Purely Relevant Logics with Contraction and Its Converse

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$RMI_{\stackrel{\sim}{\rightarrow}}$ is the logic which is obtained from the sequent calculus for $R_{\stackrel{\sim}{\rightarrow}}$, the purely intensional (or "multiplicative") fragment of the relevant logic R, by viewing sequents as consisting of *finite sets of formulas* on both sides of $\Rightarrow$ (rather than multisets or sequences of formulas). This is equivalent to adding the converse of contraction (also known as "expansion" or "anti-contraction") to the usual sequential formulation of $R_{\stackrel{\sim}{\rightarrow}}$ (the latter has exchange and contraction, but not weakening, as its structural rules, and the usual multiplicative rules for negation, conjunction, disjunction, and implication as its logical rules). Thus the logical rules of $RMI_{\stackrel{\sim}{\rightarrow}}$ for its intensional conjunction (or "fusion") are:

$$(\otimes\Rightarrow)\quad \frac{\Gamma,\varphi,\psi\Rightarrow\Delta}{\Gamma,\varphi\otimes\psi\Rightarrow\Delta}\qquad \frac{\Gamma_1\Rightarrow\Delta_1,\varphi\qquad \Gamma_2\Rightarrow\Delta_2,\psi}{\Gamma_1,\Gamma_2\Rightarrow\Delta_1,\Delta_2,\varphi\otimes\psi}\quad(\Rightarrow\otimes)$$

Now it is well known that $RMI_{\stackrel{\sim}{\rightarrow}}$ is a relevant logic, enjoying major properties like the variable-sharing property, cut-elimination, and a very natural version of the relevant deduction theorem. However, it is impossible to conservatively add to it most of the other connectives that are usually considered in substructural logics, like the multiplicative propositional constants, or the additive conjunction and disjunction. Thus by enriching it with an an additive conjunction $\wedge$ for which $\varphi\wedge\psi\to\varphi$ and $\varphi\wedge\psi\to\psi$ are valid, and $\varphi\wedge\psi$ follows from φ and ψ, the mingle[1] rule (From $\Gamma_1\Rightarrow\Delta_1$ and $\Gamma_2\Rightarrow\Delta_2$ infer $\Gamma_1,\Gamma_2\Rightarrow\Delta_1,\Delta_2$) become derivable, and we get a system which has the full power of the multiplicative fragment of the semi-relevant system RM, for which the variable-sharing property fails. It follows that the substructural logic $RMI_{\stackrel{\sim}{\rightarrow}}$ is not the logic of any class of residuated structures.

Despite the last observation, $RMI_{\stackrel{\sim}{\rightarrow}}$ has a very useful and effective semantics. In fact, already in [?] we have shown that $RMI_{\stackrel{\sim}{\rightarrow}}$ is *weakly* sound and complete with respect to a certain sequence $\mathcal{A}=\{\mathcal{A}_n\mid n\in N\}$ of finite-valued matrices (where the number of truth-values in $\mathcal{A}_n$ is $n+2$). In fact, a sentence containing exactly k propositional variables is provable in $RMI_{\stackrel{\sim}{\rightarrow}}$ iff it is valid in $\mathcal{A}_k$, iff it is valid in all elements of $\mathcal{A}$. Moreover: the matrices in $\mathcal{A}$ can all be embedded in a certain effective infinite-valued matrix $\mathcal{A}_\omega$, for which $RMI_{\stackrel{\sim}{\rightarrow}}$ is weakly sound and complete. On the other hand $RMI_{\stackrel{\sim}{\rightarrow}}$ has no finite (weakly) characteristic matrix.

[1] "mix" in the terminology of Girard.

It has turned out that the semantics provided by the family $\{\mathcal{A}_n \mid 0 \leq n \leq \omega\}$ is not sufficient for characterizing the *consequence relation* of $RMI_{\overset{\sim}{\rightarrow}}$. Thus φ follows from $\varphi \otimes \psi$ according to it, although $\varphi \otimes \psi \nvdash_{RMI_{\overset{\sim}{\rightarrow}}} \varphi$. In this paper we provide another class $\mathcal{S}$ of finite-valued matrices, for which $RMI_{\overset{\sim}{\rightarrow}}$ is finitely *strongly* sound and complete. More precisely: suppose Γ is a finite set of formulas, φ is a formula, and the number of propositional variables in $\Gamma \cup \{\varphi\}$ is k. Then $\Gamma \vdash_{RMI_{\overset{\sim}{\rightarrow}}} \varphi$ iff there is no element of $\mathcal{S}$ having less then $3k$ elements, in which all formulas of Γ are valid, but φ is not (it can be shown that the $3k$ bound cannot be improved). Moreover: again all the matrices in $\mathcal{S}$ can be embedded in one effective infinite-valued matrix for which $RMI_{\overset{\sim}{\rightarrow}}$ is *strongly* sound and complete.

The structures just described are all lattices. This fact shows that it is possible to conservatively add to $RMI_{\overset{\sim}{\rightarrow}}$ *relevant* additive connectives, that can be interpreted by the lattice operations. The Gentzen-type rules that correspond to these connectives are almost identical to those of the usual additive connectives. The only difference is that the context in the rules with two premises should not be empty. For example: it is not allowed to infer $\Rightarrow \varphi \wedge \psi$ from $\Rightarrow \varphi$ and $\Rightarrow \psi$, but if $\Gamma \cup \Delta \neq \emptyset$ then it *is* allowed to infer $\Gamma \Rightarrow \varphi \wedge \psi, \Delta$ from $\Gamma \Rightarrow \varphi, \Delta$ and $\Gamma \Rightarrow \psi, \Delta$. In the resulting logic both $\rightarrow$ and $\wedge$ have the variable-sharing property. (This is the reason why we call it "*purely* relevant".)

The Gentzen-type system just described has the drawback that the cut-elimination theorem fails for it. However, cut-elimination can be restored if the following *relevant mingle* rule is added. This rule allows to infer $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ from $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ provided that $(\Gamma_1 \cup \Delta_1) \cap (\Gamma_2 \cup \Delta_2) \neq \emptyset$. It should be noted also that in order to get completeness of the Gentzen-type system for the algebraic semantics it is necessary to turn it into a hyperesequential calculus, with some additional structural rules. This hyperesequential calculus still enjoys cut-elimination, though.

Finally, the logic and calculi just described can be extended to the first-order level by using relevant versions of the usual additive quantifiers. These relevant quantifiers are obvious counterparts of the relevant additive conjunction and disjunction described above. Thus the generalization rule for $\forall$ fails, but it is possible to derive $\Gamma \Rightarrow \Delta, \forall x \varphi$ from $\Gamma \Rightarrow \Delta, \varphi$ in case $\Gamma \cup \Delta \neq \emptyset$, and x does not occur free in $\Gamma \cup \Delta$. In addition, the rule of substitution in sequents (of terms for free variables) should be included as a primitive rule of the system.

References

[1] Avron A., *Relevant entailment - semantics and formal systems*, Journal of Symbolic Logic, 49 (1984), 334-342.