Highly Oscillatory Partial Differential Equations*

Peter A. Markowich and Christof Sparber

1 Introduction

Highly oscillatory problems frequently appear in the study and application of various linear and nonlinear partial differential equations (PDEs), most of them being either hyperbolic or dispersive. A basic example is the free semi-classical scaled Schrödinger equation with plane-wave initial data, i.e.

\[-i\varepsilon \partial_t u^\varepsilon - \frac{\varepsilon^2}{2} \Delta u^\varepsilon = 0, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R},\]

\[u^\varepsilon(x, 0) = e^{ik \cdot x / \varepsilon},\]

where \( \varepsilon \sim \hbar \) (the scaled Planck's constant), represents the small semi-classical parameter, i.e. the microscopic/macrophscopic scale ratio: 0 < \( \varepsilon \ll 1 \). Here and in the following \( \varepsilon \)-dependence is denoted by superscripts. The solution of (1), (2) is explicitly given by

\[u^\varepsilon(x, t) = \exp \left( i \left( k \cdot \frac{x}{\varepsilon} - |k|^2 \frac{t}{\varepsilon} \right) \right)\]

(3)

and hence, we observe that (1) propagates oscillations with \( O(\varepsilon) \)-wave length in time and space. In the following we deal with such kind of oscillations and we review (a highly biased selection of) problems, techniques and results in their asymptotic description. In particular, we shall focus on linear and nonlinear dispersive limits, semi-classical asymptotics, homogenization problems and Wigner transformation techniques.

To this end, we restrict ourselves to PDEs, in which a small (dimensionless) parameter \( \varepsilon \) is present already in the equation itself and ignore cases where \( \varepsilon \)-oscillations are induced only by the initial-data, since the latter case sometimes is different in several aspects from the former and also, since the former is more frequently encountered in applications. Also we shall not include stochastic models, since their description is beyond the scope of this article. From a mathematical point of view,

*This work has also been supported by the Austrian Science Foundation FWF through grant no. W8 and the Wittgenstein Award 2000 of P. M. Additional financial sponsorship has been given by the EU research network HYKE.

1Wolfgang Pauli Institute Vienna and Department of Mathematics, University of Vienna, Strudlhofgasse 4, A-1090 Vienna, Austria; e-mails: peter.markowich@univie.ac.at, christof.sparber@univie.ac.at.
it is essential for all following examples and methods that the considered systems are conservative in the sense that they allow for an a-priori $L^2$-estimate on their respective solution.

2 Oscillations in linear PDEs

2.1 The WKB-Approach for Schrödinger-type equations

In this section we consider a class of scalar IVP's for linear dispersive PDEs with fast temporal and spatial scales subject to highly oscillatory initial data. We are then interested in the high frequency asymptotics, i.e. the geometrical-optics limit of these equations. The Cauchy problem of the semi-classical Schrödinger equation serves as a typical example

$$-i\varepsilon\partial_t u^\varepsilon - \frac{\varepsilon^2}{2} \Delta u^\varepsilon + V(x) u^\varepsilon = 0, \quad x \in \mathbb{R}^d, \; t \in \mathbb{R},$$

$$u^\varepsilon(x, 0) = a_0(x)e^{i\varphi_0(x)/\varepsilon}. \quad (5)$$

In the particular case of the Schrödinger equation, the asymptotic regime $\varepsilon \to 0$ is called the semi-classical limit, cf. [66], [72]. We have seen in the introduction that equations of the form (4) propagate oscillations of wave lengths $\varepsilon$. These oscillations inhibit $u^\varepsilon$ from converging strongly in a suitable sense and hence the limit $\varepsilon \to 0$ is by no means straightforward, in particular for physical observables, which typically are quadratic in $u^\varepsilon$. The traditional way to deal with this problem is the so-called WKB-Approach, named after Wentzel [80], Kramers [52] and Brillouin [9], where one seeks an asymptotic description for the solution $u^\varepsilon$ in the following form:

$$u^\varepsilon(x, t) \sim e^{i\varphi_0(x,t)/\varepsilon}(a_0(x,t) + O(\varepsilon)). \quad (6)$$

with realvalued principal amplitude $a_0$ and realvalued phase-function $\varphi$. Formally, plugging (6) into (4), one obtains in lowest order the Hamilton-Jacobi equation or eikonal equation for the phase:

$$\partial_t \varphi + \frac{1}{2} |\nabla \varphi|^2 + V(x) = 0, \quad (7)$$

$$\varphi(x, 0) = \varphi_0(x). \quad (8)$$

In terms of the Hamiltonian function

$$H(x, \xi) := \frac{|\xi|^2}{2} + V(x) \quad (9)$$

this equivalently can be written as

$$\partial_t \varphi + H(x, \nabla \varphi) = 0, \quad (10)$$

$$\varphi(x, 0) = \varphi_0(x). \quad (11)$$

We remark that (10) is valid also for more general (scalar) dispersive PDEs than the Schrödinger equation, cf. [74]. As a next step in the asymptotic expansion, the terms of first order in $\varepsilon$ lead to a transport equation for the principal amplitude

$$\partial_t a_0 + \nabla a_0 \cdot \nabla \varphi + \frac{1}{2} a_0 \Delta \varphi = 0, \quad (12)$$

$$a_0(x, 0) = a_1(x), \quad (13)$$
which can be rewritten as a conservation law for the position density \( \rho := |a_0|^2 \):

\[
\partial_t \rho + \text{div}(\rho \nabla \varphi) = 0, \quad \rho(x, 0) = |a_1(x)|^2.
\]

For more general PDEs (defined in (18 below) one obtains, cf. [27], [74]

\[
\partial_t \rho + \text{div}(\rho \nabla \xi H(x, \nabla \varphi)) = 0, \quad \rho(x, 0) = |a_1(x)|^2.
\]

The system (10), (16) is called the WKB-System. It determines the leading order terms in the expansion (6), where the scalar-valued function \( u^\varepsilon \) is a solution of a general scalar dispersive PDE of the form

\[-i\varepsilon \partial_t u^\varepsilon + H^W(x, \varepsilon D)u^\varepsilon = 0, \quad x \in \mathbb{R}^d, \ t \in \mathbb{R}.\]

Here \( D := -i \nabla \) and \( H^W(x, \varepsilon D) \) is the Hamiltonian operator (assumed to be self-adjoint on \( L^2(\mathbb{R}^d) \)) obtained by Weyl-quantizing the symbol \( H(x, \xi) \), i.e.

\[H^W(x, \varepsilon D)f(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} H \left( \frac{x + y}{2}, \varepsilon \xi \right) f(y) e^{i \xi \cdot (x-y)} \, dy \, d\xi.\]

Here and in the following, the Weyl-quantization is used because it is closely connected [31] with the Wigner transform defined in section 5 below. Other quantization rules could be used as well.

In general, i.e. except for some special examples of initial conditions, one cannot expect the phase function \( \varphi \) to be smooth for all \( t \in \mathbb{R} \) due to the fully nonlinear structure of (10). The appearing singularities are usually called caustics and it is clear that one can rigorously justify the asymptotic expansion (6) only up to times \(|t| < t_c\), where \( t_c \) denotes the time at which the first caustic appears. More precisely, one can prove the following local-in-time result:

**Theorem 1.** Let \( u^\varepsilon(t) \) be the solution of (18), (5) and let \( a_0, \varphi \) be the sufficiently smooth solutions of (10), (11) and (16), (17). Then it holds

\[
\sup_{|t| < t_c} \| u^\varepsilon(\cdot, t) - e^{i \varphi(\cdot, t)/\varepsilon} a_0(\cdot, t) \|_2 \leq O(\varepsilon),
\]

where \( \| \cdot \|_2 \) denotes the \( L^2(\mathbb{R}^d) \)-norm.

The first rigorous result in this direction has been obtained by P. Lax [54] and we refer to [27], [44], [66] and [70] for more details on the WKB-Approach.

An alternative point of view on this problem is given by considering the, so called, Quantum Hydrodynamic System (QHD): One represents the solution of Schrödinger’s equation in the form

\[u^\varepsilon(x, t) = e^{i \varphi(\cdot, t)/\varepsilon} a^\varepsilon(x, t).\]

Inserting this ansatz into (4), defining \( \rho^\varepsilon := |a^\varepsilon|^2 \) and \( j^\varepsilon := \rho^\varepsilon \nabla \varphi^\varepsilon \), one arrives, after separating real and imaginary parts, at a closed hydrodynamic-type system for \( \rho^\varepsilon \) and \( j^\varepsilon \):

\[
\partial_t \rho^\varepsilon + \text{div} j^\varepsilon = 0, \quad \partial_t j^\varepsilon + \text{div} \left( \frac{j^\varepsilon \otimes j^\varepsilon}{\rho^\varepsilon} \right) + \rho^\varepsilon \nabla V = \frac{\varepsilon^2}{2} \rho^\varepsilon \nabla \left( \frac{1}{\sqrt{\rho^\varepsilon}} \Delta \sqrt{\rho^\varepsilon} \right),
\]
This system is exact, i.e. equivalent to the Schrödinger equation (as proved in [34]) and well posed for all \( t \in R \), due to the third order dispersive regularization term. For \( \varepsilon = 0 \) the system (at least formally) simplifies to the zero temperature Euler equations of gas dynamics, for which it is known that in general singularities will appear in finite time. From this point of view, the semi-classical limit is equivalent to a zero-dispersion limit [35], a problem which will be again encountered in section 3.1.

2.2 The stationary phase method

The results of section 2.1 show that the WKB-Method gives only local-in-time results, due to the appearance of caustics. Geometrically this corresponds to the crossing of rays, which are the characteristics corresponding to the Hamilton-Jacobi equation (10). After the first caustic and since the underlying dispersive PDE is indeed linear we expect the superposition of such rays, i.e. a superposition of WKB-modes.

Let us illustrate this by the following example: Once again, we consider the free Schrödinger equation in \( R^d \), i.e.

\[
-\varepsilon^2 \partial_t u^\varepsilon - \frac{\varepsilon^2}{2} \Delta u^\varepsilon = 0, \quad x \in R^d, \quad t \in R, \quad u^\varepsilon(x, 0) = a_I(x)e^{i\varphi_I(x)/\varepsilon}, \tag{24}
\]

Its solution is explicitly given by an oscillatory integral:

\[
u^\varepsilon(x, t) = (2\pi\varepsilon)^{-d/2} \int_{R^d} a_I(y)e^{i\varphi(x, y, \xi, t)/\varepsilon} d\xi dy, \tag{26}
\]

where

\[
\varphi(x, y, \xi, t) := (x - y) \cdot \xi + \frac{t}{2}|\xi|^2 + \varphi_I(y). \tag{27}
\]

Next, we recall the theorem of stationary phase [45], where \( \text{sgn} \ A \) denotes the number of positive eigenvalues of \( A \) minus the number of negative eigenvalues.

**Theorem 2.** Let \( a \in C_0^\infty(R^d), \Phi \in C^\infty(R^d) \) and assume that the set \( \{ y : \nabla\Phi(y) = 0, y \in \text{supp}(a) \} \) consists of finitely many points \( y_j \), with \( j = 1, \ldots, M \in N \). If the Hessians \( D^2\Phi(y_j) \) are non-singular, then for \( \varepsilon \ll 1 \) it holds:

\[
(2\pi\varepsilon)^{-d} \int_{R^d} a(y)e^{i\Phi(y)/\varepsilon} dy \sim \sum_{j=1}^M \frac{1}{\sqrt{\text{det} D^2\Phi(y_j)}} \exp \left( \frac{i\Phi(y_j)}{\varepsilon} + \frac{i\pi}{4}m_j \right) (a(y_j) + O(\varepsilon)),
\]

where \( y_j \) and \( m_j := \text{sgn} D^2\Phi(y_j) \) is the, so called, Maslov index of the the \( j \)-th ray.

This implies that locally the main contribution to the solution of the Schrödinger equation stems from stationary points w.r.t. \( y \) and \( \xi \), i.e. points at which \( \nabla_y \xi \varphi = 0 \), which gives

\[
\xi = \nabla \varphi_I(y), \quad x = y + t\xi. \tag{28}
\]

The problem of the missing compactness of \( \text{supp}(a_I) \) w.r.t. \( \xi \in R^d \) can easily be overcome by an approximation argument, since the number of stationary points
only depends on the phase. Hence, we get a ray-map, defined by the following relation
\[ x = \hat{x}(t,y) = y + t\nabla \varphi(y). \tag{29} \]
For small \( t \) the map \( y \mapsto \hat{x}(t,y) \) is single-valued. In general however, there exist (maybe infinitely) many \( y_j = y_j(x,t) \), which obey the equation (29). Note that the functions \( \varphi_j(y_j(x,t)) \) are local solutions of the Hamilton-Jacobi equation corresponding to (24)
\[ \partial_t \varphi + \frac{\nabla x \cdot \nabla \varphi}{2} = 0, \tag{30} \]
called multi-valued solutions. The search for an efficient numerical algorithm to compute such solutions has been intensively studied in recent years, cf. [5], [39], [73]. Hence, if the assumptions of theorem 2 are satisfied, we obtain (locally away from caustics), the following multi-valued WKB-approximation of \( u^\varepsilon \):
\[ u^\varepsilon(x,t) \sim \sum_{j=1}^{M} \frac{a_j(y_j(x,t))}{1 + t\partial^2 \varphi(y_j(x,t))} \exp \left( i \frac{\varepsilon}{4} \varphi_j(y_j(x,t)) + i \frac{\pi}{4} m_j \right) + O(\varepsilon). \tag{31} \]
For equations with variable coefficients, the above concepts have to be generalized, leading to the definition of Fourier-integral operators (FIO) and for more details in this direction, we refer to [1], [19], [23], [27], [45] and the references given therein. Finally, we remark that at caustic-points the FIO is an integral which can be brought into a canonical form and evaluated in terms of special integral functions, cf. [23], [60]. Nowadays the geometrical optics limit for solution of linear PDEs is well understood and there are powerful and very precise analytical machineries available to deal with these kind of problems, cf. [27]. As we shall see, the situation is quite different in the case of nonlinear PDEs, where up to know only partial results for a few types of equations are available.
Before we consider nonlinear problems we shall briefly discuss a different source of \( \varepsilon \)-oscillations in the solution of linear PDEs in the next section.

2.3 Linear Homogenization

In the mathematical modeling of microscopic effects in solid-state physics, various local characteristics, which henceforth appear as coefficients in certain PDEs, are described by functions of the form \( a(x,x/\varepsilon,t) \). Here, again \( 0 < \varepsilon \ll 1 \) represents the ratio of microscopic vs. macroscopic scales and \( a(x,y,t) \) is \( Y \)-periodic w.r.t. \( y \in R^d \), where \( Y \) denotes the unit cube in \( \mathbb{R}^d \) (to keep the discussion simple). The study of the expected high-frequency effects in these coefficients is called homogenization theory and a basic reference for it is [6].
In the setting of classical physics for crystals, temperature distributions for example are often described by elliptic PDEs of the form
\[ -\text{div} \left( a(x/\varepsilon) \nabla u^\varepsilon \right) = f(x), \quad x \in \Omega \subset \mathbb{R}^d, \tag{32} \]
\[ u^\varepsilon = 0, \quad \text{on } \partial\Omega, \tag{33} \]
where \( f : U \rightarrow \mathbb{R} \) is a given function, as is the periodic coefficients matrix \( a = (a_{i,j}(y)) \in \mathbb{R} \), with \( i,j = 1,\ldots,d \). We assume uniform ellipticity of the equation
and suppose that (at least formally) $u^\varepsilon$ admits a two-scale asymptotic expansion of the form
\[ u^\varepsilon(x) = u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + O(\varepsilon^2), \] 
(34)

Then it can be shown [6] that the leading order term $u_0$ only depends on $x$, i.e. $u_0 = v(x)$ and solves the homogenized problem:
\[-\text{div}(\bar{a} \nabla u) = f(x), \quad x \in \Omega \subset \mathbb{R}^d,\]
\[ u = 0, \quad \text{on } \partial \Omega, \] 
(35)

in the weak sense. In (35), the homogenized coefficient matrix $\bar{a} = \bar{a}_{k,j} \in \mathbb{R}$ is given by
\[ \bar{a}_{i,j} := \int_Y a_{i,j}(y) - \sum_{k=1}^d a_{j,k}(y) \partial_k \chi_i dy, \] 
(37)

where $\chi_i$ is the solution of the corrector problem, stated on the unit cell $Y$:
\[ \text{div}(a(y) \nabla \chi_i) = \sum_{j=1}^d \partial_j a_{k,j}(y), \quad y \in Y, \] 
\[ \chi_i \text{ being } Y\text{-periodic, } \forall i = 1, \ldots, d. \] 
(38)

More precisely, one obtains the following convergence result, proved in [6]:

**Theorem 3.** Let $u^\varepsilon \in H^1_0(\Omega)$ be the solution of (32). Then there exist a subsequence $\varepsilon_k$, with $\varepsilon_k \to 0$, as $k \to \infty$, and $u \in H^1_0(\Omega)$, satisfying (35), such that
\[ u^\varepsilon \rightharpoonup u \quad \text{weakly in } H^1_0(\Omega). \] 
(40)

Similar results hold for coefficients including an additional slow scale, i.e. $\alpha = a(x, x/\varepsilon)$, for almost periodic operators and also for certain classes of integral equations with highly oscillatory coefficients, cf. [6], [16], [47], [79].

The situation gets much more complicated when one passes to non-elliptic problems. A typical example is the Schrödinger equation for the motion of an electron in a crystal:
\[-i\varepsilon \partial_t u^\varepsilon - \frac{\varepsilon^2}{2} \Delta u^\varepsilon + V \left( \frac{x}{\varepsilon} \right) u^\varepsilon = 0, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \] 
\[ u^\varepsilon(x, 0) = \psi_0(x). \] 
(41)

Here $V(y) = V(y + \gamma)$, $\gamma \in \Gamma$, typically models the periodic potential of the crystal lattice ions. In the case of one spatial dimension, $\Gamma = 2\pi Z$ and the first Brillouin Zone reads $Y = [-1/2, 1/2]$, which is nothing but the Wigner-Seitz cell of the dual lattice $\Gamma^* = Z$, cf. [6].

In (41), $\varepsilon \to 0$ corresponds to the simultaneous semi-classical and homogenization limit. (See also [69] for a more detailed study on the connection of these limiting regimes.) A naive WKB-Approach along the lines of section 2.1 gives a weakly coupled system
\[ \partial_t \varphi + \frac{|\nabla \varphi|^2}{2} + V \left( \frac{x}{\varepsilon} \right) = 0, \] 
\[ \partial_t \rho + \text{div}(\rho \nabla \varphi) = 0, \] 
(43)
which still carries the small parameter $\varepsilon$. Hence, one would be forced to homogenize the Hamilton-Jacobi equation (43), a procedure, which will be discussed in section 4 below and which is known to neglect important dispersive effects of the original Schrödinger equation (41). Numerical evidence for this fact can be found in [41]. To overcome this difficulty, a generalized two-scale WKB-expansion has been introduced in [43]

$$u^x(x,t) \sim e^{i\varphi(x,t)/\varepsilon} \left(a_0(x,t,x/\varepsilon) + O(\varepsilon)\right),$$

(45)

where $a_0$ is lattice-periodic w.r.t. the fast variable, i.e. $a_0(x,t,y) = a_0(x,t,y + \gamma)$, $\gamma \in \Gamma$. At least formally [43], [41], this leads to the following Bloch-band Hamilton-Jacobi equation:

$$\partial_t \varphi + E_n(\nabla \varphi) = 0,$$

(46)

where the $n$-th band energy $E_n = E_n(k)$, $n \in N$, is obtained from the classical Bloch eigenvalue-problem [7] (see also [71]):

$$-\frac{1}{2} \Delta \psi_n(y,k) + V(y)\psi_n(y,k) = E_n(k)\psi_n(y,k), \quad y \in R, \quad k \in Y,$$

(47)

$$\psi_n \text{ and } \nabla \psi_n \text{ being } Y\text{-quasi-periodic.}$$

(48)

Here, quasi-periodic means that for some $k \in Y$, the eigenfunctions $\psi_n$ satisfy

$$\psi_n(y + \gamma, k) = e^{ik \cdot y} \psi_n(y,k), \quad \forall y \in R, \quad n \in N.$$ (49)

Now, writing $a_0(x,t,x/\varepsilon) = \tilde{a}(x,t)\sigma_n(x/\varepsilon,k)$, with $k = \nabla \varphi(x,t)$ and $\sigma_n(y + \gamma, k) = \sigma_n(y, k)$ one gets a conservation law for the $n$-th band density

$$\partial_t \tilde{\rho} + \text{div} (\tilde{\rho} \nabla E_n(\nabla \varphi)) = 0, \quad \tilde{\rho} := |\tilde{a}|^2.$$ (50)

A more rigorous approach to this is via Bloch’s theorem [7], [71], which asserts a decomposition of the original Hilbert-space into countable many orthogonal (spectral) subspaces $L^2(R^d) = \bigoplus_{n=1}^{\infty} S_n$, each of which is invariant under the lattice Hamiltonian. This leads to more sophisticated treatments of this problem, using either Wigner transforms, cf. the references in section 5.3, or adiabatic decoupling theory [46], [68]. Both of these methods do not suffer from the occurrence of caustics, in contrast to the two-scale WKB-approach above. Nevertheless, even in these more advanced approaches the problem of band-crossings is encountered and up to now resolved for some special cases only, cf. [29], [30] and the references given therein. In this case, the main obstruction for a rigorous treatment is the possible energy-exchange between different Bloch-bands, the, so-called Landau-Zener effect, [53], [83].

We finally remark that there are several recent papers in which the connection between the Bloch-decomposition and traditional homogenization theory is studied in more detail, cf. [16], [26], [41].

3 Oscillation in nonlinear PDEs

3.1 Nonlinear Dispersive limits

We have seen in section 2.1 that high-frequency asymptotics are closely related to small dispersion limits. One of the best studied examples in this field is the
quasi-linear KdV-equation:
\[
\begin{align*}
\partial_t u^\varepsilon - u^\varepsilon \partial_x u^\varepsilon &= \varepsilon^2 \partial_{xx} u^\varepsilon, \quad x, \ t \in R, \\
u^\varepsilon(x, 0) &= u_I(x).
\end{align*}
\] (51)

This equation appears in the descriptions of shallow water-waves [81] and can be seen as a regularized (due to the third order dispersive term) version of Burgers’ equation:
\[
\begin{align*}
\partial_t u - u \partial_x u &= 0, \quad x, \ t \in R, \\
u^\varepsilon(x, 0) &= u_I(x).
\end{align*}
\] (53)

Similar to the semi-classical limit for the QHD-system, the formal limit from KdV to Burgers, as \( \varepsilon \to 0 \), holds only up to caustics, i.e. for \( |t| < t_c \) where \( t_c \) is the time of formation of the first shock in Burgers’ equation. However, due to fact that the KdV-equation represents a fully integrable infinite dimensional dynamical system, one can indeed pass over the caustic and give a description of the zero-dispersion limit also for \( |t| \geq t_c \), cf. [55], [56], [57]. However, these results heavily rely on an inverse-scattering-approach, which is beyond the scope of this article and which we shall not discuss in detail, also, since this technique is precisely tuned to the fully integrable case and can not be generalized to non-integrable PDEs.

A closely related problem is the semi-classical limit for nonlinear Schrödinger-type equations (NLS):
\[
\begin{align*}
-i \varepsilon \partial_t u^\varepsilon - \frac{\varepsilon^2}{2} \Delta u^\varepsilon + V(|u^\varepsilon|^2) u^\varepsilon &= 0, \quad x \in R^d, \ t \in R, \\
u^\varepsilon(x, 0) &= a_I(x) e^{i\varphi_I(x)/\varepsilon}.
\end{align*}
\] (55)

Equations of NLS-type arise in various branches of theoretical physics, like water-wave dynamics, plasma physics and Bose-Einstein condensation cf. [78]. Locally in time, the limit \( \varepsilon \to 0 \) can be rigorously justified by passing to the corresponding QHD-system
\[
\begin{align*}
\partial_t \rho^\varepsilon + \text{div} j^\varepsilon &= 0, \\
\partial_t j^\varepsilon + \text{div} \left( \frac{j^\varepsilon \otimes j^\varepsilon}{\rho^\varepsilon} \right) + \rho^\varepsilon \nabla V(\rho^\varepsilon) &= \frac{\varepsilon^2}{2} \rho^\varepsilon \nabla \left( \frac{1}{\sqrt{\rho^\varepsilon}} \Delta \sqrt{\rho^\varepsilon} \right)
\end{align*}
\] (57)
and studying its zero-dispersion limit: The first results in this direction were given in [37], [42], where the nonlinearity is assumed to be such that: \( V \in C^\infty(R_d, R), \ V' > 0 \). For more general nonlinearities (involving also derivatives of \( u^\varepsilon \)) we refer to [35] and the references given therein.

On the other hand, there are only a few global-in-time results for small \( \varepsilon \) available, since, in contrast to the linear case, one can not expect a superposition of WKB-modes to be a valid description after the caustic. Also, the highly advanced machinery of FIO’s in general does not apply to nonlinear PDEs (for a notable exception see [50]). However, there are indeed a few particular cases which were successfully solved in recent years: For the special case \( d = 1, \ V = \text{id} \), i.e. the, so called, cubic de-focusing NLS, the equation (55) is again fully integrable leading to a similar approach as in the KdV-case, cf. [58]. For the much general case of non-fully integrable NLS, some global-in-time results are given in [11], [13] (using scattering theory), [51] and we also refer to the paper [14], which examines in more
detail the role of quadratic oscillations in NLS. Finally, we note that in the focusing case, i.e. \( V = -id \), the limiting QHD-system is not hyperbolic and a rigorous passage from (57) has not yet been found.

We conclude from the above that so far there exists only a weak mathematical theory for the description of nonlinear dispersive limits.

### 3.2 Weakly nonlinear dispersive geometrical optics

In the examples of the previous section, regularizing higher order (dispersive) effects are lost in the limit \( \varepsilon \to 0 \). On the other hand it is well known that \( \varepsilon \)-oscillations are also present in first order hyperbolic systems, which do not include any higher order terms. The first rigorously work on the corresponding geometrical optics approximation is \[54\] (see also \[70\]), which deals with linear equations. For nonlinear systems the theory has been developed in \[22\].

Consider the following class of oscillatory IVP:

\[
A_0(u^\varepsilon)\varepsilon\partial_t u^\varepsilon - \sum_{k=1}^d A_k(u^\varepsilon)\varepsilon\partial_k u^\varepsilon + g(x) = f(u^\varepsilon), \quad x \in R^d, \quad t \in R, \quad (59)
\]

\[u^\varepsilon(x, 0) = \varepsilon p v_0(x, t, \varepsilon t^p) + O(\varepsilon^{p+1}), \quad p \in Q_+, \quad (60)
\]

where either \( u^\varepsilon \in R^d \) or \( u^\varepsilon \in C^d \) and the matrices \( A_k \) are such that (59) constitutes a hyperbolic system, i.e.

\[
A_0^\varepsilon(u^\varepsilon) = A_k(u^\varepsilon), \quad A_0(0) > 0. \quad (61)
\]

The exponent \( p \), determing the amplitude in (60), will be fixed below. Further assume on the nonlinearities that \( A(u^\varepsilon), f(u^\varepsilon) \) are sufficiently smooth and homogeneous of order \( O(u^\varepsilon) \), with \( J \geq 2 \). Then, it is shown in [22] that, locally in time, it holds

\[
u^\varepsilon(x, t) \sim v^\varepsilon(t, x) := \varepsilon p v_0(x, t, \varepsilon t^p) + O(\varepsilon^{p+1}), \quad |t| < t_0, \quad (62)
\]

where \( \varphi(x, t) \in R \) is a real-valued phase-function, the principal amplitude \( v_0(x, t, \theta) \) is \( 2\pi \)-periodic w.r.t. \( \theta \) and the critical exponent \( p \) is determined such that the normalization condition:

\[pJ = p + 1 \quad (63)\]

holds. With this normalization, the nonlinearity does not enter in lowest order, i.e. in the Hamilton-Jacobi equation for the phase

\[
det \left( iA_0(0)\partial_t \varphi - \sum_{k=1}^d A_k(0)\partial_k \varphi + g(x) \right) = 0, \quad (64)
\]

\[
\varphi(x, 0) = \varphi_0(x), \quad (65)
\]

but only in the (nonlinear) transport-equation for the principal amplitude \( v_0 \). Hence, one usually refers to this asymptotic regime as weakly nonlinear geometrical optics. It can be shown that \( p \) is critical in the sense that for amplitudes \( O(\varepsilon^p) \) one can prove simultaneously existence of the approximate smooth solution for times \( t = O(1) \), i.e. on a time-scale independent of \( \varepsilon \), and nontrivial nonlinear behavior in the principal term of the approximation. More precisely, we have for all multiindices \( \alpha \)

\[
\partial_x^\alpha(u^\varepsilon - v^\varepsilon) = O(\varepsilon^n), \quad \forall n \in N, \quad (66)
\]
uniformly on $[0, T] \times \mathbb{R}^d$, where $T \in [0, t_c]$. In general the nonlinearity will produce higher order harmonics, i.e. we obtain

$$v_0(x, t, \theta) = \sum_{m \in \mathbb{Z}} v_{0,m}(x, t)e^{im\theta}, \quad (67)$$

where for every relevant physical relevant model (considered so far), one can prove that only a finite number of these harmonics are non-zero. For a particular class of first-order systems which in addition satisfy the so-called transparency condition, it is even possible to conclude the existence of $O(1)$-asymptotic solutions, using a tricky change of unknowns cf. [49]. However, global-in-time results in general are out of reach (as for dispersive systems).

Recent applications of this method include the semi-classical asymptotic description of the Dirac-Maxwell system [75] and the rigorous derivation of NLS from nonlinear hyperbolic systems [15].

4 Homogenization of nonlinear PDEs

Homogenization theory has been successfully adapted to various nonlinear PDEs. For nonlinear elliptic problems, i.e. generalizations of (32), the results are similar to the ones obtained in the linear case, assuming that certain technical conditions on the considered operators hold (monotonicity for example). We therefore omit more details on this field of research and refer the interested reader to [33], [79] and the references given therein.

Rather we shall turn our attention to the homogenization problem of fully nonlinear first order PDEs, most prominently periodic Hamilton-Jacobi equations. Hence, we consider

$$
\partial_t u^\varepsilon + H \left( \frac{x}{\varepsilon}, \nabla u^\varepsilon \right) = 0, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \quad (68)
$$

$$
\begin{align*}
&u^\varepsilon(x, 0) = u_t(x), \\
&\quad (69)
\end{align*}
$$

where the Hamiltonian $H(y, \xi)$ is assumed to be (uniformly) convex w.r.t $\xi \in \mathbb{R}^d$ and $Y$-periodic w.r.t. $y \in \mathbb{R}^d$. The first result in this direction is given in an unpublished but nevertheless famous paper by P.L. Lions, G. Papanicolaou and S. R. Varadhan in 1988. More recent references are [24], [17] and [18].

The homogenization results are based on the notion of viscosity solutions [20], which proved to be the right concept of generalized solutions for fully nonlinear PDEs of first and second order [21]. One can show that in the sense of viscosity solutions $u^\varepsilon \to u_0$ as $\varepsilon \to 0$, uniformly on compact subsets of $[0, \infty) \times \mathbb{R}^d$, where $u$ solves a homogenized Hamilton-Jacobi equation of the form

$$
\begin{align*}
&\partial_t u + \mathcal{H}(\nabla x u) = 0, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \\
&u(x, 0) = u_t(x). \\
&\quad (70)
\end{align*}
$$

Here, $\mathcal{H} : \mathbb{R}^d \to \mathbb{R}$ is the effective Hamiltonian, which is nothing but the unique real number, such that the following corrector problem

$$
\begin{align*}
&H(P + \nabla v, y) = \mathcal{H}(P), \quad P \in \mathbb{R}^d \text{ fixed}, \\
&v(P, y) \text{ is } Y\text{-periodic w.r.t. } y \in \mathbb{R}^d. \\
&\quad (72)
\end{align*}
$$

admits a viscosity solution $v = v(P, y)$, which, however, may not be unique, cf. [24].
The homogenization theory for Hamilton-Jacobi equations is applied in [25] to the eigenvalue-problem corresponding to (41). A rigorous WKB-approximation is performed under much weaker regularity assumptions than one usually needs. (For example, note that the stationary phase theorem 2 assumes \( C^{\infty} \)-regularity.) However, the obtained error estimates are not optimal.

5 Wigner transformation techniques

The Wigner transformation approach provides a rigorous tool for the description of high-frequency- and homogenization-limits of physical observables, i.e. quadratic quantities build out of the solution \( u^\varepsilon \) of some linear dispersive PDE. The foundation for this theory is laid in the early papers on \( H \)-measures and microlocal defect-measures, cf. [36], [77] (see also [10]). In the following, the basic idea is to pass from PDEs in physical space \( R^d \) to an equivalent description in phase-space \( R^d_x \times R^d_\xi \), where the problem of caustics no longer appears.

5.1 Phase-space description for dispersive equations

We again consider a linear scalar dispersive equation of the the form (18)

\[
-iz\partial_t u^\varepsilon + H^{W}(x, \varepsilon D)u^\varepsilon = 0, \quad x \in R^d, \quad t \in R,
\]

\[
u^\varepsilon(x, 0) = u^\varepsilon(x),
\]

where \( u^\varepsilon(x, t) \in C \) and the Weyl-quantized operator \( H^{W}(x, \varepsilon D) \) is assumed to be a self-adjoint (pseudo-) differential operator on \( L^2(R^d) \). Hence, from Stone’s theorem we obtain

\[
\|u^\varepsilon(\cdot, t)\|_2 = \|u^\varepsilon(\cdot)\|_2.
\]

We can now define the Wigner transform \([82]\) corresponding to \( u^\varepsilon \):

\[
w^\varepsilon(x, \xi, t) := \frac{1}{(2\pi)^d} \int_{R^d} u^\varepsilon \left( x - \frac{\varepsilon}{2} \eta, t \right) \overline{u^\varepsilon} \left( x - \frac{\varepsilon}{2} \eta, t \right) e^{i \xi \cdot \eta} d\eta.
\]

In the context of quantum mechanics, the real-valued Wigner transform is interpreted as a phase-space description of the quantum state \( u^\varepsilon \), although in general \( w^\varepsilon \) is not positive a.e., cf. [31].

As an example, we apply the Wigner transform to the Schrödinger equation (4), which yields [59]

\[
\partial_t w^\varepsilon + \xi \cdot \nabla_x w^\varepsilon - \theta^{\varepsilon}[V]w^\varepsilon = 0, \quad x, \xi \in R^d, \quad t \in R,
\]

\[
w^\varepsilon(x, \xi, 0) = w_f(x, \xi),
\]

where \( \theta^{\varepsilon}[V] \) is a (non-local) pseudo-differential operator, defined by

\[
\theta^{\varepsilon}[V]w^\varepsilon(x, \xi, t) := \frac{i}{\varepsilon(2\pi)^d} \int_{R^d} [V \left( x - \frac{\varepsilon}{2} \right) - V \left( x + \frac{\varepsilon}{2} \right)] w^\varepsilon(x, \xi, t) e^{i\xi \cdot \xi'} d\xi.d\xi'.
\]

Note that in the free motion case, i.e. \( V(x) \equiv 0 \), the Wigner equation becomes the free transport equation of classical statistical mechanics. The important feature of the Wigner transform is that it facilitates a classical computation of expectation.
values (mean values) for quantum mechanical observables $A^W(x,\varepsilon D)$ in a state $\psi^\varepsilon$, namely

$$\langle u^\varepsilon(\cdot, t), A^W(x, \varepsilon D)u^\varepsilon(\cdot, t) \rangle_2 = \int_{R^d} \int_{R^d} A(x, \xi)u^\varepsilon(x, \xi, t)dx d\xi,$$  

(81)

where the corresponding classical observable is represented by a real-valued symbol $A(x, \xi) \in \mathcal{S}(R^{2d})$. The Wigner transform can be generalized to the, so called, Wigner matrix [38]

$$W^\varepsilon(x, \xi, t) := \frac{1}{(2\pi\varepsilon)^d} \int_{R^d} u^\varepsilon(x - \frac{\varepsilon}{2}\eta, t) \otimes \overline{u^\varepsilon}(x + \frac{\varepsilon}{2}\eta, t) e^{i\eta \cdot \eta} d\eta,$$  

(82)

where $u^\varepsilon$ is a $n$-vector valued function and $\otimes$ denotes the tensor product of vectors. This allows one to treat matrix-valued first-order systems, wave equations as well as matrix-valued observables. For more details, examples and applications we refer to [31], [10], [59] and [74].

5.2 Semi-classical limit of physical observables

The following main theorem on Wigner transforms is proved in [38]:

**Theorem 4.** If $u^\varepsilon \in L^\infty(R; L^2(R^d))$, then its Wigner transform $w^\varepsilon$ is uniformly bounded in $L^\infty(R; \mathcal{S}'(R^d \times R^d_\xi))$ as $\varepsilon \to 0$. Thus, by compactness, there exists a sub-sequence $\varepsilon_k$, with $\varepsilon_k \to 0$, as $k \to \infty$, and a distribution $w \in \mathcal{S}'(R^d_\xi \times R^d_\xi)$, such that

$$w^\varepsilon \overset{k \to \infty}{\to} w \text{ in } L^\infty(R; \mathcal{S}'(R^d_\xi \times R^d_\xi)) \text{ weak-*.}$$  

Moreover $w(t) \in \mathcal{M}^+(R^d_\xi \times R^d_\xi)$, i.e. it is a bounded positive Borel-measure on phase space.

Hence, $w$ indeed can be interpreted as a classical phase-space measure, called the Wigner measure of $\psi^\varepsilon$. In particular, if $u^\varepsilon$ is given in WKB-form (6), then (up to extraction of subsequence)

$$w^\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \rho(x, t) \delta(\xi - \nabla_x \varphi(x, t)), \quad \rho(x) := |a_0(x, t)|^2.$$  

(83)

An evolution equation for the limiting Wigner measure can be derived, too.

**Theorem 5.** Let $u^\varepsilon \in L^\infty(R; L^2(R^d))$ be the solution of equation (74), then the corresponding Wigner measure is a weak solution of the classical Liouville equation

$$\partial_t w + \{H, w\} = 0, \quad x \in R^d, \quad t \in R,$$

(84)

$$w(x, \xi, 0) = w^0(x),$$

(85)

where $\{f, g\} := \nabla_x f \cdot \nabla_\xi g - \nabla_\xi f \cdot \nabla_x g$.

Note that for these results, we do not need to assume that $\psi^\varepsilon$ is of WKB-type.

In comparison to WKB-methods, the great advantage of the Wigner formalism is the fact that no singularities, i.e. caustics, appear in the solution of (85). More precisely, one can prove [74], that the solution of (85) is a so called mono-kinetic distribution.
(84), if and only if \( \rho, \varphi \) solve the WKB-system (10), (16). Moreover, assuming that only a finite number of branches appear in the multi-valued Hamilton-Jacobi solution, one can show, that locally away from caustics, \( w \) can always be decomposed into a finite sum of mono-kinetic distributions. This statement in some sense is the Wigner measure analogue of the stationary phase theorem.

The above results imply that the Wigner transform allows a rigorous and quite straightforward treatment of the semi-classical limit for physical observables, since from (81) and theorem 4 we obtain

\[
\langle u^\varphi(\cdot, t), A(x, \varepsilon D)u^\varphi(\cdot, t) \rangle_2 \xrightarrow{\varepsilon \to 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} A(x, \xi) w(x, \xi, t) dx d\xi, \quad (87)
\]

at least for \( A(x, \xi) \in \mathcal{S} \), but this also holds for more general symbols \( A \), under the additional assumption that \( u_\varphi^\varepsilon \) is \( \varepsilon \)-oscillatory and compact at infinity, cf. [38], [59]. A function \( \psi_\varphi^\varepsilon \) satisfying these properties can be seen as generalized WKB-initial data.

### 5.3 Quadratic Homogenization and Wigner transforms

A particularly important application of Wigner measures can be found in quadratic homogenization problems, i.e. homogenization of physical observables. This has been done for (acoustic) wave equations in [61] (see also [32] for a \( H \)-measure treatment) and for Schrödinger equations in [63], [69] (see also [41] for a numerical study). To this end, the more adapted concept of Wigner series has been developed, cf. [38]:

\[
w_n^\varepsilon(x, k, t) := \frac{1}{|Y|} \sum_{\gamma \in \Gamma} u^\varphi \left(x - \frac{\varepsilon}{2} \gamma, t\right) \overline{u^\varphi \left(x + \frac{\varepsilon}{2} \gamma, t\right)} e^{i k \cdot \gamma}, \quad x \in \mathbb{R}^d, \quad k \in Y, \quad (88)
\]

where \( Y \) again denotes the first Brillouin Zone of the dual lattice \( \Gamma^* \). Having in mind the example of a periodic Schrödinger equation (41) we start from a scalar periodic IVP

\[
-\varepsilon \partial_t u^\varphi + H^W \left( \frac{x}{\varepsilon}, \varepsilon D \right) u^\varphi = 0, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \quad (89)
\]

\[
u^\varphi(x, 0) = u_\varphi^\varepsilon(x) \quad (90)
\]

and, by using a Bloch-decomposition [71] \( L^2(\mathbb{R}^d) = \oplus_{n \in \mathbb{N}} S^\varphi_n \), i.e. writing

\[
u^\varphi(x, t) = \sum_{n \in \mathbb{N}} u_n^\varphi(x, t), \quad (91)
\]

we arrive at a IVP within each Bloch-band space

\[
-\varepsilon \partial_t u_n^\varphi + E_n(\varepsilon D) u_n^\varphi = 0, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \quad (92)
\]

\[
u_n^\varphi(x, 0) = u_n^\varepsilon(x) \quad (93)
\]

Here, \( E_n \) is the Fourier-multiplier pseudo-differential operator, corresponding to the Bloch-eigenvalue \( E_n(k) \), obtained in (47). One can then pass to the limit \( \varepsilon \to 0 \) within the corresponding Wigner series and since the \( u_n^\varphi \perp u_m^\varphi \), for \( n \neq m \), we can decompose the limiting measure in its Bloch-band representation [38]

\[
w_\varphi(x, k, t) = \sum_{n \in \mathbb{N}} w_n(x, k, t), \quad (94)
\]
In each band we henceforth derive the following homogenized IVP

\[ \partial_t w_n + \nabla_k E_n(k) \cdot \nabla_x w_n = 0, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \]
\[ w_n(x, k, 0) = w_n(x, k), \]

which again can connect with the corresponding two-scale WKB-approach, iff \( w_n \) is in mono-kinetic form

\[ w_n(x, k, t) = \rho(x, t) \delta_0(k - \nabla_x \varphi(x, t)), \]

where \( \delta_0 \) denotes the \( \Gamma \)-periodic Delta-distribution. The homogenization of physical observables then follows as in section 5.2 above. Of course, the above procedure can only be applied if the initial Wigner measure vanishes on sets of band-crossings. To study energy/charge-transitions at crossings a more refined analysis must be performed, cf. [30]. For completeness, we remark that the long standing problem of including simultaneously periodic and external fields (which are neglected in [38] and [63]) has been resolved recently in [68], by using a different approach.

In recent years, the Wigner transforms have been successfully applied to more advanced problems, such as the semi-classical limit of the Dirac equation [76], the derivation of effective mass theorems in crystals [69], the semi-classical limit for weakly nonlinear Schrödinger-Poisson systems [62], [84], the refraction of high-frequency waves [67] and the study Landau-Zener phenomena, cf. [28], [29], [30].

A drawback of the Wigner measure approach in comparison to WKB-type methods is the loss of phase-information. For example, the change of the Maslov indices, experienced when passing through the caustic in the multi-valued WKB-representation (31), is not tractable by using Wigner transforms, cf. [74]. Another, even more severe problem is that Wigner transforms so far can not be applied to strongly nonlinear problems, like for example semi-classical NLS. In this context, the results in [12] show that the semi-classical limit for cubic defocusing NLS is ill-posed in terms of Wigner measures.

6 Numerical challenges

The numerical solution of highly oscillatory PDEs poses formidable challenges to numerical analysis. Typical questions are:

Is it necessary to resolve all the fine-structure (oscillations) accurately by the discretisation or can a numerical scheme be devised on a relatively coarse grid such that (at least) weak limits of solutions or observables, resp. are well approximated?

How coarse can this grid be (in terms of the small parameter characterizing the oscillation wave length)? If there are no analytical results available on the accuracy of discretisations or/and on the limiting behavior of solutions or observables resp. (as is often the case for nonlinear dispersive IVPs), then how can the numerically obtained results be trusted, particularly in \( d = 3 \) spatial dimensions when, as is often the case, another grid refinement is prohibitive due to limited computer resources/time? How can analytical information on the continuous problem and on the limiting behavior of solutions be efficiently incorporated in the numerical technique? Let use start the discussion with:

1) Linear Schrödinger-type equations:

It was shown in [64], [65], that finite difference discretisations have significant drawbacks in semi-classical limit regimes for linear Schrödinger equations. In the best of
all cases, i.e. in the Crank-Nicolson scheme and leap-frog scheme, they require the
temporal and spatial grid sizes to be much smaller than the semi-classical parameter
\( \Delta t = o(\varepsilon) \), \( \Delta x = o(\varepsilon) \), in order to give accurate observables. Much more stringent
mesh size constraints are required for accurate wave-functions and for non-time re-
sversible discretisations like Euler schemes. These serious meshing restrictions of
finite difference schemes is intimately connected with the fact that they are not
time-transverse-invariant, i.e. contrary to the continuous Schrödinger equation a
constant shift of the potential changes the numerically obtained observables.
In comparison the so-called Time-Splitting-Trigonometric-Spectral Method (TSTSM),
based on splitting the kinetic term from the potential term and trigonometric spec-
tral discretisation of the Laplacian (which allows for explicit time-integration of both
splitting steps in every time step), behaves significantly better. It was shown in [3]
that the TSTSM conserves the total charge, is time-transverse-invariant, reversible
and requires only \( \Delta x = O(\varepsilon) \) and \( \Delta t = O(1) \) (but dependent on the prescribed er-
ror tolerance), for giving accurate observables. Again, more restrictive meshes are
required in order to obtain accurate wave-functions. We remark that the analysis of
the approximation of observables obtained by discretisations of linear Schrödinger-
type equations in the semi-classical regime was based on Wigner-measure techniques
and not on the classical consistency-stability concept of numerical analysis, which
gives non-sharp results.
Recently, hyperbolic technology based on the concept of \( K \)-multivalued solutions of
conservation laws [8] was used to solve the WKB-system, cf. [39],[40] and a level-set
method was presented in [48]. These methods have the advantage that they give
more direct information on the caustic structure of the underlying Schrödinger equa-
tion since they are based on solving directly for macroscopic semi-classical quanti-
ties. Their disadvantage is the bad accuracy in the reconstructed wave-function at
cautic manifolds (phase transitions).
3) Quadratic homogenization for Schrödinger equations with periodic potentials:
The main numerical difficulty is that, as \( \varepsilon \to 0 \), higher and higher Fourier-modes
of the lattice periodic potential affect the wave-function. A systematic approach,
bypassing this difficulty, was presented in [41]. At first the Bloch (spectral) decom-
position is carried out numerically by a spectral technique, then the (differentiated)
Hamilton-Jacobi equation (46) (in the n-th energy band, \( n \in N \) fixed) is solved
for the multivalued n-th band particle velocity by the method of K-multivalued
solutions of conservation laws and, as postprocessing, the transport equation (50)
for the wave-function amplitudes is solved numerically. Then the wave-function is
reconstructed by following the two-scale WKB technique. So far, the method of K-
multivalued solutions of conservation laws is restricted to one-dimensional problems,
multi-dimensional versions are being investigated at the time being.
3) Nonlinear Schrödinger equations:
The main numerical obstacles are the formation of singularities in focusing NLS and
the creation of new scales at caustics for focusing and defocusing NLS. Typically,
Krasny filters (high Fourier-mode cut-off) are needed in order to avoid artifacts
(like symmetry breaking) in focusing NLS computations. Finite difference meth-
ods typically require prohibitively fine meshes to even approximate observables well
in semi-classical defocusing and focusing NLS computations. However, due to the
creation of new oscillation scales, also the TSTSM needs more severe meshing re-
strictions for post-caustic NLS than for linear SE computations (though less severe
than in finite difference methods). For a description of TSTSM in NLS computa-
tions, an exhaustive set of test runs and a collection of references we refer to [4].
Also we remark that the TSTSM has recently been successfully applied in [2] to the numerical simulation of the Gross-Pitaevskii (quadratically confined) NLS modeling Bose-Einstein condensation (BEC).
Bibliography


