On an equivalence between integral and involutive residuated structures

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In the study of residuated lattices (RLs) it sometimes happens that an involutive RL can be reconstructed from its negative cone (consisting of the elements below the unit) via a so-called twist product construction, and moreover these two transformations – restricting to the negative cone and taking a twist product – extend to a categorical equivalence between a variety of involutive RLs and a variety of integral RLs. This enables us to reduce questions about involutive RLs to questions about integral RLs, whose structure is often better understood.

Two known instances of this phenomenon are the equivalences between Abelian ℓ-groups and commutative cancellative integral divisible RLs [1] and between odd Sugihara monoids and relative Stone algebras [4]. In this contribution we subsume these two equivalences under a single, more general equivalence, extending in particular the latter example to an equivalence between a variety of idempotent involutive residuated lattices and Brouwerian algebras.[1]

In order to provide such a unifying treatment, it is in fact natural to move from RLs, i.e. residuated ℓ-monoids, to a more general setting of what we call residuated ℓ-bimonoids. Note that all monoidal operations and RLs will be assumed to be commutative throughout.

Definition 1. *A* bimonoid *consists of two partially ordered monoids (with monoidal operations $x \cdot y$ and $x + y$ and units 1 and 0) over the same poset connected by the* hemidistributive law

$$x \cdot (y + z) \leq (x \cdot y) + z.$$

A lattice-ordered bimonoid *or* ℓ-bimonoid *is an algebra which is both a lattice and a bimonoid with respect to the lattice order and moreover satisfies the compatibility conditions*

$$x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot z) \qquad \textit{and} \qquad x + (y \wedge z) = (x + y) \wedge (x + y).$$

A residuated ℓ-bimonoid *is an ℓ-bimonoid equipped with a binary operation $x \to y$ which is the residual of multiplication, i.e. satisfies the equivalence*

$$x \cdot y \leq z \textit{ if and only if } y \leq x \to z.$$

Observe in particular that any partially ordered monoid can trivially be expanded to a bimonoid by taking $x + y = x \cdot y$. In the case of semilinear RLs (such as Abelian ℓ-groups) such an expansion in fact yields a residuated ℓ-bimonoid. More interestingly, given a pointed Brouwerian algebra with a constant 0 there is a unique way of expanding it to a residuated ℓ-bimonoid, namely by taking $x + y = (0 \to x) \wedge (0 \to y) \wedge (x \vee y)$.

Bimonoids in fact form a natural setting for the study of Boolean-like complementation. Indeed it turns out that complemented ℓ-bimonoids are nothing but involutive RLs in disguise.

[1] Brouwerian algebra are essentially Heyting algebras without the assumption of the bottom element, while relative Stone algebras are semilinear Brouwerian algebras, i.e. Gödel algebras without the bottom element.

Definition 2. *Elements x and y of a bimonoid are* complements *if*

$$x \cdot y \leq 0 \qquad \text{and} \qquad 1 \leq x + y.$$

A bimonoid is complemented *if every element x has a complement $\overline{x}$.*

Every element of a bimonoid has at most one complement. Each complemented bimonoid can therefore be expanded by the map by $- : x \mapsto \overline{x}$ to a unique *bimonoid with complementation.*

Proposition 3. *Involutive residuated lattices are termwise equivalent to ℓ-bimonoids with complementation, the residual being defined as $x \to y = -x + y$ and the complement as $-x = x \to 0$.*

The crucial observation is that although by restricting in an involutive RL to the negative cone we lose complementation, addition remains a well-defined operation on the negative cone, provided only that $1 + 1 = 1$. The presence of two in principle distinct monoidal operations on the negative cone is, however, obscured in some varieties, which is why no need arises to talk about residuated ℓ-bimonoids rather than RLs: in ℓ-groups we have $x + y = x \cdot y$ in the whole algebra, while in odd Sugihara monoids we have $x + y = x \wedge y = x \cdot y$ in the negative cone.[2]

Proposition 4. *The negative cone of an involutive RL which satisfies $1 + 1 \approx 1$ is a residuated ℓ-bimonoid, the residual being defined as $x \to y = 1 \wedge (-x + y)$.*

The problem is now the following: given a residuated ℓ-bimonoid, is it isomorphic to the negative cone of an involutive RL? We divide this problem into two parts: first, to embed an ℓ-bimonoid into a complemented one, and second, to show that in some cases this construction reduces to a suitable twist product. Characterizing those cases where the negative cone of this twist product yields the original residuated ℓ-bimonoid then turns out to be an easy task.

The first part of the problem can be solved for every ℓ-bimonoid, as we can show using the technology of so-called involutive residuated frames [3].

Definition 5. *A (complete) Δ-extension of an ℓ-bimonoid $\mathbf{A}$ is a (complete) complemented ℓ-bimonoid $\mathbf{B}$ which contains $\mathbf{A}$ as a subalgebra such that every element of $\mathbf{B}$ is a finite (arbitrary) join of elements of the form $a \cdot \overline{b}$ for $a, b \in \mathbf{A}$, or equivalently meet of elements of the form $\overline{a} + b$.*

Proposition 6. *Every ℓ-bimonoid $\mathbf{A}$ has a unique complete Δ-extension, denoted $\mathbf{A}^{\Delta}$.*

To deal with the second part of the problem, we introduce some auxiliary notions.

Definition 7. *Let $\mathbf{A}$ be an integral residuated ℓ-bimonoid. Then $\langle a, b \rangle$ for $a, b \in \mathbf{A}$ is a* representation *of $x \in \mathbf{A}^{\Delta}$, symbolically $\langle a, b \rangle \sim x$, if $x = a \cdot \overline{b}$. It is a* normal representation *of x if moreover $a = 1 \wedge x$ and $b = 1 \wedge \overline{x}$. If x has a representation, we call it* representable.

Definition 8. *An integral residuated ℓ-bimonoid is called* twistable *if it satisfies the inequalities:*

$$x \to ((x \to 0) + y) \leq x \to y$$
$$(x \to (y + z)) \wedge z \leq (x \to y) + z$$
$$(xy \to 0) \cdot ((y \to x) \to (y + z)) \cdot ((x \to y) \to (x + w)) \leq z + w$$

Both cancellative integral divisible RLs and Boolean-pointed Brouwerian algebras are twistable. In the twistable case we can provide an intrinsic characterization of normal representations.

[2] A known example where formulations in terms of RLs are insufficient is provided by Sugihara monoids: the equivalence between Sugihara monoids and their negative cones is best formulated in terms of pointed RLs [2]. This is because the operation $x + y$ may in this case be recovered from the operation $x \cdot y$ and the constant 0.

Proposition 9. *Let* $\mathbf{A}$ *be a twistable residuated* ℓ*-bimonoid. Then* $\langle a, b\rangle$ *with* $a, b \in \mathbf{A}$ *is a normal representation of some* $x \in \mathbf{A}^{\Delta}$ *if and only if*

$$a \to b = b \qquad \text{and} \qquad b \to a = a \qquad \text{and} \qquad a \cdot b \leq 0.$$

Twistability is also helpful when computing with representations and comparing them.

Lemma 10. *Let* $\mathbf{A}$ *be a twistable residuated* ℓ*-bimonoid and suppose that* $\langle a, b\rangle$ *and* $\langle c, d\rangle$ *are normal representations of* x *and* y*. Then* $x \leq y$ *in* $\mathbf{A}^{\Delta}$ *if and only if* $a \leq c$ *and* $d \leq b$ *in* $\mathbf{A}$.

Lemma 11. *Let* $\mathbf{A}$ *be a twistable residuated* ℓ*-bimonoid and suppose that* $\langle a, b\rangle$ *and* $\langle c, d\rangle$ *are normal representations of* x *and* y*. Then:*

$$\begin{array}{cccc} x \cdot y \sim \langle a \cdot c, b + d\rangle & x \wedge y \sim \langle a \wedge c, b \vee d\rangle & 1 \sim \langle 1, 0\rangle & \overline{x} \sim \langle b, a\rangle \\ x + y \sim \langle a + c, b \cdot d\rangle & x \vee y \sim \langle a \vee c, b \wedge d\rangle & 0 \sim \langle 0, 1\rangle & \overline{y} \sim \langle d, c\rangle \end{array}$$

The following notion now allows us to transform arbitrary representations into normal ones.

Definition 12. *A* normalizing term $\tau = \langle \tau_{+}(x, y), \tau_{-}(x, y)\rangle$ *for a residuated* ℓ*-bimonoid is a pair of terms in the signature of residuated* ℓ*-bimonoids such that* $\tau\langle a, b\rangle = \langle \tau_{+}(a, b), \tau_{-}(a, b)\rangle$ *is normal,* $\langle a, b\rangle \sim \tau\langle a, b\rangle$*, and* $\tau\langle a, b\rangle = \langle a, b\rangle$ *if* $\langle a, b\rangle$ *is normal. A class of residuated* ℓ*-bimonoids is called* τ-normalizable *if* τ *is a normalizing term for each algebra in the class.*

For example, $\langle (x \to xy0) \to x0, x \to y0\rangle$ is a normalizing term for both cancellative integral divisible RLs and pointed Brouwerian algebras, reducing to $\langle x \to y, y \to x\rangle$ in the former case.

Putting the above lemmas and definitions together yields (most of) the following result. The algebra $\mathbf{A}^{\bowtie}$ is in fact the unique Δ-extension of $\mathbf{A}$.

Theorem 13. *Let* K *be a* τ*-normalizable variety of twistable residuated* ℓ*-bimonoids with a complemented unit interval (i.e.* $[0, 1]$ *is complemented as a bimonoid). Then for each* $\mathbf{A} \in \mathsf{K}$ *the representable elements of* $\mathbf{A}^{\Delta}$ *form a subalgebra* $\mathbf{A}^{\bowtie}$ *with the operations:*

$$\begin{array}{cccc} \langle a, b\rangle \cdot^{\bowtie} \langle c, d\rangle = \tau\langle ac, b + d\rangle & \langle a, b\rangle \wedge^{\bowtie} \langle c, d\rangle = \tau\langle a \wedge c, b \vee d\rangle & 1^{\bowtie} = \langle 1, 0\rangle & -^{\bowtie}\langle a, b\rangle = \langle b, a\rangle \\ \langle a, b\rangle +^{\bowtie} \langle c, d\rangle = \tau\langle a + c, bd\rangle & \langle a, b\rangle \vee^{\bowtie} \langle c, d\rangle = \tau\langle a \vee c, b \wedge d\rangle & 0^{\bowtie} = \langle 0, 1\rangle & -^{\bowtie}\langle c, d\rangle = \langle d, c\rangle \end{array}$$

The class $\mathsf{K}^{\bowtie} = \{\mathbf{A}^{\bowtie} \mid \mathbf{A} \in \mathsf{K}\}$ *is a variety of involutive RLs which satisfies* $x \approx (1 \wedge x) \cdot (0 \vee x)$*. The varieties* K *and* $\mathsf{K}^{\bowtie}$ *are equivalent via a twist functor and a negative cone functor.*

This theorem in fact subsumes both of the equivalences mentioned in the introduction, and moreover it yields the following new result. Here an involutive RL is called *strongly idempotent* if it satisfies both $x \cdot x \approx x$ and $x \cdot (1 \wedge x) \approx x$, and *odd* if $0 \approx 1$.

Theorem 14. *The category of pointed Brouwerian algebras with a complemented unit interval is equivalent to the category of strongly idempotent involutive RLs. The category of Brouwerian algebras is equivalent to the category of odd strongly idempotent involutive RLs.*

References

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