limit analysis for a model with algebraically nonlinear diffusivities. This is done for the first time for non vanishing doping profiles. In contrast to the corresponding problem in the linear drift diffusion model we employed multiplier techniques in order to obtain the crucial a.e.-estimates. However, the most interesting case of a doping profile with change of sign is still unsolved.
Appendix 1

Proof of Lemma 2.1 Multiplying \((1,1)\) by \(\frac{1}{\mu_n}(n^\lambda)^{q-1}\) and integrating the resulting equation over \(\Omega\), we have, by integration by parts, that

\[
\frac{d}{dt} \int_{\Omega} \frac{(n^\lambda)^q}{\mu_{nq}} \, dx = -\int_{\Omega} \gamma_n(q-1)(n^\lambda)^{q+q-3} |\nabla n^\lambda|^2 \, dx - (q-1) \int_{\Omega} (n^\lambda)^{q-1} E^\lambda \cdot \nabla n^\lambda \, dx
\]

\[
= -\int_{\Omega} \gamma_n(q-1)(n^\lambda)^{q+q-3} |\nabla n^\lambda|^2 \, dx + \frac{2-1}{Q} \int_{\Omega} (n^\lambda)^q \text{div} E^\lambda \, dx
\]

\[
= -\int_{\Omega} \gamma_n(q-1)(n^\lambda)^{q+q-3} |\nabla n^\lambda|^2 \, dx - \frac{2-1}{Q} \int_{\Omega} (n^\lambda)^q (n^\lambda - p^\lambda - C) \, dx.
\]

I. e.,

\[
\frac{d}{dt} \int_{\Omega} \frac{(n^\lambda)^q}{\mu_{nq}} \, dx + \int_{\Omega} \gamma_n(q-1)(n^\lambda)^{q+q-3} |\nabla n^\lambda|^2 \, dx = -\frac{q-1}{Q \lambda^2} \int_{\Omega} (n^\lambda)^q (n^\lambda - p^\lambda - C) \, dx.
\]

Similarly, we have

\[
\frac{d}{dt} \int_{\Omega} \frac{(p^\lambda)^q}{\mu_{p^q}} \, dx + \int_{\Omega} \gamma_p(q-1)(p^\lambda)^{q+q-3} |\nabla p^\lambda|^2 \, dx = \frac{q-1}{Q \lambda^2} \int_{\Omega} (p^\lambda)^q (n^\lambda - p^\lambda - C) \, dx.
\]

Thus, we have

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{(n^\lambda)^q}{\mu_{nq}} + \frac{(p^\lambda)^q}{\mu_{p^q}} \right) \, dx
\]

\[
+ \int_{\Omega} \gamma_n(q-1)(n^\lambda)^{q+q-3} |\nabla n^\lambda|^2 + \gamma_p(q-1)(n^\lambda)^{q+q-3} |\nabla p^\lambda|^2 \, dx
\]

\[
= -\frac{q-1}{Q \lambda^2} \int_{\Omega} (n^\lambda)^q + (p^\lambda)^q (n^\lambda - p^\lambda) \, dx + \frac{q-1}{Q \lambda^2} \int_{\Omega} \text{C}((n^\lambda)^q - (p^\lambda)^q) \, dx
\]

Since \(q > 1\) and \((x^q - y^q)(x - y) \geq 0\) for any \(x, y \geq 0\), we have

\[
\frac{d}{dt} \int_{\Omega} ((n^\lambda)^q + (p^\lambda)^q) \, dx \leq \frac{(q-1) \max \{\mu_n, \mu_p\} \|C\|_{\infty}}{\lambda^2} \int_{\Omega} ((n^\lambda)^q + (p^\lambda)^q) \, dx.
\]

Thus, we have, for any \(0 \leq t \leq T\),

\[
\int_{\Omega} ((n^\lambda)^q + (p^\lambda)^q) \, dx \leq e^{\frac{(q-1) \max \{\mu_n, \mu_p\} \|C\|_{\infty} T}{\lambda^2}} \int_{\Omega} ((n_0^\lambda)^q + (p_0^\lambda)^q) \, dx,
\]

which gives the results. The proof of Lemma 2.1 is complete.
Appendix 2

Proof of Lemma 2.2 We only consider the case $\gamma_n > 1, \gamma_p > 1$. For the other case, the proof is similar. Using (1.1) and (1.2), we have, by integration by parts, that

$$
\frac{d}{dt} \int_{\Omega} \frac{(n^\lambda n^\gamma - n^\lambda)}{\gamma_n - 1} + \frac{(p^\lambda p^\gamma - p^\lambda)}{\gamma_p - 1} \, dx
$$

$$
= \frac{1}{\gamma_n - 1} \int_{\Omega} (\gamma_n (n^\lambda)_{\gamma_n - 1} - 1) n^\lambda_t \, dx + \frac{1}{\gamma_p - 1} \int_{\Omega} (\gamma_p (p^\lambda)_{\gamma_p - 1} - 1) p^\lambda_t \, dx
$$

$$
= - \frac{\mu_n}{\gamma_n - 1} \int_{\Omega} \nabla (\gamma_n (n^\lambda)_{\gamma_n - 1}) \cdot (\nabla (n^\lambda)_{\gamma_n} + n^\lambda E^\lambda) \, dx - \frac{\mu_p}{\gamma_p - 1} \int_{\Omega} \nabla (\gamma_p (p^\lambda)_{\gamma_p - 1}) \cdot (\nabla (p^\lambda)_{\gamma_p} - p^\lambda E^\lambda) \, dx
$$

$$
= - \mu_n \int_{\Omega} \frac{\nabla (n^\lambda)_{\gamma_n}}{n^\lambda} \cdot (\nabla (n^\lambda)_{\gamma_n} + n^\lambda E^\lambda) \, dx - \mu_p \int_{\Omega} \frac{\nabla (p^\lambda)_{\gamma_p}}{p^\lambda} \cdot (\nabla (p^\lambda)_{\gamma_p} - p^\lambda E^\lambda) \, dx
$$

and

$$
\frac{d}{dt} \int_{\Omega} \frac{\lambda^2}{2} |E^\lambda|^2 \, dx = \int_{\Omega} \lambda^2 E^\lambda \cdot E^\lambda_t \, dx = \int_{\Omega} \lambda^2 \Phi^\lambda div E^\lambda_t \, dx
$$

$$
= - \int_{\Omega} \Phi^\lambda (n^\lambda_t - p^\lambda) \, dx
$$

$$
= - \int_{\Omega} E^\lambda \cdot [\mu_n (\nabla (n^\lambda)_{\gamma_n} + n^\lambda E^\lambda) - \mu_p (\nabla (p^\lambda)_{\gamma_p} - p^\lambda E^\lambda)] \, dx
$$

$$
= - \mu_n \int_{\Omega} n^\lambda E^\lambda \cdot \frac{\nabla (n^\lambda)_{\gamma_n} + n^\lambda E^\lambda}{n^\lambda} \, dx + \mu_p \int_{\Omega} p^\lambda E^\lambda \cdot \frac{\nabla (p^\lambda)_{\gamma_p} - p^\lambda E^\lambda}{p^\lambda} \, dx.
$$

Thus, we have

$$
\frac{d}{dt} e^\lambda (t) = \frac{d}{dt} \int_{\Omega} \frac{(n^\lambda)_{\gamma_n - 1}}{\gamma_n - 1} + \frac{(p^\lambda)_{\gamma_p - 1}}{\gamma_p - 1} \, dx + \frac{d}{dt} \int_{\Omega} \frac{\lambda^2}{2} |E^\lambda|^2 \, dx
$$

$$
= - \mu_n \int_{\Omega} |\nabla (n^\lambda)_{\gamma_n} + n^\lambda E^\lambda|^2 \, dx - \mu_p \int_{\Omega} |\nabla (p^\lambda)_{\gamma_p} - p^\lambda E^\lambda|^2 \, dx
$$

This completes the proof of Lemma 2.2.

References


