# Convergence Rates of Galerkin FEM for elliptic SPDEs

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# Outline

1. Stochastic BVP
2. Karhunen-Loève (KL) Expansion
3. Reduction to High Dimensional Deterministic BVP
4. Stochastic Regularity
5. Convergence Rates of sGFEM - analytic Covariance
6. Convergence Rates of sGFEM - $H^{t,t}_{pw}(D \times D)$ Covariance
7. Conclusions
8. References

## Problem Formulation

$$D \subset \mathbb{R}^d \quad \text{bounded, Lipschitz,} \quad d = 2, 3.$$

deterministic Boundary Value Problem: given

$$a \in L^\infty(D), \operatorname{ess\,inf}_{x \in D} a(x) \geq a_0 > 0, \quad f \in H^{-1}(D) = (H_0^1(D))',$$

find $u \in H_0^1(D)$ such that

$$B(u, v) := \int_D a(x) \nabla_x u \cdot \nabla_x v dx = \int_D f(x) v(x) dx \text{ in } D \quad \forall v \in H_0^1(D). \tag{1}$$

Existence, Uniqueness, Regularity, $hp$-FEM, AFEM, ....:

## What to do if $a(x)$ is "uncertain" ?

- Accurate numerical solutions for *one* $a(x)$ are of limited use.
- Assume statistical information (joint pdf's) on data $a(x)$ available.
- Reformulate (1) as sPDE.
- Reconsider numerical solution methods for (1):
  - Given statistics of random input data (KL-expansion)
  - compute statistics of random solution (WPC-expansion)

## Stochastic BVP

Given:

- probability space $(\Omega, \Sigma, P)$ on data space $X(D) \subseteq L^\infty(D)$, $V \subseteq H^1(D)$,
- random diffusion coefficient $a(x,\omega) \in L^2(\Omega, dP; X(D))$,
- deterministic source term $f \in H^{-1}(D) = (H_0^1(D))'$,

(sBVP) Find $u(x,\omega) \in L^2(\Omega, dP; H_0^1(D))$ satisfying

$$\mathbb{E}\left[\int_D a(x,\omega)\nabla_x u \cdot \nabla_x v dx\right] = \mathbb{E}\left[\int_D f(x) v dx\right] \quad \forall v \in L^2(\Omega, dP; H_0^1(D))$$

if $a(\cdot,\omega) \in L^\infty(D)$ and if $\text{essinf} a(\cdot,\omega) \geq a_0 > 0$ a.s., then ex. unique $u \in L^2(\Omega, dP; H_0^1(D))$.

# Monte Carlo

Sampling (sBVP): Each 'sample' = 1 deterministic BVP

1. Generate (in parallel) $N_\Omega$ data "samples" $\{a_j(x)\}_{j=1}^N$,

2. Solve (in parallel) the $N_\Omega$ dBVPs

$$-\nabla_x \cdot (a_j(x) \nabla_x u_j) = f \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad \partial D$$

3. **Estimate** $k$-point correlations $\mathcal{M}^k u$ from solution ensemble $\{u_j(x)\}_{j=1}^N$ ($k = 1$: estimate mean field $\mathbb{E}[u]$ from sample average $\mathbb{E}^{N_\Omega}[u]$).

Assume $u \in L^\alpha(\Omega, V)$ for some $\alpha \in (1, 2]$ with $V = H_0^1(D)$.

Then ex. $C(\alpha)$ such that for every $N_\Omega \geq 1$ and every $0 < \varepsilon < 1$

$$P\left( \|\mathbb{E}[u] - \mathbb{E}^{N_\Omega}[u]\|_V \leq C \frac{\|u\|_{L^\alpha(\Omega,V)}}{\varepsilon^{1/\alpha} N_\Omega^{(\alpha-1)/2}} \right) \geq 1 - \varepsilon$$

# Karhunen-Loève expansion

- separation of deterministic and stochastic variables -

**Proposition 1 (Karhunen-Loève)**
If $a \in L^2(\Omega, dP; L^2(D))$ then

$$a(x,\omega) = \mathbb{E}[a](x) + \sum_{m\geq 1} \sqrt{\lambda_m}\varphi_m(x)Y_m(\omega)$$

in $L^2(\Omega, dP; L^2(D))$ where
$(\lambda_m, \varphi_m)_{m\geq 1}$ eigenpair sequence of **covariance operator**

$$\mathcal{C}[a] : L^2(D) \to L^2(D) \qquad (\mathcal{C}[a]v)(x) := \int_D C[a](x,x')v(x')dx' \quad \forall v \in L^2(D)$$

$(Y_m)_{m\geq 1}$ vanishing mean, pairwise uncorrelated rv's

$$Y_m(\omega) = \frac{1}{\sqrt{\lambda_m}} \int_D (a(x,\omega) - \mathbb{E}[a](x))\varphi_m(x)dx : \Omega \to \Gamma_m \subseteq \mathbb{R} \quad m = 1, 2, ...$$

## Karhunen-Loève expansion

- convergence -

$$a(x,\omega) = E_a(x) + \sum_{m\geq 1} \sqrt{\lambda_m}\varphi_m(x)Y_m(\omega)$$

KL expansion converges in $L^2(D \times \Omega)$, not necessarily in $L^\infty(D \times \Omega)$

To ensure $L^\infty(D \times \Omega)$ convergence, must

- estimate decay rate of KL eigenvalues $\lambda_m$: Sc & Todor JCP (2006),

- bound $||\varphi_m||_{L^\infty(D)}$: (Todor Diss ETH (2005), SINUM (2006))

- **assume**: bounds for $||Y_m||_{L^\infty(\Omega)}$

## Karhunen-Loève expansion

- eigenvalue estimates -

Regularity of $C_a$ ensures decay of KL-eigenvalue sequence $(\lambda_m)_{m\geq 1}$

$C[a](x,x') : D \times D \to \mathbb{R}$ is

- **piecewise analytic on** $D \times D$ if ex. **smoothness partition** $\mathcal{D} = \{D_j\}_{j=1}^{J}$ of $D$ into a finite sequence of simplices $D_j$ such that

$$\overline{D} = \bigcup_{j=1}^{J} \overline{D_j} \tag{2}$$

  and such that $C[a](x,x')$ is analytic in an open neighbourhood of $\overline{D_j} \times \overline{D_{j'}}$ for any pair $(j,j')$.

- **piecewise $H^{t,t}$ on $D \times D$** if

$$V_a \in H^{t,t}_{pw}(D \times D) := \bigcap_{i,j \leq J} L^2(D_i, H^t(D_j))$$

## Karhunen-Loève expansion

- eigenvalue estimates -

- $(H, \langle \cdot, \cdot \rangle)$ Hilbert space,
- $\mathcal{C} \in \mathcal{K}(H)$ compact, s.a.,
- eigenpair sequence $(\lambda_m, \phi_m)_{m \geq 1}$.

If $\mathcal{C}_m \in \mathcal{B}(H)$ is any operator of rank at most $m$,

$$\lambda_{m+1} \leq \|\mathcal{C} - \mathcal{C}_m\|_{\mathcal{B}(H)}. \tag{3}$$

## Karhunen-Loève expansion

- eigenvalue estimates -

**Theorem 2** (KL-eigenvalue decay)

- ($>$ exponential KL decay: *Gaussian* $C_a(x, x')$)

$$C_a(x, x') := \sigma^2 \exp(-\gamma|x - x'|^2) \Longrightarrow \quad 0 \leq \lambda_m \leq c(\gamma, \sigma)/m! \quad \forall m \geq 1$$

- (exponential KL decay: *Piecewise analytic* $C_a(x, x')$)

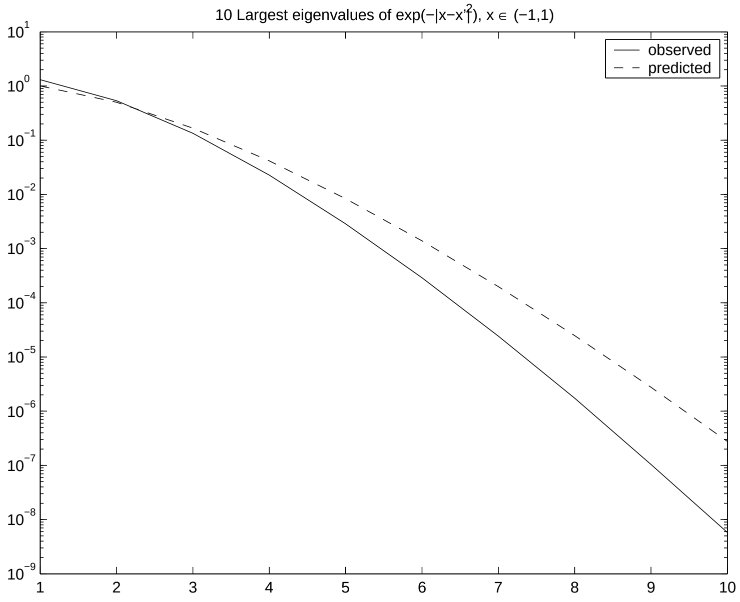
$$C_a \text{ pw analytic on } D \times D \Longrightarrow \exists c > 0 \quad 0 \leq \lambda_m \leq c \exp(-bm^{1/d}) \quad \forall m \geq 1$$

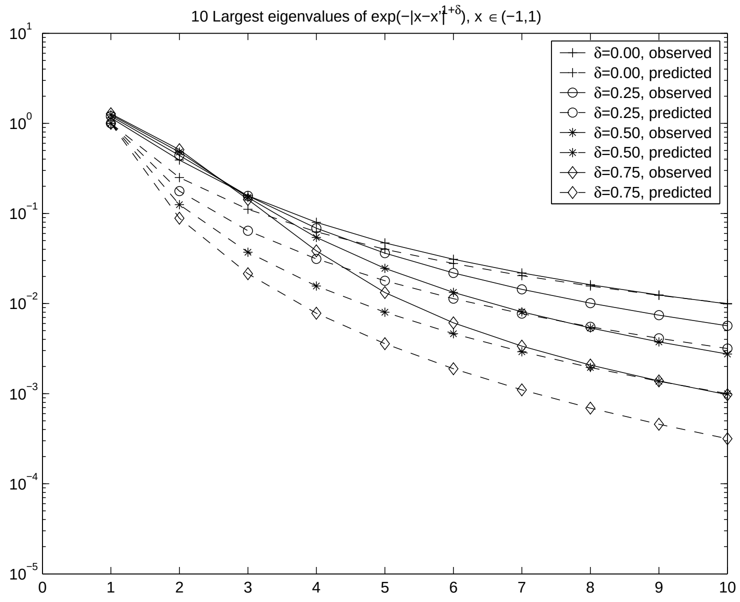
- (Algebraic KL-eigenvalue decay for p.w. $H^t(D)$-kernels)

$$C_a \in H^{t,t}_{pw}(D \times D) \ (t \geq d/2) \Longrightarrow \quad 0 \leq \lambda_m \leq cm^{-t/d} \quad \forall m \geq 1$$

# Karhunen-Loève expansion

- eigenvalue estimates -



# Karhunen-Loève expansion

- eigenfunction estimates -

Regularity of $C_a$ ensures $L^\infty$ bounds for $L^2$-scaled eigenfunctions $(\varphi_m)_{m\geq 1}$

**Theorem 3 (C.S. & Todor JCP 2006)**
Assume

$$C_a \in H^{t,t}_{pw}(D \times D) \quad \text{for} \quad t > d.$$

Then

$$\forall \delta > 0 \quad \text{ex.}\, C(\delta) > 0 \quad s.t. \quad \forall m \geq 1 : \|\phi_m\|_{L^\infty(D)} \leq C(\delta)\lambda_m^{-\delta}.$$

Hence:

$$\forall \delta > 0 \quad \text{ex. } C(\delta) > 0 \quad s.t. \quad \forall m \geq 1 : b_m := \lambda_m^{1/2}\|\phi_m\|_{L^\infty(D)} \leq C(\delta)\lambda_m^{1/2-\delta}$$

## Karhunen-Loève expansion

- convergence rate -

Conclusion:
KL expansion of

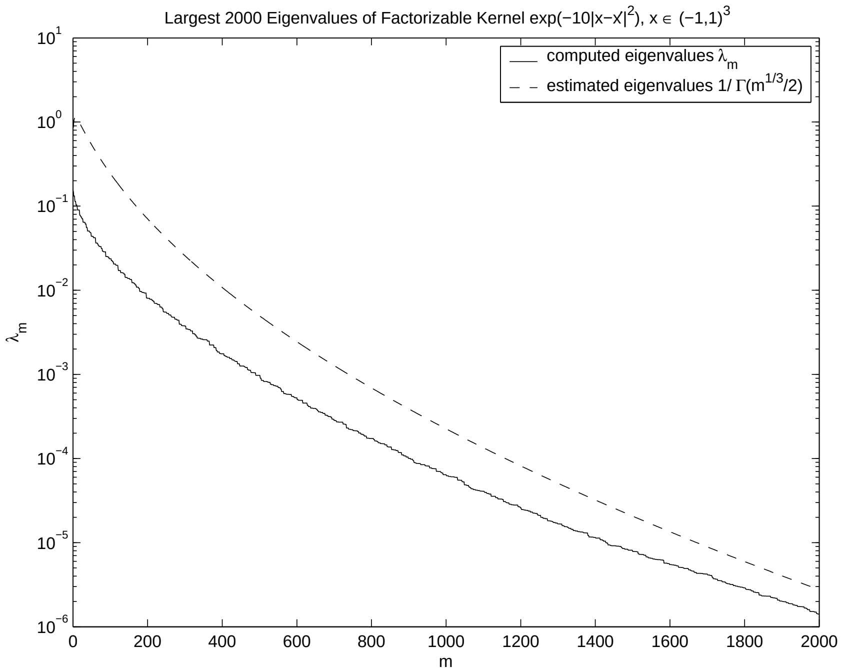
$$a(x,\omega) \in L^2(\Omega, dP; L^\infty(D))$$

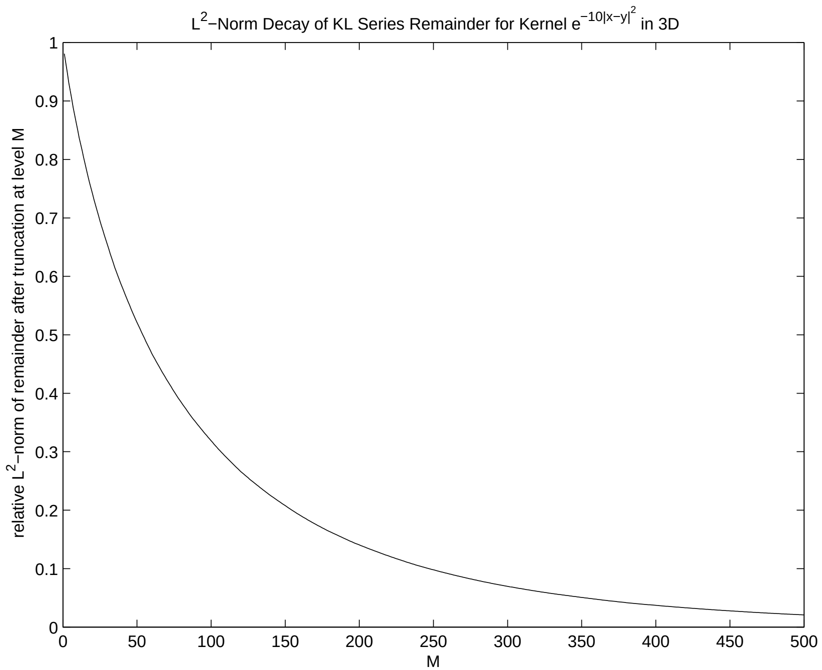
converges uniformly and exponentially on $D \times \Omega$ if
- $C_a(x,x')$ piecewise analytic
- $(Y_m(\omega))_{m\geq 1}$ uniformly bounded on $\Omega$

(e.g. $Y_m(\omega)$ uniformly distributed in $(-1,1)$)

# Karhunen-Loève expansion

- convergence rate -



# Karhunen-Loève expansion

- truncation from infinite to finite dimension $M$ -

$\infty > M \in \mathbb{N}$ KL-truncation order

$$a_M(x,\omega) := E_a(x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\varphi_m(x)Y_m(\omega)$$

$(SBVP)$ with stochastic coefficient $a(x,\omega)$

$$-\mathrm{div}(a(x,\omega)\nabla_x u(x,\omega)) = f(x) \quad \text{in } L^2(\Omega, dP; H^{-1}(D))$$

$(SBVP)_M$ with truncated stochastic coefficient $a_M(x,\omega)$

$$-\mathrm{div}(a_M(x,\omega)\nabla_x u_M(x,\omega)) = f(x) \quad \text{in } L^2(\Omega, dP; H^{-1}(D))$$

**Theorem 4** If $C_a$ pw analytic and $(Y_m)_{m\geq 1}$ uniformly bounded, then $\forall \delta > 0$ ex. $b, C(\delta), M_0 > 0$ such that $(SBVP)_M$ well-posed and, for $M \geq M_0$,

$$\|u - u_M\|_{L^2(\Omega;H_0^1(D))} \leq \begin{cases} C\exp(-bM^{1/d}) & \forall M \geq M_0 \quad \text{if } C_a \text{ pw analytic} \\ C(\delta)M^{-t/2d+1-\delta} & \forall M \geq M_0 \quad \text{if } C_a \in H_{pw}^{t,t}(D\times D) \end{cases}$$

## Reduction to high dimensional deterministic bvp

$$a_M : D \times \Omega \to \mathbb{R}, \quad a_M(x,\omega) = \mathbb{E}[a](x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\varphi_m(x)Y_m(\omega)$$

**Assumption**
$(Y_m)_{m\geq 1}$ independent, uniformly bounded family of rv's
(e.g. $Y_m$ uniformly distributed in $\Gamma_m = I = (-1/2, 1/2)$, $m = 1, 2, 3, ...$)

$$\begin{array}{rcl} \text{Random variable } Y_m & \longrightarrow & \text{Parameter } y_m \in I \\ (Y_1, Y_2, \ldots, Y_M) & \longrightarrow & y = (y_1, y_2, \ldots, y_M) \in I^M \\ dP \quad = \rho(y)dy & = & \bigotimes_{m\geq 1} \rho_m(y_m)dy_m \end{array}$$

$$\tilde{a}_M : D \times I^M \to \mathbb{R}, \quad \tilde{a}_M(x,y) = \mathbb{E}[a](x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\varphi_m(x)y_m$$

## Reduction to high dimensional deterministic bvp

stochastic bvp

$$-\text{div}(a_M(x,\omega)\nabla_x u_M(x,\omega)) = f(x) \quad \text{in } H^{-1}(D), \quad P-\text{a.e. } \omega\in\Omega$$

parametric deterministic bvp

$$-\text{div}(\tilde{a}_M(x;y_1,y_2,...,y_M)\nabla_x \tilde{u}_M(x,y)) = f(x) \quad \text{in } H^{-1}(D), \quad \forall y\in I^M$$

**Proposition 5**
Under **Assumption**, the parametric deterministic bvp is well-posed and

$$u_M(x,\omega) = \tilde{u}_M(x, Y_1(\omega), Y_2(\omega), \ldots, Y_M(\omega))$$

## Reduction to high dimensional deterministic bvp

- stochastic semi-discretization -

$$\tilde{a}_M(x,y) = E_a(x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\varphi_m(x)y_m$$

parametric deterministic bvp:
find $\tilde{u}_M \in L^2_\rho(I^M; H^1_0(D))$ such that $\forall v \in L^2_\rho(I^M; H^1_0(D))$ :

$$\int_{I^M}\left(\int_D \tilde{a}_M(x,y)\nabla_x \tilde{u}_M(x,y)\cdot\nabla_x v(x,y)dx\right)\rho(y)dy = \int_{I^M}\int_D f(x)v(x)dx\rho(y)dy$$

Galerkin semi-discretization in $y$ (sGFEM):

$$V^M \subset L^2(I^M), \quad \hat{N} = \mathrm{dim} V^M < \infty \quad dBVPs$$

find $\tilde{U}_M \in V^M \otimes H^1_0(D)$ such that $\forall v \in V^M \otimes H^1_0(D)$:

$$\int_{I^M}\left(\int_D \tilde{a}_M(x,y)\nabla_x \tilde{U}_M(x,y)\cdot\nabla_x v(x,y)dx\right)\rho(y)dy = \int_{I^M}\int_D f(x)v(x)dx\rho(y)dy$$

## sGFEM for high dimensional deterministic bvp

- stochastic semi-discretization -

**Quasi-Optimality**:

$$\|u - \tilde{U}_M\|_{L^2(\Omega, dP; H_0^1(D))} \leq C \inf_{v \in V^M \otimes H_0^1(D)} \|u - v\|_{L^2(\Omega, dP; H_0^1(D))}$$

$\tilde{a}_M(x, y)$ affine in $y \Rightarrow \tilde{u}_M(x, y)$ analytic in $y \Rightarrow V^M$polyn. space w.r.to $y$

task: solve dbvp with KL-accuracy* $O(\exp(-cM^{1/d}))$ in "low complexity"**

*how to choose the polynomial space $V^M = \mathcal{P}(I^M)$ in $y = (y_1, y_2, \ldots, y_M)$ ?

**how to choose a basis $\mathcal{B}$ of $\mathcal{P}$ ?

# sGFEM for high dimensional deterministic bvp

- stochastic semi-discretization: p.w. analytic $C_a$ -

**'ANOVA' type Product Spaces in $I^M$:**

For $M, \mu \geq 0, \nu << M \in \mathbb{N}_0$ define index set

$$\Lambda^M_{\mu,\nu} := \{\alpha \in \mathbb{N}_0^M \mid |\alpha|_1 \leq \mu, \quad |\alpha|_0 \leq \nu\} \subset \mathbb{N}_0^M, \tag{4}$$

**polynomial subspace** (Wiener's "Polynomial Chaos", N. Wiener (1938))

$$\widehat{V}^M = \mathcal{P}_{\Lambda^M_{\mu,\nu}}(I^M) := \text{span}\{\mathbf{y}^\alpha \mid \alpha \in \Lambda^M_{\mu,\nu}\} \subset L^2(I^M), \tag{5}$$

**MRA subspace** ($1-d$ polynomial multiwavelets of degree $p \geq 0$):

$$1d\text{-MRA}: \quad V^\ell = W^0 \oplus W^1 \oplus W^2 \oplus ... W^\ell \subset L^2(-1,1), \quad \ell = 0,1,2,...$$

$$\widehat{V}^M = \widehat{V}^M_{\mu,\nu} := \bigoplus_{\alpha \in \Lambda^M_{\mu,\nu}} \bigotimes_{i=1}^{M} W^{\alpha_i} \subset L^2(I^M). \tag{6}$$

## sGFEM for high dimensional deterministic bvp

- stochastic semi-discretization: p.w. analytic $C_a$ -

**Theorem 6 (Todor + Sc IMA Journ Numer. Anal. (2007))**
If ex. $b, C, \kappa > 0$ s.t.

$$\lambda_m \leq C \exp(-bm^\kappa) \qquad m \to \infty,$$

ex. $c_3, c_4, c_r > 0$ such that for **MRA subspace** $\widehat{V}^M_{\mu,\nu}$ with degree $p \geq 0$ and

$$\mu = \lceil c_3 M^\kappa \rceil, \quad \nu = \lceil c_4 M^{\kappa/(\kappa+1)} \rceil \tag{7}$$

holds, as $M \to \infty$

$$N = \dim \widehat{V}^M_{\mu,\nu} \leq C \exp(\frac{c_r}{p+1} M^\kappa + o(M^\kappa)). \tag{8}$$

and

$$\|\tilde{u}_M - P_{\widehat{V}^M_{\mu,\nu}} \tilde{u}_M\|_{L^2(I^M, H^1_0(D))} \leq C \exp(-c_r M^\kappa + o(M^\kappa)) \leq C N^{-(p+1)} \tag{9}$$

Note the same constant $c_r$ appears in (9) and (8) respectively.

## sGFEM for high dimensional deterministic bvp

- stochastic semi-discretization: p.w. analytic $C_a$ -

For the **polynomial subspace** $\mathcal{P}_{\Lambda^M_{\mu,\nu}}(I^M) \otimes H^1_0(D) = \mathcal{P}_{\Lambda^M_{\mu,\nu}}(I^M, H^1_0(D))$
i. ($\mathcal{P}$): ex. $b, \widehat{c} > 0$ s.t.

$$\inf_{v \in \mathcal{P}_{\Lambda^M_{\mu,\nu}} \otimes H^1_0(D)} \|\tilde{u}_M - v\|_{L^\infty(I^M; H^1_0(D))} \lesssim \exp(-bM^{1/d})$$

$$N := \dim \mathcal{P}_{\Lambda^M_{\mu,\nu}} \lesssim \exp(\widehat{c} M^{1/(d+1)} \log(M)) \qquad (10)$$

ii. sGFEM converges w. **spectral rate**:

$$\forall s > 0 : \quad ex.C(s) \quad s.t. \inf_{v \in \mathcal{P}^M_{\mu,\nu} \otimes H^1_0(D)} \|\tilde{u}_M - v\|_{L^\infty(I^M; H^1_0(D))} \lesssim C(s) N^{-s}$$

iii. ($\mathcal{B}$): In $L^2$-ONbasis of $\mathcal{P}_{\Lambda^M_{\mu,\nu}}$ the stiffness matrix of $(sBVP)_M$ in $I^M$ is well-conditioned and sparse (at most $O(M)$ nontrivial "entries" / row)

## High dimensional deterministic bvp

- stochastic semi-discretization: $C_a \in H^{t,t}_{pw}(D \times D)$ -

Recall: $C_a \in H^{t,t}_{pw}(D \times D)$ yields:

$$\lambda_m \lesssim m^{-s}, \qquad 1 < s = t/d, \qquad m = 1, 2, \ldots$$

KL - convergence rate: if $t > 2d$ then ex. $M_0 > 0$ such that

$$\|u - u_M\|_{L^2(\Omega; H^1_0(D))} \lesssim \|a - a_M\|_{L^2(\Omega; L^\infty(D))} \le CM^{-s} \quad \forall M \ge M_0, \quad 0 < s < t/2d + 1.$$

Convergence rate of sGFEM of $O(N^{-s'})$ possible? Which $s' > 0$?

# High dimensional deterministic bvp

- stochastic semi-discretization: $C_a \in H^{t,t}_{pw}(D \times D)$ -

Notation:

- $\Lambda = \{$all sequences $\mu = (\mu_n)_{n=1}^{\infty} \subset \mathbb{N}_0$ w. finite support $\} \subset \mathbb{N}_0^{\mathbb{N}}$
- For $\mu \in \Lambda$, denote $\operatorname{supp} \mu = \{n \in \mathbb{N} : \mu_n \neq 0\}$
- $$|\mu|_0 := \# \operatorname{supp} \mu < \infty, \quad \mu \in \Lambda. \tag{11}$$
- $\Lambda$ countable: for each $K = 1, 2, \ldots,$ define
$$\Lambda(K) := \{\mu \in \Lambda : \operatorname{supp} \mu \subset \{1, \ldots, K\}\} \subset \Lambda.$$
Then $\Lambda = \bigcup_{K=1}^{\infty} \Lambda(K)$.
- $$|\mu|_1 = \sum_{n \geq 1} \mu_n < \infty \quad \text{since} \quad \mu \in \Lambda.$$

## High dimensional deterministic bvp

- stochastic semi-discretization: $C_a \in H^{t,t}_{pw}(D \times D)$ -

If $C_a \in H^{t,t}_{pw}(D \times D)$, then for any $s < t/2d$,

$$b_m := \lambda_m^{1/2} \|\varphi\|_{L^\infty(D)} \lesssim m^{-s}, \quad m = 1, 2, ...$$

For each $\mu \in \Lambda$,

$$b^\mu := \prod_{m \geq 1} b_m^{\mu_m} = b_1^{\mu_1} b_2^{\mu_2} \cdots .$$

# High dimensional deterministic bvp

- stochastic semi-discretization: $C_a \in H^{t,t}_{pw}(D \times D)$ -

## Stochastic Regularity

Let

$$U = (-1,1)^\infty = B_1(\ell_\infty), \qquad V = H^1_0(D).$$

If $u(y,\cdot) : U \to V$ solves (sBVP), then exists $c > 0$ (depending only on the ellipticity constant $\gamma > 0$ in $0 < \gamma < a(x,\omega) < 1/\gamma$) such that

$$\forall \mu \in \Lambda : \quad \sup_{\vec{y} \in U} \|\partial_y^\mu u(\vec{y},\cdot)\|_V \le c^{|\mu|_1 + 1} |\mu|_1! b^\mu \|f\|_{L^2(D)}$$

## High dimensional deterministic bvp

- stochastic semi-discretization: $C_a \in H^{t,t}_{pw}(D \times D)$ -

**Goal:** convergence rate of sGFEM.

find finite sets $\Lambda_0 \subset \Lambda$ of "relevant" monomials $y^\mu$, $\mu \in \Lambda_0$ of cardinality $N = \#\Lambda_0 \to \infty$, and "coefficients" $\psi_\mu \in V$ and estimate

$$\|u - \sum_{\mu \in \Lambda_0} y^\mu \psi_\mu\|_{L^\infty(U,V)} \leq C(s) N^{-s(t)},$$

with, hopefully, $s(t) > 1/2$ (MCM).

## High dimensional deterministic bvp

- stochastic semi-discretization: $C_a \in H^{t,t}_{pw}(D \times D)$ -

Assume:

$$b_n \leq \gamma, \quad n = 1, 2, \ldots, \tag{12}$$

where $\gamma < 1$ and

$$b_n \leq C_0 n^{-s}, \quad n = 1, 2, \ldots, \tag{13}$$

where $s > 0$ and $C_0 > 0$ are fixed constants.

**Theorem 7** Under assumptions (12) and (13), for any $\tau > 1/s$ the sequence $\{b^{\tau\mu} : \mu \in \Lambda\}$ is in $\ell_1(\mathbb{N})$, i.e. there exists $C(\gamma, s)$ such that

$$\sum_{\mu \in \Lambda} b^{\tau\mu} = \sum_{\mu \in \Lambda} \phi^{\tau}_{\mu} \leq C(\gamma, s) \tag{14}$$

where $C(\gamma, s)$ depends only on $\gamma$ (as $\gamma \to 1$), $s$, and on $\tau$ (as $\tau \to 1/s$).

## High dimensional deterministic bvp

- stochastic semi-discretization: $C_a \in H^{t,t}_{pw}(D \times D)$ -
$\ell_1$-approximation

$\Sigma_N$ set of all sequences which have at most $N$ non-zero coordinates. Given $x = (x_j)_{j=1}^{\infty}$, define error of $N$-term approximation

$$\sigma_N(x) := \inf_{y \in \Sigma_N} \|x - y\|_{\ell_1}. \tag{15}$$

Recall: $\sigma_N(x) \leq CN^{-r}$ iff $x \in \ell^w_\tau$ with $\tau := (r+1)^{-1}$.

If moreover $x \in \ell_\tau$, then

$$\sigma_N(x) \leq \|x\|_{\ell_\tau} N^{-r}, \quad N = 1, 2, \ldots \tag{16}$$

with $r = 1/\tau - 1$.

# High dimensional deterministic bvp

- stochastic semi-discretization: $C_a \in H^{t,t}_{pw}(D \times D)$ -
$\ell_1$-approximation

**Corollary 8**

Suppose sequence $b$ satisfies assumptions of Theorem 7 for some $s > 1$.

If $r < s - 1$, then for any $N \in \mathbb{N}$, there is $\Lambda_0 \subseteq \Lambda$ of size at most $N$ such that

$$\sum_{\mu \in \Lambda \setminus \Lambda_0} b^\mu \leq C(r, b) N^{-r} \qquad (17)$$

where $C(r, b)$ is independent of $N$.

**Remark** For $C_a \in H^{t,t}_{pw}(D \times D)$ may take any

$$s < t/2d, \quad \text{resp.} \quad r < t/2d - 1.$$

## High dimensional deterministic bvp

- stochastic semi-discretization: $C_a \in H^{t,t}_{pw}(D \times D)$ -

### Rate of Convergence of sGFEM

Consider $u : U \to V$ of the form

$$u(y) = \sum_{\mu \in \Lambda} y^\mu \psi_\mu, \quad y \in U, \tag{18}$$

where $\psi_\mu \in V$ and

$$\|\psi_\mu\|_V \le b^\mu, \quad \mu \in \Lambda. \tag{19}$$

**Theorem 9** Suppose that $u$ is a function of the form (18) satisfying (19).

If $b = (b_1, b_2, \dots)$ satisfies the assumptions of Theorem 7 for some $s > 1$, then for each $r < s - 1$, and each $N \ge 1$, there is $\Lambda_0 \subset \Lambda$ of cardinality $N$ such that

$$\|u - \sum_{\mu \in \Lambda_0} y^\mu \psi_\mu\|_{L^\infty(U;V)} \le C(r) N^{-r}. \tag{20}$$

Note: for $C_a \in H^{t,t}_{pw}(D \times D)$, have assumptions in Theorem 7 with $s < t/2d$.

Hence *best* $N$*-term adaptive sGFEM* converges, *assuming exact solution of the deterministic problems*, with any rate $r < t/2d - 1$ in terms of $N_\Omega$, the number of deterministic problems to be solved.

## High dimensional deterministic bvp

- stochastic semi-discretization: $C_a \in H^{t,t}_{pw}(D \times D)$ -

Issues:

- Localization of $\Lambda_0$ in $O(\#\Lambda_0)$.
- Selection of polynomial basis.
- So far, only semidiscrete approximation.
  *Fully Discrete sGFEM* needs (A)FEM in $D$: $\psi_\mu \to \psi^L_\mu \in V^L \subset V$.

## High dimensional deterministic bvp

- stochastic semi-discretization: Fully discrete sGFEM -

Regularity of deterministic problem:

- Smoothness Scale of approximation spaces:

$$V = \mathcal{A}^0 \supset \mathcal{A}^1 \supset \mathcal{A}^2 \supset \mathcal{A}^s \dots$$

  Examples:

  1. (Isotropic Sobolev Scale, $h$-FEM in $D$ on quasiuniform meshes)

  $$\mathcal{A}^0 = H_0^1(D), \quad \mathcal{A}^s = H^s(D) \cap H_0^1(D), \quad s > 1.$$

  2. (Weighted Kondrat'ev Scale, $h$-FEM in $D$ on graded meshes)

  $$\mathcal{A}^0 = H_0^1(D), \quad \mathcal{A}^s = V_\beta^s(D) \cap H_0^1(D), \quad s > 1 \quad \text{integer.}$$

  3. (Besov Scale, $h$-AFEM in $D$)

  $$\mathcal{A}^0 = H_0^1(D), \quad \mathcal{A}^s = B_{2,\infty}^s(D) \cap H_0^1(D), \quad s > 1.$$

- Spatial regularity at order $s' > 1$:

$$u(y) = \sum_{\mu \in \Lambda} y^\mu \psi_\mu, \quad y \in U, \quad \psi_\mu \in \mathcal{A}^{s'} \subset V. \tag{21}$$

# High dimensional deterministic bvp

- stochastic semi-discretization: Fully discrete sGFEM -

**Proposition 10** (Spatially discete sGFEM /full tensor approximation):

Assume

1. spatial regularity (21) of order $s' > 1$: $\psi_\mu \in \mathcal{A}^{s'}$,

2. stochastic regularity of order $t$:

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t > 2d,$$

3. spatial hierarchical approximation scale:

$$V_0 \subset V_1 \subset V_2 \subset ... \subset V$$

with uniformly bounded, $V$-stable and quasioptimal projectors $P_\ell : V \to V_\ell$ and

$$N_{D,\ell} := \dim V_\ell = O(2^{\ell d}), \qquad \ell \to \infty.$$

Then, for every $N_\Omega \in \mathbb{N}$ ex. $\Lambda_0 \subset \Lambda$ with $\#\Lambda_0 \lesssim N_\Omega$ and

$$\|u - \sum_{\mu \in \Lambda_0} y^\mu P_\ell \psi_\mu\|_{L^\infty(U;V)} \lesssim N_\Omega^{-r} + N_D^{-s'/d}$$

with *total 'number of DOF'*

$$N_{total} = N_\Omega N_D$$

## Conclusion

- Elliptic PDE with stochastic coefficients:
  Variational Formulation, Existence, Uniqueness

- Karhúnen - Loève Expansion of $L^2$-Random Input Data:
  Exponential pointwise convergence for p.w. analytic $C_a$,
  Algebraic pointwise convergence for $C_a \in H^{t,t}_{pw}$,

- Fast Computation of KL-expansion in general domains $D \subset \mathbb{R}^3$
  by gFMM ala Rokhlin and Greengard, $\mathcal{H}$-Matrix techniques, ...

- Transform SPDE into parametric, deterministic PDE on $U = (-1, 1)^\infty$

- Truncation to $M < \infty$ dimensions; conditional expectation; error estimates.

- Convergence Rates of $h$-, $p$- type sGFEM for sparse tensor approximations of

$$B_1(\ell_\infty) \ni y \to u(y, \cdot) \in V$$

- Stochastic Regularity of random solution = domain of analyticity of parametric, deterministic problem

- Algebraic Conv. of $h$-sGFEM and Spectral Conv. of $p$-sGFEM as

$$h \to 0, \quad p \to \infty, \quad M \to \infty.$$

- Diffusion problems in physical dimension $d = 2$, with $M = 80$ on PC

- Sparse collocation on input-KL adapted 'lattices' of integration points in $(-1,1)^M$ (C.S. and Todor IMAJNA (2007))

- Sparse tensorization of $x$- and $y$-Galerkin discretizations:

$$N_{total} \simeq N_\Omega \log N_D + N_D \log N_\Omega$$

- Learning $a(x,\omega)$ from measurements and forward solves?

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