Long-time Asymptotics via Entropy Methods
for Diffusion Dominated Equations

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Abstract. Uniform convergence rates of diffusion dominated equations
towards their asymptotic profiles are quantified via entropy methods for
bounded integrable non-negative initial data with finite entropy. Conver-
gence rates are sharp since they coincide with the purely diffusive ones. The
approach is applied to both convection- and absorption-diffusion equations.
Finally, Wasserstein metrics are used to control the expansion of the support
for the convection-diffusion case.

Key words: convection-diffusion, absorption-diffusion, entropy methods,
large time behaviour, convergence rate.

AMS subject classification: 35B40, 35K65.

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1 Introduction and main result

This article contributes to the analysis of the long-time behaviour of non-negative solutions of diffusion dominated equations, particularly: the convection-diffusion equation

\[ u_t = (u^m)_{xx} - (u^q)_x, \quad (1) \]
\[ u(t = 0, x) = u_0(x) \geq 0, \quad (2) \]

for \((t, x) \in [0, \infty) \times \mathbb{R}\) and for the range of exponents

\[ m \geq 1, \quad q > m + 1, \quad (3) \]

and the absorption-diffusion equation

\[ u_t = (u^m)_{xx} - u^r, \quad (4) \]
\[ u(t = 0, x) = u_0(x) \geq 0, \quad (5) \]

for \((t, x) \in [0, \infty) \times \mathbb{R}\) and for the range of exponents

\[ m \geq 1, \quad r > m + 2. \quad (6) \]

Applications of (1) and (4) occur, for instance, in the modeling of transport processes through a porous medium (see e.g. [Kal]).

Regarding the convection-diffusion equation (1), the well-posedness of the Cauchy problem (1)-(2) as well as the long-time behaviour has been investigated by several authors (see e.g. [EZ, EVZ, Zua, CL] for the parabolic case \(m = 1\) and [BT, Gil89, LS97, LS98, RV] for the degenerate parabolic case \(m > 1\)). It is well-known that solutions of (1) with appropriate data (2) (see section 2 for the precise integrability assumptions) preserve the initial mass

\[ \int_{\mathbb{R}} u \, dx = \int_{\mathbb{R}} u_0 \, dx =: M \]

and, for large times, tend towards an asymptotic self-similar profile \(U_M\), which is uniquely determined by the mass \(M\). More precisely, for \(p \in [1, \infty]\) asymptotic estimates of the form

\[ \lim_{t \to \infty} t^{r_p(q,m)} \|u(t) - U_M(t)\|_p \leq C(M, p, q, m), \quad (7) \]

are established, where \(r_p(q,m) > 0\) denotes the \(L^p\)-convergence rate and \(C(M, p, q, m)\) is a non-negative constant. Considering the assumed exponents \(q > m + 1\), we discuss the so called diffusion dominated range of (1), since for these exponents the asymptotic profiles \(U_M\) are the source-type solutions
of the purely diffusive heat- \((m = 1)\) respectively porous medium \((m > 1)\) equation parameterized by the mass \(M\) (see (22) below). The self-similarity of the source-type solutions implies the invariance of \(U_M\) under the rescalings

\[
u_\lambda(t, x) = \lambda^{\frac{1}{m+1}} u(\lambda t, \lambda^{\frac{1}{m+1}} x),
\]

for all \(\lambda > 0\). By computing the equation satisfied by the rescaled functions \(u_\lambda\), it can be formally guessed that (7) is equivalent to

\[
u_\lambda(\cdot, t) \to U_M(\cdot, t) \quad \lambda \to \infty.
\]

This idea of proving (7) was made rigorous by establishing precise \(\lambda\)-invariant a-priori estimates on the decay of the solutions (see e.g. [EZ] resp. [RV]). Another approach [LS97, LS98] used directly the decay estimates of the solution to measure the \(L^p\)-distance to the asymptotic profile.

Similarly to equation (1), the large time behaviour of the absorption-diffusion equation (4) is dominated by the diffusion term. The precise statement of the convergence, the relevant references and their discussion are postponed to the fourth section.

In this paper, we discuss an entropy-entropy dissipation approach to analyse the long-time behaviour of the Cauchy problems (1)-(2) and (4)-(5). Entropy methods have already been used by several authors in the purely diffusive case [CT00, CJMTU, DE, DP, FM, LM, Ott]. In a preliminary step, a time-dependent rescaling is used, which transforms (1) into a nonlinear Fokker-Planck type equation with quadratic confining potential. Moreover, the time-dependent asymptotic profiles \(U_M\) (shifted in time) are rescaled into stationary profiles.

The actual entropy approach quantifies the dissipation process, which morphs the solutions towards the asymptotic profiles. The main tools are the distinguished Liapunov functional of the purely diffusive case, called entropy functional, and its corresponding entropy dissipation. A quantitative estimate, which bounds the entropy dissipation below by the relative entropy with respect to the asymptotic stationary profiles, was already obtained in previous works [CJMTU, DP, Ott]. Exploiting this entropy-entropy dissipation estimate, one establishes firstly convergence in entropy, which secondly implies convergence in norms for the purely diffusive case.

In our context, after applying the above mentioned rescaling, we shall think about equations (1) and (4) as asymptotic perturbations of the purely diffusive nonlinear Fokker-Planck equations. We show that the behaviour of the relative entropy for large times is controlled by the purely diffusive case reflecting the perturbation character of these models. This control ultimately gives an exponential decay in the \(L^1\)-norm towards the stationary profiles.
Backward rescaling leads finally to algebraic $L^1$-convergence rates in (7) and the $L^p$-rates follow by interpolation with a $L^\infty$-bound.

It is interesting to study how the convergence rates in (7) depend especially on the order of bounded initial moments. For the IVP (1)-(2) with $L^1$-integrable data it was shown in [EZ] ($m = 1$) respectively [LS98] ($m > 1$, non-negative data) that there is $L^1$-convergence but without actual rate in (7) (i.e. $C = 0$, $r_1(q, m) = 0$). On the other hand, assuming additionally a bounded first order moment leads in the parabolic case $m = 1$ [EZ, Zua] to the rates

$$r_1(q, 1) = \begin{cases} 
\frac{1}{2} & \text{for } q > 3, \\
\frac{1}{2} - \varepsilon & \text{for } q = 3 \text{ and for all } \varepsilon > 0, \\
\frac{q}{2} - 1 & \text{for } 2 < q < 3.
\end{cases}$$

(9)

For $q > 3$, the rate $\frac{1}{2}$ is optimal in the sense that it is indeed the convergence rate of solutions of the heat equation towards the heat kernel (see [Zua] for second order terms in the asymptotic expansion (7)). For $2 < q < 3$, the rates $\frac{q}{2} - 1$ are sharp in the class of integrable initial data with bounded first order moment as they can’t be improved by the second order term.

The initial data in the proposed entropy approach will have to guarantee the natural conditions of propagation of finite initial entropy, that means bounded second order moments. Improving the decay estimates by assuming boundedness of higher-order moments is a quite natural question in terms of existing results for the heat equation [DZ94], it is an open problem to use the ideas of the entropy method to obtain such a result even in the pure diffusive case. Our main result therefore assumes

**Theorem 1.1** Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be non-negative with bounded second order moment $\int_\mathbb{R} u_0 |x|^2 \, dx < \infty$. Then,

$$r_1(q, m) = \begin{cases} 
\frac{1}{m+1} & \text{for } 1 \leq m < 2, \\
\frac{1}{m(m+1)} & \text{for } 2 \leq m.
\end{cases}$$

(10)

As main improvement via the entropy method, we obtain the optimal rates (i.e. the convergence rates of the purely diffusive equation) uniformly in the diffusion dominated range $q > m + 1$. This is indeed a consequence of the bounded second order initial moment as will become clear in section 3, where we will prove the optimal rates (10) in the range $m + 1 < q < m + 3$ via an iteration process, which relies crucially on the bounded second order moment.

A similar result is obtained for the absorption-diffusion equation (4). The crucial point here is to tackle the loss of mass due to the absorption term in
the entropy method and thus, to quantify it carefully in the entropy inequalities. The results are generalized both for the convection-diffusion and the absorption-diffusion equations to any dimension. Here, the main difficulty is to find good approximation procedures to overcome some technical difficulties in passing to the limit the entropy dissipation estimate. We refer to the relevant section for details.

The remainder of the paper is organised as follows: In section 2, we restate the used existence theory of (1)-(2) and the a-priori estimates on the solutions. In section 3, we firstly demonstrate formally the entropy method before we prove theorem 3.1 rigorously as well as we discuss more general initial data and the extension to higher dimensions. Section 4 generalizes the procedure to obtain decay rates to the absorption-diffusion equation (4). Finally, in the one dimensional case of the convection diffusion equation, we illustrate the convergence in Wasserstein metrics in section 5. As a consequence, we deduce precise asymptotic estimates of the support growth.

**Notations:** for all $p \in [1, \infty]$, we use $\| \cdot \|_p$ for the usual $L^p(\mathbb{R})$-norm. $(\cdot)_x^2$ shall be the short-form of $((\cdot)_x)^2$. Beside commonly denoted function spaces, $C_b(\mathbb{R} \times (0, \infty))$ specifies the space of continuous and bound functions on $\mathbb{R} \times (0, \infty)$. Finally, $C(\cdots, \cdot)$ or $C_i(\cdots, \cdot)$ $i = 1, 2, 3 \ldots$ will denote various constants depending on the stated arguments as well as on $q$ and $m$, which we shall suppress.

## 2 Preliminaries and rescaling

In this section, we cite results on the initial value problem (IVP)

$$ u_t = \Delta_x (u^m) - d \cdot \nabla_x u^q, \quad u(t = 0, x) = u_0(x), $$

with $(t, x) \in [0, \infty) \times \mathbb{R}^N$, $N \geq 1$ and $d \in \mathbb{R}^N$. Parts of these results are probably not restated in full generality but matching to the exponents (3) and the theorems below (which is e.g. $N \geq 1$ for $m = 1$ and $N = 1$ for $m > 1$).

### 2.1 Existence, apriori estimates, source-type solutions

**Proposition 2.1 (Existence for the linear diffusion case) [EZ, Zua]**

Let $u_0$ be non-negative in $L^1(\mathbb{R}^N)$ for $N \geq 1$. Then, the IVP (11)-(12) for $m = 1$, $q > 2$ has a unique classical solution $u \in C([0, \infty), L^1(\mathbb{R}^N)) \cap C((0, \infty), W^{2,p}(\mathbb{R}^N)) \cap C^1((0, \infty), L^p(\mathbb{R}^N))$ for every $p \in (1, \infty)$ satisfying the
conservation of mass
\[ \int_{\mathbb{R}} u(t, x) \, dx = \int_{\mathbb{R}} u_0(x) \, dx = M, \]
as well as, provided that \( u_0 \in L^p(\mathbb{R}^N) \), the propagation of the \( L^p \)-norm \([CL]\)
\[ \|u(t)\|_p = \|u_0\|_p, \quad t \geq 0. \] (13)

Proposition 2.2 (Existence for the degenerate diffusion case)

(i) Let \( u_0 \) be non-negative in \( L^1(\mathbb{R}) \). Then, the IVP (11)-(12) for \( N = 1 \) with \( m > 1, q > m+1 \) has a unique mild solution \( u \in C([0, \infty), L^1(\mathbb{R})) \) in the sense of nonlinear semigroup theory \([BT]\). Moreover, for \( u_{0_1}, u_{0_2} \in L^1(\mathbb{R}) \) and \((\cdot)_+ \) denoting \( \max\{0, \cdot\} \), we have the \( L^1 \)-contraction property
\[ \|u(t_1) - u(t_2)\|_1 \leq \|u_{0_1} - u_{0_2}\|_1, \] (14)
which ensures also that \( u(t) \) remains non-negative for \( t > 0 \).

(ii) If \( u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R}) \) for \( p \in (1, \infty) \), then \([RV]\)
\[ \|u(t)\|_p \leq \|u_0\|_p, \quad t \geq 0. \] (15)

(iii) Let \( u_0 \) be additionally in \( L^\infty(\mathbb{R}) \) \([LS98]\). Then, \( u \in L^\infty((0, \infty) \times \mathbb{R}) \) satisfies the conservation of mass
\[ \int_{\mathbb{R}} u(x, t) \, dx = \int_{\mathbb{R}} u_0(x) \, dx = M, \]
and
\[ \|u(t)\|_\infty \leq \|u_0\|_\infty, \quad t \geq 0. \] (16)

(iv) Let \( u_0 \) be non-negative in \( C_b(\mathbb{R}) \). Then, there exists a unique generalised solution of the IVP (1), (2) in the sense of \([Gil89]\).

(v) Let \( u_0 \) be non-negative in \( \chi_{m,q} \) with
\[ \chi_{m,q} = \{ v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) : v^m \in W^{1,\infty}_{loc}(\mathbb{R}), (v^m)_x - v^a \in W^{1,1}(\mathbb{R}) \}. \]
Then, \( u(t) \) is in \( C_b([0, \infty), L^1(\mathbb{R})) \) \([BT]\), the mild solution is the generalised solution and there exists \( p_0 \in (1, 2) \) such that for any \( \tau > 0 \) and any compact set \( K \subset \mathbb{R} \times (0, \infty) \),
\[ \|u^a\|_L^\infty(K) + \|(u^m)_x\|_L^{p_0}(K) + \|u_t\|_L^{p_0}(K) \leq C(K, \|u(\tau)\|_L^\infty(\mathbb{R})), \] (17)
where \( C \) is a constant just depending on \( K, \|u(\tau)\|_L^\infty(\mathbb{R}) \) and \( \tau \) \([LS98]\).

(vi) Solutions with compactly supported initial data remain compactly supported for all \( t \geq 0 \) \([Gil88]\).
Remark 2.1 The space $\chi_{m,q}$ is dense in $L^1(\mathbb{R})$ [BT].

Proposition 2.3 (Temporal decay estimates)
Let $m = 1$ and $u_0 \in L^1(\mathbb{R}^N)$ for $N \geq 1$, then for $t_0 > 0$ [CL, EZ]
\[
\|u(t)\|_\infty \leq C(M) t^{-\frac{N}{2}} \quad t > 0. \tag{18}
\]
\[
\|\nabla_x u(t)\|_\infty \leq C(M, t_0) t^{-\frac{N}{2} - \frac{1}{2}} \quad t \geq t_0 > 0, \tag{19}
\]
Let $m > 1$ and $u_0 \in L^1(\mathbb{R})$, then there holds [LS98]
\[
\|u(t)\|_\infty \leq C(M) t^{-\frac{1}{m+1}} \quad t > 0. \tag{20}
\]
Moreover, for $u_0 \in L^1(\mathbb{R})$, it follows from [Rey, Theorem 3.1], that for $t_0 > 0$
\[
\|(u^{m-1})_x(t)\|_\infty \leq C(M, t_0) t^{-\frac{m}{m+1}} \quad t \geq t_0 > 0. \tag{21}
\]

Proposition 2.4 (Source-type solutions as asymptotic profiles)
The source-type solutions in the case $m = 1$ are mass-multiples of the heat kernel, while in the case $m > 1$, they are the Barenblatt-Pattle solutions of the porous medium equation:
\[
U_M(t, x) = \begin{cases} 
M (4\pi t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{4t}}, & m = 1, \\
t^{-\frac{1}{m+1}} \left( \beta_M^2 - c_m^2 |x|^2 t^{-\frac{2}{m+1}} \right)^{\frac{1}{m-1}} & m > 1,
\end{cases} \tag{22}
\]
where $c_m^2 = \frac{m-1}{2m(m+1)}$ and $\beta_M^2$ is a constant such that $\|U_M\|_1 = M$.

2.2 Rescaling
As mentioned in the introduction, it is convenient to rescale the variables and the solution as
\[
x = yR(t), \quad s = \frac{1}{\lambda} \ln (1 + \lambda t), \quad \rho(s, y) = R^N(t) u, \tag{23}
\]
where $R(t)$ is defined as
\[
R(t) := (\lambda t + 1)^{\frac{1}{\lambda}}, \quad R(s) = e^s. \tag{24}
\]
We emphasise that (23) leaves the initial data $\rho_0 = u_0$ unchanged and preserves the $L^1$-norm $\|\rho(t)\|_1 = \|u(t)\|_1$ for all $t \geq 0$.

The rescaling (23) transforms the convection-diffusion equation (11) into a nonlinear Fokker-Planck type equation
\[
\rho_s = \nabla_y (\rho y) + \Delta_y (\rho^m) - R^{-\delta} d \cdot \nabla_y (\rho^q), \quad \rho(s = 0, y) = u_0(y), \tag{25}
\]
where we have set 
\[
\lambda = 2 + N(m - 1), \quad \delta = N(q - (m + 1)) + (N - 1) > 0.
\]

**Proposition 2.5 (Rescaled existence results and decay estimates)**

Let \( m = 1 \) and \( u_0 \) be non-negative in \( L^1(\mathbb{R}^N) \). Then, \( \rho \in C([0, \infty), L^1(\mathbb{R}^N)) \cap C((0, \infty), W^{2, p}(\mathbb{R}^N)) \cap C^4((0, \infty), L^p(\mathbb{R}^N)) \) for every \( p \in (1, \infty) \), preserves the mass \( \int_{\mathbb{R}} \rho(s, y) \, dy = M \) and satisfies, for \( s_0 > 0 \) the bounds
\[
\begin{align*}
\|\rho(s)\|_{\infty} & \leq C(M, s_0) s^{-\frac{N}{2}} \quad 0 < s \leq s_0, \quad (27) \\
\|\rho(s)\|_{\infty} & \leq C(M, s_0) \quad s \geq s_0 > 0, \quad (28) \\
\|\nabla \rho(s)\|_{\infty} & \leq C(M, s_0) \quad s \geq s_0 > 0. \quad (29)
\end{align*}
\]

Provided \( u_0 \in L^p(\mathbb{R}^N) \), the \( L^p \)-norms propagate for \( p \in [1, \infty] \) as
\[
\|\rho(s)\|_p \leq \|u_0\|_p e^{\frac{M}{1-p} \int_0^s \frac{1}{2} \, ds} \quad s \geq 0. \quad (30)
\]

Let \( m > 1 \). Then, the statements (i), (iv)-(vi) of proposition 2.2 remain for \( \rho \) replacing \( u \). The propagation of the \( L^p \)-norms (ii), (iii) changes to
\[
\begin{align*}
\|\rho(s)\|_p & \leq \|u_0\|_p e^{\frac{M}{m+1} \int_0^s \frac{1}{2} \, ds} \quad s \geq 0. \quad (31)
\end{align*}
\]

for \( u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R}) \) with \( p \in (1, \infty] \). The rescaled bounds of proposition 2.3 guarantee still \( \rho \in L^\infty((0, \infty) \times \mathbb{R}) \). For \( p \in [1, \infty) \), we have
\[
\begin{align*}
\|\rho(s)\|_{\infty} & \leq C(M, s_0) s^{-\frac{1}{m+1}} \quad 0 < s \leq s_0, \quad (32) \\
\|\rho(s)\|_{\infty} & \leq C(M, s_0) \quad s \geq s_0 > 0, \quad (33) \\
\|\rho^{m-1}(s)\|_{\infty} & \leq C(M, s_0) \quad s \geq s_0 > 0. \quad (34)
\end{align*}
\]

**Remark 2.2** It is worthwhile to mention that the proof of the \( L^1-L^\infty \) smoothing effect (20) leading to (33) (based on a technique of [BB] and adapted in [LS97]) works only for the \( u \) variables, but not in the rescaled setting.

**Proposition 2.6 (Rescaled asymptotic profiles)**

The source-type solutions (22) rescales to the profiles \( U_M(x, t) \rightarrow \)
\[
\rho_M(s, y) = \begin{cases} 
M(2\pi)^{-\frac{N}{2}} (1 - e^{-2s})^{-\frac{1}{2}} e^{-\frac{|y|^2}{2}} (1 - e^{-2s})^{-1} & m = 1, \\
\lambda^{\frac{1}{2}} (1 - e^{-\lambda s})^{-\frac{1}{2}} \left( \beta_M^2 - \frac{C}{2} \lambda^{\frac{1}{2}} (1 - e^{-s\lambda})^{-\frac{1}{2}} |y|^2 \right)^{\frac{1}{m-1}} & m > 1,
\end{cases}
\]
\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (35)
\]
which tend, as \( s \to \infty \), towards stationary profiles, which are also directly obtained by rescaling the time shifted profiles

\[
U_M(x, t + \lambda^{-1}) \to \rho_{M, \infty}(y) = \begin{cases} 
M(2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} & m = 1, \\
\lambda^{\frac{1}{2}} \left( \beta_M^2 - c_m^2 \lambda^2 \right) \frac{m^2}{m-1} & m > 1.
\end{cases}
\] (36)

Direct expansion of the time shifted profiles yields the convergence rates

\[
\|u_M(t) - u_M(t + \lambda^{-1})\|_1 = O(\lambda t + 1), \quad \|\rho_M - \rho_{M, \infty}\|_1 = O(e^{-\lambda s}). \tag{37}
\]

3 Convergence Rates

Our approach uses the entropy functionals

\[
H(\rho) = \begin{cases} 
\int_{\mathbb{R}} (\rho |\nabla \rho|^2 + \rho \ln \rho) \, dy & m = 1, \\
\int_{\mathbb{R}} (\rho |\nabla \rho|^2 + \frac{1}{m-1} \rho^m) \, dy & m > 1,
\end{cases}
\] (38)

for which the following results are proved in [CJMTU, DP, Ott].

**Proposition 3.1 (Entropy-entropy dissipation estimates),**

(i) The stationary profile \( \rho_{M, \infty} \) is the minimiser of the entropy functional \( H \) in the class of all admissible comparison functions \( \chi_M \)

\[
\chi_M := \left\{ \rho \in L^1(\mathbb{R}^N) : \rho \geq 0, \int_{\mathbb{R}} \rho(y) \, dy = M \right\}
\]

(ii) For \( \rho, \rho_{M, \infty} \in \chi_M \) and with the relative entropy

\[
H(\rho |\rho_{M, \infty}) = H(\rho) - H(\rho_{M, \infty}) \geq 0,
\] (39)

the following Csiszar-Kullback type inequality is valid with a constant \( C \).

\[
\|\rho - \rho_{M, \infty}\|_1 \leq C H(\rho |\rho_{M, \infty})^\alpha \quad \text{with} \quad \alpha = \begin{cases} 
\frac{1}{2} & 1 \leq m \leq 2, \\
\frac{1}{m} & 2 < m.
\end{cases}
\] (40)

(iii) The generalised logarithmic Sobolev inequality

\[
H(\rho |\rho_{M, \infty}) \leq \frac{1}{2} \int_{\mathbb{R}} \rho \xi(y, \rho)^2 \, dy =: D(\rho),
\] (41)

where

\[
\xi(y, \rho) := \begin{cases} 
\frac{|y|^2}{2} + \ln(\rho) & m = 1, \\
\frac{|y|^2}{2} + \frac{m}{m-1} \rho^{m-1} & m > 1,
\end{cases}
\] (42)

holds as long as the right hand side of (41), the so called entropy dissipation term \( D(\rho) \), is finite.
Idea of the proof

To demonstrate formally our approach, it will be useful to rewrite (25) as

$$\rho_s = \left( \rho \xi(y, \rho) - R^{-\delta} \rho^\theta \right)_y.$$  \(43\)

Multiplication of (43) with \(\xi(y, \rho)\) and integration by parts yields

$$\frac{d}{ds} H(\rho) = - \int_{\mathbb{R}} \rho \xi(y, \rho)_y^2 \, dy + R^{-\delta} \int_{\mathbb{R}} \rho^\theta y \, dy.$$  \(44\)

Let \(u_0(1 + |x|^2) \in L^1(\mathbb{R})\) and \(u_0 \in L^\infty(\mathbb{R})\) and consider firstly \(0 < s_0\). Then, by (30) resp. (31) and by Young’s inequality, it follows that

$$\|\rho\|_{\infty}^{q-1} \int_{\mathbb{R}} \rho |y| \, dy \leq C(s_0, \|u_0\|_{\infty}) \left( M + \int_{\mathbb{R}} \rho y^2 \, dy \right) \quad 0 \leq s \leq s_0.$$  

Since the second order moment is bound in terms of the entropy

$$\int_{\mathbb{R}} \rho y^2 \, dy \leq C H(\rho),$$  \(45\)

with a constant \(C\) (which is well-known also for \(m = 1\), see e.g. [CJMTU]) a Gronwall lemma applied on (44) yields the propagation of the initial entropy:

$$H(\rho(s)) \leq C(s_0, M, \|u_0\|_{\infty}) H(u_0) \quad 0 \leq s \leq s_0.$$  \(46\)

Next, for \(s_0 \leq s \leq s_1\), the global \(L^\infty\)-bounds (28), resp. (33) and the global \(W^{1,\infty}\)-bounds (29) resp. (34) justify the logarithmic Sobolev inequality (41) for all \(s_1 > s_0\). This implies further that

$$\frac{d}{ds} H(\rho) \leq -2H(\rho M, \|u_0\|_{\infty}) + R^{-\delta} \int_{\mathbb{R}} \rho^\theta y \, dy \quad s \geq s_0,$$  \(47\)

$$\leq -2H(\rho) + 2H(\rho M, \|u_0\|_{\infty}) + R^{-\delta}\|\rho\|_{\infty}^{q-1} \frac{1}{2} (M + CH(\rho)),$$

by (39), (45) and by Young’s inequality. Moreover, using the global \(L^\infty\)-bounds of (28) resp. (33), we estimate that

$$\frac{d}{ds} H(\rho) \leq \left( -2 + R^{-\delta} C_1(s_0, M) \right) H(\rho) + C_2(s_0, M) \quad s_0 \leq s \leq s_1.$$  

By choosing \(s_0\) such that \(R^{-\delta}(s_0) C_1(s_0, M) < 1\), a Gronwall lemma yields

$$H(\rho(s)) \leq C(s_0, M, H(\rho(s_0))) \quad s_0 \leq s \leq s_1,$$
for all \( s_1 > s_0 \). Hence, in combination with (46), the entropy and therefore the second order moment is globally bound:

\[
\int_{\mathbb{R}} \rho y^2 \, dy \leq C H(\rho(s)) \leq C(s_0, M, \|u_0\|_{\infty}, H(u_0)) \quad s \geq 0. \tag{48}
\]

Considering the relative entropy, we may now estimate the second term on the r.h.s. (47) using the global entropy bound (48) such that we obtain

\[
\frac{d}{ds} H(\rho|\rho_{M,\infty}) \leq -2 H(\rho|\rho_{M,\infty}) + R^{-\delta} C(s_0, M, \|u_0\|_{\infty}, H(u_0)) \quad s \geq s_0,
\]

and therefore, by a Gronwall lemma,

\[
H(\rho|\rho_{M,\infty}) \leq C(s_0, M, \|u_0\|_{\infty}, H(u_0)) e^{-\min\{\delta, 2\} s} \quad s \geq s_0,
\]

\[
\|\rho - \rho_{M,\infty}\|_1 \leq C_1(s_0, M, \|u_0\|_{\infty}, H(u_0)) e^{-\min\{\delta, 2\} \alpha s} \quad s \geq s_0,
\]

where we have used the Csiszár-Kullback type inequality (40).

In the cases \( \delta < 2 \), the convergence rate can be improved iteratively by the following argument: Since \( \int_{\mathbb{R}} \rho_{M,\infty}^2 \, dy = 0 \), the Taylor expansion \( \rho^\delta = \rho_{M,\infty}^\delta + q \zeta^{\delta-1}(\rho - \rho_{M,\infty}) \) with \( |\zeta| \leq \max\{|\rho(t)|_{\infty}, |\rho_{M,\infty}|_{\infty}\} \) gives

\[
\int_{\mathbb{R}} \rho^\delta y \, dy \leq \|q\zeta\|_{\infty}^{\delta-1} \|\rho - \rho_{\infty}\|^2 \|\rho - \rho_{\infty}\|_1^{\delta-1} \leq C(s_0, M, \|u_0\|_{\infty}, H(u_0)) e^{-\frac{\delta}{\alpha + 1} s},
\]

where we have used the \( L^\infty \)-bounds (28) resp. (33) together with the entropy bound (48). Hence the last term on the r.h.s. of (47) decays actually with the rate \( \delta_1 = \delta(1 + \frac{2}{\alpha}) \). Iterating this argument, the improvement of the \( L^1 \)-convergence rate stops after finitely many iterations at the optimal rate \( 2\alpha \) as soon as \( \delta_k = \delta_{k-1}(1 + \frac{2}{\alpha}) \geq 2 \).

Finally, rescaling backwards together with (37) gives the \( L^1 \)-convergence rates stated in theorem 3.1. The convergence rates \( r_p \) in other \( L^p \)-norms follow by interpolation with the \( L^\infty \)-bounds (28) resp.(33)

\[
\|u - u_M\|_p \leq R^\frac{1}{p}(t) \|\rho - \rho_M\|_{\infty}^{\frac{1}{p}} \|u - u_M\|_1^{\frac{1}{p}} \leq C(1 + t)^{-\frac{p-1}{\alpha(m+1)} - \frac{r_1}{p}}. \tag{49}
\]

### 3.1 Proof of the main result

**Theorem 3.1** Let \( u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) be non-negative with bounded second order moment \( \int_{\mathbb{R}} u_0 |x|^2 \, dx < \infty \). Then, the solutions \( u \) of (1)-(2) satisfies

\[
\|u - U_M\|_1 \leq C(M, \|u_0\|_{\infty}, H(u_0))(1 + t)^{-r_1},
\]
with a constant $C$ depending on $M, \|u_0\|_\infty$ and $H(u_0)$ and the $L^1$-convergence rates

$$r_1(q, m) = \begin{cases} \frac{1}{m+1} & 1 \leq m < 2, \\ \frac{2}{m(m+1)} & 2 \leq m. \end{cases}$$

(50)

Moreover, for $p \in (1, \infty)$,

$$\|u - U_M\|_p \leq C(p, M, \|u_0\|_\infty, H(u_0))(1 + t)^{-r_p},$$

with

$$r_p = \frac{p - 1}{p(m + 1)} + \frac{r_1}{p}.$$  (51)

**Remark 3.1** We obtain the $L^1$-rates (50) as a benefit of our assumptions on the initial data, especially of the bounded second order moment. The rates are uniform in the diffusion dominated range $q > m + 1$ as well as sharp and optimal (i.e. the same as for the purely diffusive equations). For general non-negative $L^1$-data, no $L^1$-rates hold [EZ, LS98]. For $L^1$-data with bounded first order moment, weaker rates as (50) are sharp in the case $m = 1$ for $2 < q < 3$ [Zua]. The $L^p$-rates (51) are improvements of the general $L^p$-rates (i.e. the first term on the r.h.s. of (51) [EZ, LS98]) by the new $L^1$-rates.

**Proof of theorem 3.1:** In the linear diffusion case $m = 1$, the regularity of the classical solution (proposition 2.5) justifies the formal integration by parts and therefore the global bound of the second order moment (48). Together with the $L^\infty$-bound (28) and the $W^{1,\infty}$-bound (29) this ensures that the entropy dissipation $D(\rho)$ is globally bound. Hence, the logarithmic Sobolev inequality (41) is justified, which completes the proof in this case.

In the degenerate diffusion case $m > 1$, we restrict firstly to compactly supported initial data in $\chi_{m,q}$. Then, for $s_0 > 0$, the mild solution $u(s_0)$ is compactly supported in $C_b(\mathbb{R})$. Taking $u(s_0)$ as initial data, we consider an approximating sequence $u(s_0)_k$ introduced in [Gil89, pp. 182-183] (cf. also [LS98]) with

$$\frac{1}{k} \leq u(s_0)_k \leq 2\|u(s_0)\|_\infty, \quad u(s_0)_{k+1} \leq u(s_0)_k,$$

and $u(s_0)_k$ converges to $u(s_0)$ in $C(\mathbb{R})$. By denoting $u_k$ the classical solution of (1) with initial data $u(s_0)_k$, it follows that

$$\frac{1}{k} \leq u_k \leq 2\|u(s_0)\|_\infty \quad t \geq 0,$$  (52)

and that the limit

$$\lim_{k \to \infty} u_k(x, t) = \overline{u}(x, t)$$  (53)
exists for all \((x, t) \in \mathbb{R} \times [0, \infty)\) such that \(\overline{u}(x, t) = u(x, t + s_0)\) [Gil89]. We denote in the following with \(\rho_k\) the rescaled approximative solutions. Since the approximative solutions are no longer integrable, we use a sequence of cutoff-functions \(\theta_n = \theta(\frac{x}{n})\) for \(n \in \mathbb{N}\), where \(\theta \in \mathcal{D}(\mathbb{R})\) with \(0 \leq \theta \leq 1\) and \(\theta(y) = 1\) for \(|y| \leq 1\) and \(\theta(y) = 0\) for \(|y| \leq 2\). Since the approximation alters only the initial data, the approximative solutions satisfy the equation

\[
(\rho_k)_s = \left(\rho_k \left(\frac{y^2}{2} + \frac{m}{m-1} \rho_k^{m-1}\right)_y + R^{-\delta} \rho_k^q\right)_y,
\]  

(54)

for which multiplication with \(\left(\frac{y^2}{2} + \frac{m}{m-1} \rho_k^{m-1}\right)\theta_n\) and integration by parts gives

\[
\frac{d}{ds} \int_{\mathbb{R}} \left(\rho_k \left(\frac{y^2}{2} + \frac{1}{m-1} \rho_k^m\right)_y \theta_n dy = - \int_{\mathbb{R}} \rho_k \left(\frac{y^2}{2} + \frac{m}{m-1} \rho_k^{m-1}\right)_y \theta_n dy \right.
\]

\[
- \int_{\mathbb{R}} \rho_k \left(\frac{y^2}{2} + \frac{m}{m-1} \rho_k^{m-1}\right) (\theta_n)_y dy
\]

\[
- R^{-\delta} \int_{\mathbb{R}} \rho_k^q \theta_n dy + R^{-\delta} \int_{\mathbb{R}} \rho_k^q + \frac{m}{q + m - 1} \int_{\mathbb{R}} \rho_k^q + \frac{m}{q + m - 1} \int_{\mathbb{R}} \rho_n dy.
\]  

(55)

Expanding the first term on the right hand side of (55) leads to

\[
- \int_{\mathbb{R}} \rho_k y^2 \theta_n dy - 2 \int_{\mathbb{R}} \rho_k^m y \theta_n dy - \frac{m^2}{(m-1)^2} \int_{\mathbb{R}} \rho_k (\rho_k^{m-1})^y \theta_n dy.
\]  

(56)

For the first term, the pointwise convergence (53) of \(\rho_k \to \rho\) and the uniform \(L^\infty\)-bound (52) on \(\rho_k\) justify to pass to the limit \(k \to \infty\) for all \(n \in \mathbb{N}\). For the second term of (56), (17) implies since \(q > m + 1\) that \(\rho_k^m\) is uniformly bound in \(W^{2,p_0}(K) \cap W^{1,\infty}(K)\) [LS98]. By a compactness argument [LS98] and by the pointwise convergence (53), there exists a subsequence \(\rho_k\) such that

\[
\lim_{k \to \infty} \rho_k = \rho \quad \text{in} \quad C(\mathbb{R} \times (0, \infty)),
\]  

(57)

\[
\lim_{k \to \infty} (\rho_k^m)_y(s) = (\rho^m)_y(s) \quad \text{in} \quad L^p_{\text{loc}}(\mathbb{R}) \quad \text{for almost every} \quad s > 0,
\]  

(58)

and therefore

\[
\lim_{k \to \infty} \int_{\mathbb{R}} (\rho_k^m)_y y \theta_n dy = \int_{\mathbb{R}} (\rho^m)_y y \theta_n dy \quad \forall n \in \mathbb{N},
\]  

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for almost every \( s > 0 \). For the third term of (56), the uniform \( L^\infty \)-bound (34) ensures weak convergence of \( (\rho_k^{m-1})_y \) in \( L^2_{\text{loc}} \) to some limit \( \Psi \) and hence
\[
\lim_{k \to \infty} \int_{\mathbb{R}} \rho_k(\rho_k^{m-1})^2_y \theta_n \, dy = \int_{\mathbb{R}} \rho \Psi \theta_n \, dy \quad \forall n \in \mathbb{N}, \tag{59}
\]
such that it remains to identify \( \Psi \) as \( (\rho^{m-1})^2_y \) in integral on the r.h.s. of (59). By using (34), then
\[
\lim_{k \to \infty} \int_{\{\rho=0\}} \rho_k(\rho_k^{m-1})^2_y \theta_n \, dy = 0 \quad \forall n \in \mathbb{N},
\]
and thus, contributions to this integral come only from the open set \( \{\rho > 0\} \) (since \( \rho \) is continuous), on which (57) implies \( \rho_k(s, y) \searrow \rho(s, y) \) for all \( (s, y) \) and (58) yields, for another subsequence,
\[
(\rho_k^m)_y(s, y) = \frac{m}{m-1} \rho_k(\rho_k^{m-1})_y \to \frac{m}{m-1} \rho(\rho^{m-1})_y = (\rho^m)_y(s, y) \quad \text{a.e.}.
\]
Hence, we conclude almost everywhere convergence of \( (\rho_k^{m-1})^2_y \to (\rho^{m-1})^2_y \) on the set \( \{\rho > 0\} \) and the Lebesgue dominated convergence theorem yields
\[
\lim_{k \to \infty} \int_{\mathbb{R}} \rho_k(\rho_k^{m-1})^2_y \theta_n \, dy = \int_{\mathbb{R}} \rho(\rho^{m-1})^2_y \theta_n \, dy \quad \forall n \in \mathbb{N}.
\]
Returning to (55), the second term on the right hand side expands with further integration by parts to
\[
- \int_{\mathbb{R}} \left( \rho_k \frac{y^3}{2} + \frac{1}{m-1} \rho_k^m y \right) (\theta_n)_y \, dy + \int_{\mathbb{R}} \left( \rho_k^m \frac{y^2}{2} + d_m \rho_k^{2m-1} \right) (\theta_n)_{yy} \, dy, \tag{60}
\]
with \( d_m = m^2/(2m^2 - 3m + 1) \) and (52) allows to pass to the limit in \( k \) for all \( n \in \mathbb{N} \). Similarly, it is justified to drop the subscript \( k \) in the remainder terms of (55).

For the limit \( n \to \infty \) of
\[
\frac{d}{ds} \int_{\mathbb{R}} \left( \rho \frac{y^2}{2} + \frac{m}{m-1} \rho^m \right) \theta_n \, dy = - \int_{\mathbb{R}} \rho \left( \frac{y^2}{2} + \frac{m}{m-1} \rho^{m-1} \right)_y^2 \theta_n \, dy
\]
\[
+ R^{-\delta} \int_{\mathbb{R}} \rho^\delta y \theta_n \, dy + \int_{\mathbb{R}} O(\rho y^\delta, \rho^m y, \rho^{m+1}, \rho^\delta y^2)(\theta_n)_y \, dy
\]
\[
+ \int_{\mathbb{R}} O(\rho^m y^2, \rho^{2m-1})(\theta_n)_{yy} \, dy,
\]

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the bounds (33) and (34) imply again, since \( \rho \) is compactly supported, that each integrant is bounded uniformly in \( n \) by integrable functions and hence, by the dominated convergence theorem, that

\[
\frac{d}{ds} H(\rho) = -\int_{\mathbb{R}} \rho \left( \frac{y^2}{2} + \frac{m}{m-1}\rho^{m-1} \right) dy + R^{-\delta} \int_{\mathbb{R}} \rho^q y dy \quad s \geq s_0,
\]

with a global bound on the entropy dissipation \( D(\rho) \) justifying the generalised logarithmic Sobolev inequality (41). Finally, Gronwall’s lemma yields with (48)

\[
H(\rho) \leq C(M, \|u_0\|_\infty, H(u_0)) e^{-\min\{2,\delta\} s} + H(\rho_\infty), \quad s \geq s_0, \quad (61)
\]

for \( \rho \) compactly supported in \( \chi_{m,q} \) and therefore in \( L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) by a density argument (cf. remark 2.1).

In order to obtain the result for all \( u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) with finite entropy, we consider smooth truncations \( u_{0k} \) with compact support and reduced mass \( M_k = \int_{\mathbb{R}} u_{0k} \, dx \) such that

\[
u_{0k} \to u_0 \quad \text{in} \quad L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),
\]

\[
H(u_{0k}) \to H(u_0) \quad \text{in} \quad \mathbb{R}, \quad M_k \neq M \quad \text{in} \quad \mathbb{R},
\]

and therefore \( H(\rho_{M_k,\infty}) \neq H(\rho_{M,\infty}) \). Then, the \( L^1 \)-contraction (14) implies that, for a subsequence, \( \rho_k \) converges to \( \rho \) almost everywhere and hence

\[
\lim_{k \to \infty} \left( \rho_k \frac{y^2}{2} + \frac{1}{m-1}\rho_k^m \right) = \left( \rho \frac{y^2}{2} + \frac{1}{m-1}\rho^m \right) \quad \text{almost everywhere}.
\]

Since the r.h.s. of (61) (for \( \rho_k \) instead of \( \rho \)) depends continuously on the approximation, we have the uniform bound

\[
H(\rho_k) \leq 2C(M, \|u_0\|_\infty, H(u_0)) e^{-\min\{2,\delta\} s} + H(\rho_{M,\infty}),
\]

for \( k \) large enough and Fatou’s lemma implies

\[
H(\rho|\rho_{M,\infty}) \leq \liminf_{k \in \mathbb{N}} H(\rho_k|\rho_{M,\infty}) \leq 2C(M, \|u_0\|_\infty, H(u_0)) e^{-\min\{2,\delta\} s},
\]

and hence the decay of \( H(\rho|\rho_{M,\infty}) \) as stated in the theorem. The improvement of the decay rates follows from the formal argument. ■
3.2 Generalisations

Initial data

In the above section, the assumption \( u_0 \in L^\infty(\mathbb{R}) \) was only necessary to prove the propagation of the initial entropy until a positive time \( s_0 > 0 \), since after that, the propagation of entropy follows from (28), (29) resp. (33), (34).

**Lemma 3.1** Let \( u_0 \in L^1(\mathbb{R}) \cap L^{2q-1}(\mathbb{R}) \) be non-negative with bounded second order moment \( \int_\mathbb{R} u_0 |x|^2 dx < \infty \). Then, for \( s_0 > 0 \), holds

\[
H(\rho(s)) \leq C(M, \|u_0\|_{2q-1}) H(u_0) \quad 0 < s \leq s_0.
\]

**Proof:** It is sufficient to show the propagation of the second order moment:

\[
\frac{d}{ds} \int_\mathbb{R} \rho \frac{y^2}{2} dy = -2 \int_\mathbb{R} \rho \frac{y^2}{2} dy + \int_\mathbb{R} \rho^m dy + R^{-\delta} \int_\mathbb{R} \rho^q y dy,
\]

Using Young’s inequality for the last term on the r.h.s., we estimate

\[
\frac{d}{ds} \int_\mathbb{R} \rho \frac{y^2}{2} dy \leq -2(1 - R^{-\delta}) \int_\mathbb{R} \rho \frac{y^2}{2} dy + \int_\mathbb{R} \rho^m dy + R^{-\delta} \int_\mathbb{R} \rho^{2q-1} dy,
\]

and the lemma follows by the propagation of the \( L^{2q-1} \)-norm (30) resp. (31) and a Gronwall lemma. \( \square \)

**Corollary 3.1** Let \( u_0 \in L^1(\mathbb{R}) \cap L^{m} (\mathbb{R}) \) be non-negative with bounded second order moment \( \int_\mathbb{R} u_0 |x|^2 dx < \infty \). Then, theorem 3.1 holds with

\[
\|u - U_M\|_1 \leq C(M, \|u_0\|_{2q-1}, H(u_0))(1 + t)^{-r_1},
\]

**Theorem 3.2** Let \( q < m + 2 \) and \( u_0 \in L^1(\mathbb{R}) \cap L^{m}(\mathbb{R}) \) be non-negative with bounded second order moment \( \int_\mathbb{R} u_0 |x|^2 dx < \infty \). Then theorem 3.1 holds with

\[
\|u - U_M\|_1 \leq C(M, \|u_0\|_{m}, H(u_0))(1 + t)^{-r_1},
\]

**Proof:** For \( s_0 > 0 \), we estimate for (44) that

\[
\frac{d}{ds} H(\rho) \leq \left| \int_\mathbb{R} \rho^q y dy \right| \leq \|\rho\|_{q-1}^{q-1} \left( \int_\mathbb{R} \rho dy + \int_\mathbb{R} \rho y^2 dy \right) \quad 0 < s \leq s_0,
\]

and therefore, using the sharp bounds (27) resp. (32) as well as (45),

\[
\frac{d}{ds} H(\rho) = \frac{d}{ds} (M + H(\rho)) \leq C(M) s^{\frac{q-1}{m+2}} (M + H(\rho)) \quad 0 < s \leq s_0.
\]

Since \( q < m + 2 \), \( s^{\frac{q-1}{m+2}} \) is integrable on the interval \( (0, s_0) \) and a general Gronwall lemma implies the propagation of entropy and the theorem. \( \blacksquare \)
Higher dimensions

The entropy method of section 3 is essential independent of the space dimension and applies formally straight-forward to the multidimensional IVP’s (11), (12), i.e. to the rescaled IVP’s

\[ \rho_s = \nabla \cdot (\rho y + \nabla (\rho^m)) - R^{-\delta} d \cdot \nabla \rho^g, \]

\[ \rho(s = 0, y) = u_0(y) \geq 0. \]

The objective of the next theorem is to generalize rigorously the result of theorem 3.1 to any dimension.

**Theorem 3.3** Let \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) for \( N > 1 \) be non-negative with bounded second order moment \( \int_{\mathbb{R}} u_0 |x|^2 \, dx < \infty \). Then, the solutions \( u \) of (11)-(12) satisfies

\[ \|u - U_M\|_1 \leq C(M, \|u_0\|_\infty, H(u_0))(1 + t)^{-r_1}, \]

with a constant \( C \) depending on \( M, \|u_0\|_\infty \) and \( H(u_0) \) and the \( L^1 \)-convergence rates

\[ r_1(q, m) = \begin{cases} \frac{1}{N(m-1)+2} & 1 \leq m < 2, \\ \frac{1}{mN(m-1)+2} & 2 \leq m. \end{cases} \]

Moreover, for \( p \in (1, \infty) \),

\[ \|u - U_M\|_p \leq C(p, M, \|u_0\|_\infty, H(u_0))(1 + t)^{-r_p}, \]

with

\[ r_p = \frac{N}{N(m-1)+2} \frac{p-1}{p} + \frac{r_1}{p}. \]

**Proof:** The proof of the theorem 3.3, in the case \( m = 1 \), is analog to theorem 3.1, since the existence theory (proposition 2.1) and the temporal decay estimates (proposition 2.3) have been equally well established for all \( N \geq 1 \) [EZ, CL].

For \( m > 1 \) and \( N > 1 \), existence theory, growth of the support and sufficient \( L^\infty \)-decay estimates have been shown in [RV]. Nevertheless, to prove theorem 3.3, we lack a multi-dimensional analog of (21) in order to control the entropy dissipation (see [LJ]). Therefore, we cannot pass to the limit in the entropy dissipation term in the strong sense and thus, there is no way rather than finding a good approximation argument based on regularized problems from which we can get an approximated entropy decay estimate of the type

\[ \frac{d}{ds}H(\rho) \leq -2H(\rho |\rho_{M,\infty}|) + R^{-\delta} \int_{\mathbb{R}} \rho^g y \, dy \quad s \geq 0 \]
written in integral form. This can be done by using carefully the compactness arguments, growth support estimates and the approximation procedure by homogeneous Dirichlet problems in [RV]. Other regularizing procedures as the no-flux boundary conditions have been used for the porous-medium case in [Ott, CJMTU]. However, they introduce, in this particular case, nasty boundary conditions that we do not know how to control. Thus, we will refer for most of the details to [RV] and work here out only the main differences.

**Step 0. Main objective.** We first remark that the result is a consequence of proving the decay of the entropy functional in the form

\[ H(\rho_M, \rho_{M, \infty}) \leq C(s_0, M, \|u_0\|_\infty, H(u_0)) e^{-2s} \]

with \( s_0 \) large enough for solutions of (62)-(63).

**Step 1. Density argument.** It suffices to prove (69) for continuous compactly supported initial data \( u_0 \) by density arguments, therefore by the growth support results in [RV, Theorem 1.3], we can assume that the solution of the rescaled Cauchy problem (62)-(63) remains always inside a fixed euclidean ball \( B_r \). As a consequence the solution of the initial boundary value problem

\[ \rho_s = \nabla \cdot (\rho \nabla + \nabla (\rho^m)) - R^{-\delta} d \cdot \nabla \rho^d, \quad \text{for } y \in B_r \text{ and } s > 0 \]

\[ \rho(s = 0, y) = u_0(y) \geq 0, \quad \text{for } y \in B_r \]

\[ \rho = 0, \quad \text{for } y \in \partial B_r \text{ and } s \geq 0 \]

is the same as the Cauchy problem (62)-(63).

**Step 2. Approximated problems.** We now define following the ideas in [Vaz92, RV] an approximated problem to (70)-(72)

\[ \frac{\partial \rho_n}{\partial s} = \nabla \cdot (\rho_n \nabla + \nabla (\rho_n^m)) - R^{-\delta} d \cdot \nabla (d \rho_n^d), \quad \text{for } y \in B_r \text{ and } s > 0 \]

\[ \rho_n(s = 0, y) = u_0(y) + \frac{1}{n} > 0, \quad \text{for } y \in B_r \]

\[ \rho_n = \frac{1}{n}, \quad \text{for } y \in \partial B_r \text{ and } s \geq 0 \]

which has a smooth positive non mass-preserving solution in \([0, S] \times B_r\) for any \( S > 0 \) since the equation for this initial data is non degenerate and standard classical results apply. The solutions \( \rho_n \) satisfies by maximum principle arguments bounds from above uniform in \( n \) on \([0, S] \times \overline{B}_r\) for any \( S > 0 \) (although depending on \( S \)) and the solutions are positively bounded away from zero. The first is just a consequence of the simple remark that \( Ce^{Ns} \) are supersolutions of the problem for certain choices of \( C \). Moreover, they form a decreasing sequence of functions, i.e. \( \rho_{n+1}(s, y) \leq \rho_n(s, y) \) on \([0, S] \times \overline{B}_r\).

It is quite standard how to pass to the limit from the problem (73)-(75) to
the problem (70)-(72). The main ingredients [RV] are the energy estimate, that is, a uniform bound in $n$ on
\[ \int_0^S \int_{B_r} |\nabla \rho_n^m|^2 \, dx \, ds \]
for any $S > 0$ and the general result of equicontinuity of degenerate quasilinear parabolic equations in [DiB] that proves that the sequence $\rho_n$ is equicontinuous in $C([0, S] \times \overline{B_r})$. For the first one, it suffices to multiply the equation by $\rho_n^m - (\frac{1}{n})^m$, to integrate by parts and to estimate some of the terms using the uniform $L^\infty$ bounds. We refer all these details to [Vaz92, RV] and we will use that the limit function $\rho$ is the weak solution in the standard sense for (70)-(72) and therefore, for (62)-(63). Let us remind that the limit is in fact a strong limit in $C([0, S] \times \overline{B_r})$.

**Step 3. Approximated entropy decay estimate.** - Let us multiply (73) by $\xi(y, \rho_n) - A_n$ with $\xi$ as defined in (42) and $A_n$ the value of $\xi(y, \rho_n)$ on the boundary of $B_r$, that is
\[ A_n = \frac{r^2}{2} + \frac{m}{m-1} \left( \frac{1}{n} \right)^{m-1} . \]
Integrating the result by parts we find (with the outer normal $\nu$ and the surface measure $d\sigma_r$)
\[ \frac{d}{ds} H(\rho_n) - A_n \frac{d}{ds} M_n(s) = -D(\rho_n) + R^{-\delta} \int_{B_r} \rho_n^q(y \cdot d) \, dy 
- R^{-\delta} \frac{m}{m+q-1} \int_{\partial B_r} \rho_n^{m+q-1}(d \cdot \nu) \, d\sigma_r \]
where $M_n(s)$ is the mass of the solution $\rho_n$ at time $s$ and $D(\rho_n)$ is the entropy dissipation at the solution $\rho_n$. Using the generalized logarithmic Sobolev inequality applied to $\rho_n$ [CJMTU] and substituting the value of $\rho_n$ at the boundary, we deduce
\[ \frac{d}{ds} H(\rho_n) - A_n \frac{d}{ds} M_n(s) \leq -H(\rho_n | \rho_{M_n(s), \infty}) + R^{-\delta} \int_{B_r} \rho_n^q(y \cdot d) \, dy \n- R^{-\delta} \frac{m}{m+q-1} \int_{\partial B_r} (d \cdot \nu) \, d\sigma_r \]
where $\rho_{M_n(s), \infty}$ is the Barenblatt-Pattle profile with mass $M_n(s)$.

**Step 4. Limit of the entropy decay estimate.** - Working in the integrated over the time interval $[0, S]$ version of the last inequality, it is
straightforward to pass to the limit \( n \to \infty \). Due to the convergence of \( \rho_n \) towards \( \rho \) in \( C([0,S] \times \overline{B_R}) \) and the conservation of mass of \( \rho \) solution of (70)-(72), we conclude

\[
\frac{d}{ds} H(\rho) \leq -2H(\rho_{M,\infty}) + R^{-\delta} \int_R \rho^\delta y dy \quad s \geq 0,
\]

in integral form. At this stage, all the formal arguments done in the proof of theorem 3.1 applies and (69) is proven for the initial data in Step 1 and the general result is deduced taking into account Step 0. \( \blacksquare \)

4 Absorption-di\'usion equation

The entropy approach, which was presented in section 3, applies in a modified way to the absorption-di\'usion equation

\[
\begin{align*}
  u_t &= (u^m)_{xx} - u^r, \\
  u(t = 0, x) &= u_0(x) \geq 0,
\end{align*}
\]

for \((t, x)\) in \([0, \infty) \times \mathbb{R}\) and for the range of exponents

\[ m \geq 1, \quad r > m + 2, \]

for which the large-time behaviour is also dominated by the di\'usion term (see e.g. [GV84, KP, Vaz98]). For non-negative \( u_0 \in L^1(\mathbb{R}) \), it was shown that

\[
\lim_{t \to \infty} t^{-\frac{1}{m+1}} |u(t) - U_{M(\infty)}(t)| \to 0,
\]

uniformly on the sets \( \{|x| \leq at^{-\frac{1}{m+1}}\} \) for \( a \geq 0 \) and the positive limiting mass

\[
M(\infty) := \|u_0\|_1 - \int_0^\infty \int_{\mathbb{R}} u^r dx dt > 0.
\]

Assuming non-negative integrable initial data, existence and uniqueness of non-negative bounded and continuous weak solutions of (4) can be found, for instance, in [Kal, GV91] and the references therein. We consider non-negative \( u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). Then, in order to perform the entropy approach, we obtain analog \( L^\infty\)-decay to (18) respectively (20) by comparison with the heat porous media equation. The \( W^{1,\infty\ast}\)-decay analog to (19) respectively (21) of proposition 2.3 has been shown in [HV].

The rescaling (23) transforms (4) into

\[
\begin{align*}
  \rho_s &= (\rho y)_y + (\rho^m)_{yy} - R^{-\delta} \rho^r, \\
  &= (\rho \xi(y, \rho)_y - R^{-\delta} \rho^r, \quad (76)
\end{align*}
\]

\[
\begin{align*}
  \rho_s &= (\rho y)_y + (\rho^m)_{yy} - R^{-\delta} \rho^r, \\
  &= (\rho \xi(y, \rho)_y - R^{-\delta} \rho^r, \quad (77)
\end{align*}
\]

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where \( \delta = r - (m + 2) > 0 \) and \( \xi(y, \rho) \) as in (42) and the solution \( \rho(s) \) satisfies in particular
\[
\|\rho(s)\|_\infty \leq C(\|u_0\|_1, \|u_0\|_\infty) \quad s > 0 .
\] 
(78)

Moreover, the absorption term leads obviously to a decay of the mass \( M(s) := \int_R \rho(s, y) \, dy \):
\[
\frac{d}{ds} M(s) = -R^{-\delta} \int_R \rho^r \, dy .
\] 
(79)

By (78) and (79), the \( L^p \)-norms are globally bound for all \( p \in [1, \infty] \). Hence, by multiplying (76) with \( y^{2n} \) for \( n \in \mathbb{N} \), integration by parts yields
\[
\frac{d}{ds} \|\rho y^{2n}\|_1 \leq (-2n + R^{-\delta} \|\rho\|_\infty^{r-1}) \|\rho y^{2n}\|_1 + 2n(2n - 1)\|\rho\|_\infty^{n-1} \|\rho y^{2(n-1)}\|_1
\] 
and therefore, by iteration \( 2(n-1) \rightarrow 2n \), the propagation of the moments
\[
\|\rho y^{2n}\|_1(s) \leq C(\|u_0\|_1, \|u_0\|_\infty) \|u_0(y)y^{2n}\|_1 \quad s > 0, n \in \mathbb{N} .
\] 
(80)

The entropy approach is based on the generalised logarithmic Sobolev inequality (41) and the Csiszár-Kullback type inequality (40), which are valid on the set of admissible comparison functions \( \chi_M \) with the same mass \( M \) (see proposition 3.1). We shall therefore consider the relative entropy with respect to the asymptotic profile \( \rho_{M(s),\infty} \) depending the actual mass \( M(s) \)
\[
H(\rho|\rho_{M(s),\infty}) := H(\rho) - H(\rho_{M(s),\infty}) \geq 0 \quad s \geq 0 .
\] 
(81)

For the time-derivative of the entropy \( H(\rho) \), we multiply (77) with \( \xi(y, \rho) \) as defined in (42) and integrate by parts. We obtain (with (79) for \( m = 1 \))
\[
\frac{d}{ds} H(\rho) = -D(\rho) - R^{-\delta} \int_R \rho^r \times \left\{ \begin{array}{l}
\left( \frac{y^2}{2} + \ln \rho + 1 \right) \, dy \quad m = 1 , \\
\left( \frac{y^2}{2} + \frac{m}{m-1} \rho^{m-1} \right) \, dy \quad m > 1 ,
\end{array} \right.
\] 
(82)

and, as a direct consequence, the propagation of the entropy
\[
H(\rho(s)) \leq C(\|u_0\|_1, \|u_0\|_\infty) H(u_0) \quad s > 0 .
\] 
(83)

In order to calculating the time-derivative of the asymptotic entropy \( H(\rho_{M(s),\infty}) \), we observe that \( D(\rho_{M(s),\infty}(y)) = 0 \) implies \( \rho_{M(s),\infty} \xi(\rho_{M(s),\infty})y = 0 \) with \( \xi(y, \rho) \) as in (42), and therefore, on the support of \( \rho_{M(s),\infty} \),
\[
\xi_M(s) := \xi(\rho_{M(s),\infty}) = \begin{cases}
\ln \left( \frac{M(s)}{\sqrt{2\pi}} \right) & m = 1 , \\
\frac{m}{m-1} \lambda \beta_M(s)^{m-1} & m > 1 ,
\end{cases}
\] 
where \( \rho_{M(s),\infty}(y) > 0 \).
\] 
(84)
In the case \( m = 1 \), we calculate easily \( H(\rho_{M(s),\infty}) = M\xi_{M(s)} \), while in the case \( m > 1 \), we use \( H(\rho_{M(s),\infty}) = M(s)\xi_{M(s)} - \int (\rho_{M(s),\infty})^m dy \) and obtain by straightforward calculations

\[
\frac{d}{ds} H(\rho_{M(s),\infty}) = \frac{dM}{ds} \times \begin{cases} 
\ln \left( \frac{M(s)}{\sqrt{2\pi}} \right) + 1 & m = 1, \\
\xi_{M(s)} & m > 1.
\end{cases}
\] (85)

By combining (82) with (85) (and recalling (36) for \( m = 1 \)), we obtain for the time-derivative of the relative entropy

\[
\frac{d}{ds} H(\rho|\rho_{M(s),\infty}) = -D(\rho) - R^{-\delta} \int R^\times \left( \ln \rho - \ln \rho_{M(s),\infty} \right) dy - \int R^\times \left( \frac{y^2}{2} + \frac{m}{m-1}\rho^{m-1} - \xi_{M(s)} \right) dy \quad m = 1,
\] (86)

By the generalised logarithmic Sobolev inequality (41) and since the solution \( \rho \) decays exponentially with rate \( \min\{2, \delta\} \alpha \). Analog to section 3, we use a bootstrapping argument to improve this rate: Considering firstly the case \( m = 1 \), where

\[
\rho^2 \ln \rho - \rho^2 \ln(\rho_{M,\infty}) = \rho^2 \ln \rho - \rho^2_{M,\infty} \ln(\rho_{M,\infty}) + \ln(\rho_{M,\infty}) (\rho^2_{M,\infty} - \rho^2)
\]

\[
= \left( \zeta(y)(1 + 2 \ln \zeta(y)) + \left( -\ln \left( \frac{M}{\sqrt{2\pi}} \right) + \frac{y^2}{2} \right) (\rho + \rho_{M,\infty}) \right) (\rho - \rho_{M,\infty}),
\]

with \( \zeta(y) \) is in the interval \( (\rho(y), \rho_{M,\infty}(y)) \), we estimate with Hölder’s inequality that

\[
\int R^\times \left( \zeta(1 + 2 \ln \zeta) + (\rho + \rho_{M,\infty}) \left( -\ln \left( \frac{M}{\sqrt{2\pi}} \right) + \frac{y^2}{2} \right) \right) (\rho - \rho_{M,\infty}) dy
\]

\[
\leq \left( \|\rho\|_{\infty}^2 \|\zeta(1 + 2 \ln \zeta)\|_{\infty} + \|\rho\|_{\infty}^2 \|\rho + \rho_{M,\infty}\|_{\infty} \ln \left( \frac{M}{\sqrt{2\pi}} \right) \right) \|\rho - \rho_{M,\infty}\|_1
\]

\[
+ \|\rho\|_{\infty}^2 \|\rho + \rho_{M,\infty}\|_{\infty} \frac{1}{4} (\rho - \rho_{M,\infty}) y^4 \|\frac{1}{4} \|\rho - \rho_{M,\infty}\|_1^4.
\]

In the case \( m > 1 \), we use \( \rho^r_{M(s),\infty}\xi_{M(s)} = \rho^r_{M(s),\infty} \left( \frac{y^2}{2} + \frac{m}{m-1}\rho^{m-1}_{M(s),\infty} \right) \) to expand the r.h.s. of (86) as

\[
\int R^\times \left( (\rho^r - \rho^r_{M,\infty}) \frac{y^2}{2} + \frac{m}{m-1}(\rho^{r+m-1} - \rho^{r+m-1}_{M,\infty}) - \xi_{M}(\rho^r - \rho^r_{M,\infty}) \right) dy
\]

\[
\leq C(p, \|\zeta\|_{\infty}, \|(\rho - \rho_{M,\infty}) y^4\|_1^{\frac{1}{2}}) \|\rho - \rho_{M,\infty}\|_1^{\frac{1}{2}} + C(p, \|\zeta\|_{\infty}) \|\rho - \rho_{M,\infty}\|_1
\]

We conclude that the integral on the r.h.s of (86) decays as $\|\rho - \rho_{M(s),\infty}\|_1^\frac{1}{2}$ provided that the initial fourth order moment $\|u_0 x^4\|_1$ is bounded. Analog to section 3, this improves for $\delta < 2$ the exponential decay rate of $\|\rho - \rho_{M(s),\infty}\|_1$ to $\min\{2, \delta_1\} \alpha$ with $\delta_1 = \delta(1 + \frac{2}{\alpha})$ and, after finitely many steps, we obtain the optimal rate $2\alpha$ as stated in

**Theorem 4.1 (Decay to the time-dependent asymptotic profile)**

Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be non-negative with bounded fourth order moment $\int_{\mathbb{R}} u_0|x|^4 \, dx < \infty$. Let $U_{M(t)}$ denote the rescaled time-dependent asymptotic profile $\rho_{M(s),\infty}$. Then, the solutions $u$ of (4)-(5) satisfies

$$
\|u - U_{M(t)}\|_1 \leq C(\|u_0\|_1, \|u_0\|_\infty, \|u_0 x^4\|_1)(1 + t)^{-r_1},
$$

with a constant $C$ depending on $\|u_0\|_1$, $\|u_0\|_\infty$ and $\|u_0 x^4\|_1$ and with the $L^1$-convergence rates

$$
\begin{align*}
r_1(q, m) &= \begin{cases} 
\frac{1}{m+1} & 1 \leq m < 2, \\
\frac{2}{m(m+1)} & 2 \leq m.
\end{cases}
\end{align*}
$$

Moreover, for $p \in (1, \infty)$,

$$
\|u - U_{M(t)}\|_p \leq C(p, \|u_0\|_1, \|u_0\|_\infty, H(u_0))(1 + t)^{-r_p},
$$

with

$$
r_p = \frac{1}{m + 1} \frac{p-1}{p} + \frac{r_1}{p}. \tag{88}
$$

**Remark 4.1** The assumption $\int_{\mathbb{R}} u_0|x|^4 \, dx < \infty$ in relation to the propagation of the even moments (80) has been made for the sake of simplicity of the above arguments and can easily be relaxed to $\int_{\mathbb{R}} u_0|x|^{2+\varepsilon} \, dx < \infty$ for $\varepsilon > 0$.

**Proof:** The proof of theorem 4.1 applies similarly to theorem 3.3 the logarithmic Sobolev inequalities directly to the regularised problems, i.e. (4) on a bounded interval $(-n, n)$ with homogeneous Dirichlet data (see also [KPV]) and a sequence of data $\rho_{0,n} \searrow \rho_0$ pointwise and monotonely from above such that

$$
\|\rho_{0,n}\|_1 = \left(1 + \frac{1}{n}\right) \|\rho_0\|_1.
$$

Then, the solutions $\rho_n$ are positive and smooth on $(-n, n)$ and satisfy the mass decay

$$
\frac{d}{ds} M_n(s) \leq R^{-\delta} \int_{-n}^{n} \rho_n' \, dy,
$$

\[23\]
since $\int_{-\infty}^{\infty} (\rho_n^m)_{yy} \, dy \leq 0$. Proceeding analog to the above formal calculations, it is straightforward to show that the approximating relative entropy $H_n(\rho_n | \rho_{n,M_n(s),\infty})$ decays equally or less as stated in (86), such that a generalized logarithmic Sobolev inequality on $(-n,n)$ [CJMTU] and a bootstrapping argument analog to above imply convergence of $\rho_n$ to $\rho_{n,M_n(s),\infty}$ as stated in the theorem, which follows by the limit $n \to \infty$. □

**Remark 4.2 (Multidimensional absorption diffusion)**

The proof of theorem 4.1 can be as in theorem 3.3 generalized to the $N$-dimensional absorption-diffusion equation

$$u_t = \Delta (u^m) - u^r,$$

with $(t,x) \in [0,\infty) \times \mathbb{R}^N$ and $N > 1$, and with the convergence rates

\[
\begin{align*}
    r_1(q,m) &= \begin{cases} 
        \frac{1}{N(m-1)+2} & 1 \leq m < 2, \\
        \frac{1}{mN(m-1)+2} & 2 \leq m,
    \end{cases} \\
    r_p(q,m) &= \frac{N}{N(m-1)+2} \left( \frac{p-1}{p} + \frac{r_1}{p} \right) .
\end{align*}
\]

**Corollary 4.1 (Decay to the stationary asymptotic profile)**

Let $N \geq 1$. Let $U_{M(\infty)}$ denote the stationary asymptotic profile with mass $M(\infty)$. Then, since $\int_{\mathbb{R}} \rho^r_{M(\infty)} \, dy > 0$, it follows from (79) that

$$\|u - U_{M(\infty)}\|_1 \leq C(\|u_0\|_1, \|u_0\|_\infty, \|u_0 x^{2+\varepsilon} \|_1)(1 + t)^{-r_1},$$

with $\varepsilon > 0$ and

\[
    r_1(q,m) = \begin{cases} 
        \min \{ 1, \delta \} \frac{1}{N(m-1)+2} & 1 \leq m < 2, \\
        \min \{ \frac{2}{m}, \delta \} \frac{1}{N(m-1)+2} & 2 \leq m.
    \end{cases} \tag{89}
\]

## 5 Convergence in Wasserstein metric

The results of this section are proven formally in such a way that it should remain only a matter of adapting the existence theory by providing appropriate smoothings of the solutions such as in [RV, CGT, CT03, LT] to make them rigorous.

We are going to compare two solutions $\rho_i$ with $i = 1,2$ of

\[
\begin{align*}
    \frac{\partial \rho_i}{\partial s} &= (\rho_i y)_y + (\rho_i^m)_{yy} - \theta_i R^{-\delta} (\rho_i^q)_y, \\
    \rho_i(s = 0, y) &= u_{0,i}(y) \geq 0,
\end{align*}
\]

\[
\begin{align*}
    \theta_i \in \{ 0, 1 \}, \quad i = 1,2. \tag{90, 91}
\end{align*}
\]

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subject to non-negative initial data \( u_{0,i} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) with normalised masses \( \|u_{0,i}\|_1 = 1 \) and with all moments bound, i.e. \( \int_\mathbb{R} u_{0,i}|x|^k \, dx < \infty \) for all \( k \in \mathbb{N} \). By setting \( \theta_i = 0 \), we drop the convection term of (25) (i.e. \( \theta_i = 1 \)) and consider mass-preserving solutions \( \rho_i \) of the rescaled heat \( (m = 1) \) respectively porous medium equation \( (m > 1) \).

The propagation of the \( k \)-th order moment follows formally by multiplying (90) with \(|y|^k\), provided that the \((k-1)\)-th and the \((k-2)\)-th order moment are bound:

\[
\frac{d}{ds} \int_\mathbb{R} \rho_i |y|^k \, dy = -k \int_\mathbb{R} \rho_i |y|^k \, dy + k(k-1) \int_\mathbb{R} \rho_i^m |y|^{k-2} \, dy \\
- k \int_\mathbb{R} \rho_i^m |y|^{k-1} \delta(y) \, dy + \theta_i k R^{-\delta} \int_\mathbb{R} \rho_i^q |y|^{k-1} \text{sign}(y) \, dy,
\]

where the third term on the r.h.s. – involving the \( \delta \)-distribution – is bounded for \( k = 1 \) by the \( L^\infty \)-bound on the solution. Hence, by iteration and via a smoothing argument, all moments of \( \rho_i \) remain bounded:

\[
\int_\mathbb{R} \rho_i(s, y)|y|^k \, dy < \infty \quad s > 0, \quad k \in \mathbb{N}, \quad i = 1, 2. \tag{92}
\]

The Wasserstein \( p \)-distances of two one-dimensional probability measures \( \rho_1, \rho_2 \) with bounded moments (92) can be written as [Vil]

\[
W_p(\rho_1(s), \rho_2(s))^p = \int_0^1 |F_i^{-1}(s, z) - F_i^{-1}(s, z)|^p \, dz \quad p \in [1, \infty),
\]

in terms of the generalised inverses \( F_i^{-1} \) of the \( \rho_i \)-distribution functions \( F_i \):

\[
F_i^{-1}(s, z) = \inf \{ y : F_i(y) > z \}, \quad F_i(s, y) := \int_{-\infty}^y \rho_i \, dy, \quad i = 1, 2.
\]

**Theorem 5.1 (Decay of the Wasserstein \( p \)-distances)**

Let \( 0 \leq u_{0,i} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) with normalised masses \( \|u_{0,i}\|_1 = 1 \) and bounded moments \( \int_\mathbb{R} u_{0,i}|x|^k \, dx < \infty \) for all \( k \in \mathbb{N} \) and \( i = 1, 2 \). Then, for the solutions \( \rho_1, \rho_2 \) of (90) and independent of \( \theta_i \in \{0,1\} \)

\[
W_p(\rho_1(s), \rho_2(s)) \leq C(\|u_{0,1}\|_\infty, \|u_{0,2}\|_\infty) W_p(u_{0,1}, u_{0,2}) e^{-s},
\]

for all \( p \in [1, \infty) \) and with a constant \( C \) depending on \( \|u_{0,1}\|_\infty, \|u_{0,2}\|_\infty \). Backward rescaling yields

\[
W_p(u_1(t), u_2(t)) \leq C(\|u_0,1\|_\infty, \|u_{0,2}\|_\infty) W_p(u_{0,1}, u_{0,2}).
\]
Formal proof: Note firstly that taking derivatives of \( F_i(s, F_i^{-1}(s, z)) = z \) with respect to \( s \) and \( z \) yields the relations

\[
\frac{\partial F_i}{\partial s}(s, y) = -\rho_i(s, y) \frac{\partial F_i^{-1}}{\partial s}(s, z), \quad \rho_i(s, y) = \left( \frac{\partial F_i^{-1}}{\partial z}(s, z) \right)^{-1}, \tag{93}
\]

where \( y = F_i^{-1}(s, z) \). Next, integrating \( \int^y (90) \, dy \) yields

\[
\frac{\partial F_i}{\partial s} = y\rho_i(s, y) + (\rho^m_i)(s, y) - \theta_i R^{-\delta} \rho_i^q(s, y)
\]

and using (93) as well as \( \frac{\partial \rho_i}{\partial y} = -\frac{\partial^2 F_i^{-1}}{\partial s^2}(s, z)(\frac{\partial F_i^{-1}}{\partial z}(s, z))^{-3} \) we derive an evolution equation for \( F_i^{-1}(s, z) \):

\[
\frac{\partial F_i^{-1}}{\partial s} = -F_i^{-1} - \frac{\partial}{\partial z} \left( \frac{\partial F_i^{-1}}{\partial z} \right)^{-m} + \theta_i R^{-\delta} \left( \frac{\partial F_i^{-1}}{\partial z} \right)^{-1-q} . \tag{94}
\]

The evolution of the Wasserstein \( p \)-distances follows by subtracting (94) for \( i = 1, 2 \), multiplying with \( p |F_1^{-1} - F_2^{-1}|^{p-1} \) \( \text{sign}(F_1^{-1} - F_2^{-1}) \) and integrating:

\[
\frac{d}{ds} \int_0^1 |F_i^{-1} - F_2^{-1}|^p \, dz = -p \int_0^1 (F_1^{-1} - F_2^{-1})^p \, dz \tag{95}
\]

\[
+ p(p - 1) \int_0^1 \left( \left( \frac{\partial F_1^{-1}}{\partial z} \right)^{-m} - \left( \frac{\partial F_2^{-1}}{\partial z} \right)^{-m} \right) |F_1^{-1} - F_2^{-1}|^{p-2} \left( \frac{\partial F_1^{-1}}{\partial z} - \frac{\partial F_2^{-1}}{\partial z} \right) \, dz
\]

\[
+ pR^{-\delta} \int_0^1 \left( \theta_1 \frac{\partial F_1^{-1}}{\partial z}^{1-q} - \theta_2 \frac{\partial F_2^{-1}}{\partial z}^{1-q} \right) |F_1^{-1} - F_2^{-1}|^{p-1} \text{sign}(F_1^{-1} - F_2^{-1}) \, dz
\]

The second term on the r.h.s. of (95) is non-positive since \((x^{-m} - y^{-m})(x - y) \leq 0\), while the third term can be estimated with the \( L^\infty \)-bound (33) since \( \dot{\rho}^{-1} = \frac{\partial F_i^{-1}}{\partial z} \). With Hölder’s inequality for the third term, we obtain

\[
\frac{d}{ds} \int_0^1 (F_1^{-1} - F_2^{-1})^p \, dz \leq -p \int_0^1 (F_1^{-1} - F_2^{-1})^p \, dz
\]

\[
+ pR^{-\delta} \left\| \theta_1 \dot{\rho}_1^{q-1} - \theta_2 \dot{\rho}_2^{q-1} \right\| \infty \left( \int_0^1 (F_1^{-1} - F_2^{-1})^p \, dz \right)^{\frac{p-1}{p}} . \]

Hence, by a Gronwall lemma, \( W_p(\rho_1, \rho_2)^p \) is globally bounded for all \( p \in [1, \infty) \), which implies further that \( W_p(\rho_1, \rho_2)^p \) decays actually exponentially with rate \( \min\{p, \delta\} \). Improving iteratively the rate, we conclude the optimal rate \( p \) for the decay of \( W_p(\rho_1, \rho_2)^p \). The result stated in the theorem follows using

\[
W_p(u_1(t, x), u_2(t, x)) = W_p(\rho_1(s, y), \rho_2(s, y)) e^s ,
\]
which is easily shown via a change of variables.

Concerning our last result, we recall the finite speed of propagation in the case \( m > 1 \), which is that initially compactly supported solutions of (90) for either \( \theta_i = 0,1 \) remain compactly supported in time [Vaz98, Gil88]. Moreover, we observe that the Wasserstein p-distances \( W_p(\rho_1, \rho_2) \) are increasing with respect to \( p \) such that the limit

\[
W_\infty(\rho_1, \rho_2) := \lim_{p \to \infty} W_p(\rho_1, \rho_2) = \sup_{z \in (0,1)} \text{ess } |F_1^{-1}(s, z) - F_2^{-1}(s, z)|,
\]

is well-defined and \( W_\infty \) satisfies the same decay estimates as \( W_p \).

**Corollary 5.1 (Estimate on the propagation of the support)**

Let \( m > 1 \) and \( 0 \leq u_{0,i} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) with \( \|u_{0,i}\|_1 = 1 \) be compactly supported. Let \( \Omega_i(s) \) resp. \( \Omega_i(t) \) denote to supports of \( \rho_i(s, y) \) resp. \( u_i(t, x) \). Moreover, define \( I_i(\cdot) := \inf\{\Omega_i(\cdot)\} \) as well as \( S_i(\cdot) := \sup\{\Omega_i(\cdot)\} \). Then,

\[
\max\{|I_1(s) - I_2(s)|, |S_1(s) - S_2(s)|\} \leq C(\|u_{0,1}\|_\infty, \|u_{0,2}\|_\infty) W_\infty(u_{0,1}, u_{0,2}) e^{-s},
\]

respectively

\[
\max\{|I_1(t) - I_2(t)|, |S_1(t) - S_2(t)|\} \leq C(\|u_{0,1}\|_\infty, \|u_{0,2}\|_\infty) W_\infty(u_{0,1}, u_{0,2}).
\]

**Proof:** Since the inverse \( F_i^{-1}(s, z) \) of distributions \( F_i = \int_{-\infty}^y \rho_i dy \) with integrable compactly supported densities \( \rho_i(s, y) \) is continuous at \( z = 0,1 \), we estimate

\[
\sup_{z \in (0,1)} \text{ess } |F_1^{-1}(s, z) - F_2^{-1}(s, z)| \geq \max \{|F_1^{-1}(s, 0) - F_2^{-1}(s, 0)|, |F_1^{-1}(1) - F_2^{-1}(1)|\} \geq \max \{|I_1(s) - I_2(s)|, |S_1(s) - S_2(s)|\}.
\]

**Remark 5.1** By taking \( \theta_1 = 1 \) such that \( \rho_1 \) is the solution of the convection-diffusion equation (25) and by choosing \( \rho_2 = \rho_{1,\infty} \) the Barenblatt-Pattle solution for \( \theta_2 = 0 \), corollary 5.1 implies exponential decay of the boundaries of the support of \( \rho_1 \) towards the boundaries of the support of the Barenblatt-Pattle profile. In the original variables, this implies that the boundaries of the support remain bounded to each-other, while the solutions spread out in time at the rate given by the Barenblatt-Pattle profile.
References


