An elementary proof of the fact that tb & rot are the only classical invariants

recall: combinatorial data of Legendrian knot: (in front projection)

$$(p, n, c_{L\uparrow}, c_{L\downarrow}, c_{R\uparrow}, c_{R\downarrow})$$

p = #positive crossings, n = #negative crossings, $c_{L\uparrow}$ = # $\overset{\leftarrow}{<}$, $c_{L\downarrow}$ = # $\underset{\leftarrow}{<}$, $c_{R\uparrow}$ = # $\overset{\rightarrow}{>}$, $c_{R\downarrow}$ = # $\underset{\rightarrow}{>}$

Proposition: Let L_1, L_2 be Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ s.t.

$$tb(L_1) = tb(L_2), \quad rot(L_1) = rot(L_2)$$

Then there exist front projections of L_1 & L_2 s.t.

$$(p^1, n^1, c^1_{L\uparrow}, c^1_{L\downarrow}, c^1_{R\uparrow}, c^1_{R\downarrow}) = (p^2, n^2, c^2_{L\uparrow}, c^2_{L\downarrow}, c^2_{R\uparrow}, c^2_{R\downarrow})$$

Corollary: Let F be a Legendrian Reidemeister-move invariant function in $p, n, c_{L\uparrow}, c_{L\downarrow}, c_{R\uparrow}, c_{R\downarrow}$.
Then, if $tb(L_1) = tb(L_2)$ & $rot(L_1) = rot(L_2)$,

$$F(L_1) = F(L_2)$$

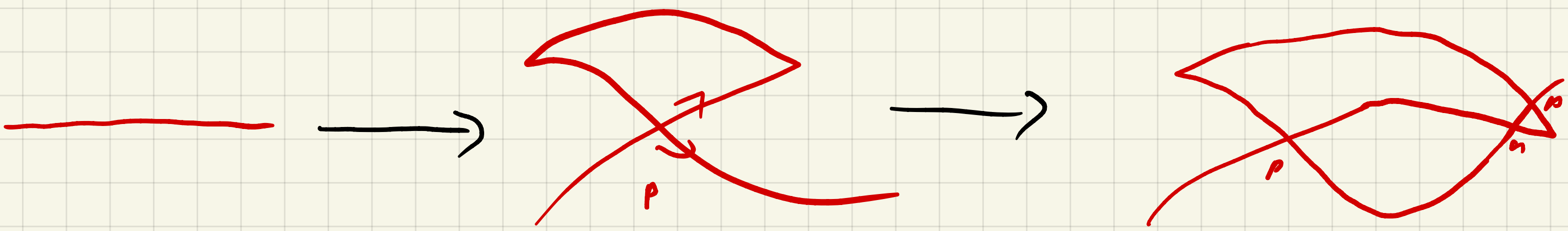
Moral: tb & rot are the only "classical invariants" !

proof of proposition:

$$tb(L_1) = tb(L_2), \qquad rot(L_1) = rot(L_2)$$
$$p^1 - n^1 - \tfrac{1}{2}c^1 \qquad p^2 - n^2 - \tfrac{1}{2}c^2$$

$$c_\downarrow = c_{L\downarrow} + c_{R\downarrow}$$
$$c_\uparrow = c_{L\uparrow} + c_{R\uparrow}$$
$$c_L = c_{L\uparrow} + c_{L\downarrow}$$
$$c_R = c_{R\uparrow} + c_{R\downarrow}$$

by using Reidemeister I & II we can achieve that $p^1 = p^2$ & $n^1 = n^2$ since



using this, achieve that $n^1 = n^2$

then use just (RI) to achieve $p^1 = p^2$

$\Rightarrow$ get equations (i) $c^1 = c^2 \Leftrightarrow c_L^1 = c_L^2 \Leftrightarrow c_R^1 = c_R^2 \Leftrightarrow c_\downarrow^1 + c_\uparrow^1 = c_\downarrow^2 + c_\uparrow^2$

& (ii) $c_\downarrow^1 - c_\uparrow^1 = c_\downarrow^2 - c_\uparrow^2$

(i) + (ii) : 1) $c_\uparrow^1 = c_\uparrow^2$

(i) − (ii) : 2) $c_\downarrow^1 = c_\downarrow^2$ also $p^1 = p^2$ & $n^1 = n^2$

3) $c_L^1 = c_L^2$

4) $c_R^1 = c_R^2$

seems like we are done ...

but we just have $(p^1, n^1, c_\uparrow^1, c_\downarrow^1, c_L^1, c_R^1) = (p^2, n^2, c_\uparrow^2, c_\downarrow^2, c_L^2, c_R^2)$

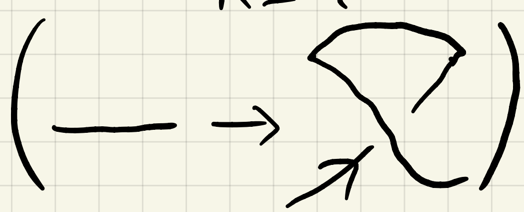
<u>but</u>: it is still possible that $\exists\, k \in \mathbb{Z}$ s.t.

1)–4) $\Rightarrow$
$$\begin{aligned}
c^1_{L\uparrow} &= c^2_{L\uparrow} + k \\
c^1_{L\downarrow} &= c^2_{L\downarrow} - k \\
c^1_{R\uparrow} &= c^2_{R\uparrow} - k \\
c^1_{R\downarrow} &= c^2_{R\downarrow} + k
\end{aligned}$$

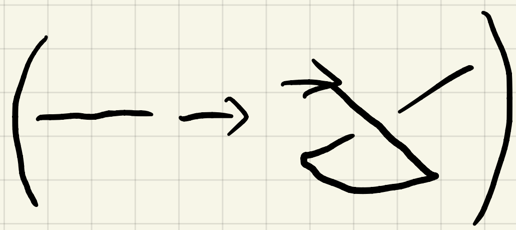
to fix this problem we do different LRI-moves on L_1 & L_2

$$\begin{aligned}
c^1_{L\uparrow} &= c^2_{L\uparrow} + k \\
c^1_{L\downarrow} &= c^2_{L\downarrow} - k \\
c^1_{R\uparrow} &= c^2_{R\uparrow} - k \\
c^1_{R\downarrow} &= c^2_{R\downarrow} + k
\end{aligned}
\quad \xrightarrow[\text{LRIb for } L_2]{\text{LRIa for } L_1}$$

LRI a



$+1\, c^2_{R\uparrow}$
$+1\, c^2_{L\downarrow}$



$+1\, c^1_{R\downarrow}$
$+1\, c^1_{L\uparrow}$

LRI b

<u>Important</u>: after this move, since LRIa & LRIb are performed simultaneously, we again have that for $\tilde{L}_1$ & $\tilde{L}_2$

$$(\tilde{p}^1, \tilde{n}^1, \tilde{c}^1_{\uparrow}, \tilde{c}^1_{\downarrow}, \tilde{c}^1_{L}, \tilde{c}^1_{R}) = (\tilde{p}^2, \tilde{n}^2, \tilde{c}^2_{\uparrow}, \tilde{c}^2_{\downarrow}, \tilde{c}^2_{L}, \tilde{c}^2_{R})$$

therefore we know that, again:

$$\widetilde{c}^1_{L\uparrow} = \widetilde{c}^2_{L\uparrow} + \tilde{k}$$
$$\widetilde{c}^1_{L\downarrow} = \widetilde{c}^2_{L\downarrow} - \tilde{k}$$
$$\widetilde{c}^1_{R\uparrow} = \widetilde{c}^2_{R\uparrow} - \tilde{k}$$
$$\widetilde{c}^1_{R\downarrow} = \widetilde{c}^2_{R\downarrow} + \tilde{k}$$

we now determine $\tilde{k}$

$$\Longrightarrow c^2_{L\uparrow} + k = c^1_{L\uparrow} = \widetilde{c}^1_{L\uparrow} = \widetilde{c}^2_{L\uparrow} + \tilde{k} = c^2_{L\uparrow} + 1 + \tilde{k}$$

so $\boxed{\tilde{k} = k-1}$

so we can repeat this process until $k=0$

$\Rightarrow$ provided $tb(L_1) = tb(L_2)$ & $rot(L_1) = rot(L_2)$ $\exists$ projections for L_1 & L_2 s.t.

$$(p^1, n^1, c^1_{L\uparrow}, c^1_{L\downarrow}, c^1_{R\uparrow}, c^1_{R\downarrow}) = (p^2, n^2, c^2_{L\uparrow}, c^2_{L\downarrow}, c^2_{R\uparrow}, c^2_{R\downarrow})$$

$\square$